Generalized principal extensions in topological algebras

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Abstract. The "*generalized principal extension*" of a *subset of a topological algebra* is, by definition, a subset of another topological algebra. Hence, the *generalized spectrum of an element*, "local spectrum", of a topological algebra is thus a subset of a topological algebra, not necessarily of $\mathbb{C}$. We also give a criterion for the *continuity of* the (generalized) *Newburg map* (Theorem 3.1).

1. Introduction and preliminaries

The present paper is a continuation of previous work of the author [4-7], pertaining to an extension in topological algebras, in general (*not normed* ones) [10], of the classical work of S.T.M. Ackermans on "*principal extensions of Riemann surfaces*" in Banach algebras (see e.g. [1], [2]). Precisely, our aim is to obtain within that context, the standard notion of the point-spectrum of an element of a topological algebra [10], to the case one considers as its "range", no more a subset of $\mathbb{C}$, but of another topological algebra, in general.

The topological algebras considered have the complexes as scalars, and in principle have separately continuous multiplication, unless otherwise specified. Our standard reference thereat is [10]. Thus, we start with the following.

Definition 1.1. Let $A, \mathbb{E}$ be topological algebras and $x \in A$. The set,

$$\sigma_A^{\mathbb{E}}(x) := \hat{x}(Hom_{\mathbb{C}}(A, \mathbb{E})) := \{f(x) : f \in Hom(A, \mathbb{E})\} \subseteq \mathbb{E}, \tag{1.1}$$

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is called the *generalized spectrum* of x.

Here $Hom(A, \mathbb{E})$ is the set of *generalized characters* of A: continuous algebra morphisms of A in $\mathbb{E}$ (cf. [10: p. 177]). More generally, for any *subset* B of A, one defines its *generalized spectrum* by the relation,

$$\sigma_A^{\mathbb{E}}(B) := \bigcup_{x \in B} \sigma_A^{\mathbb{E}}(x). \tag{1.2}$$

We come now to the following basic notion.

Definition 1.2. Let $A, \mathbb{E}$ be topological algebras and $S \subseteq \mathbb{E}$. The set

$$M(S) := \{x \in A : \sigma_A^{\mathbb{E}}(x) \subseteq S\} \subseteq A, \tag{1.3}$$

is called the *generalized principal extension of S in A.*

Furthermore, we consider the ("extension") map ($\mathfrak{P}(\cdot)$ denotes "*power set*"),

$$M : \mathfrak{P}(\mathbb{E}) \to \mathfrak{P}(A), \quad \textit{such that} \;\; S \mapsto M(S) := \{x \in A : \sigma_A^{\mathbb{E}}(x) \subseteq S\}. \tag{1.4}$$

The following *properties of generalized principal extensions* are easily verified. They represent analogous instances of the classical case, $\mathbb{E} = \mathbb{C}$ (see e.g. [4: Chapt. I; Section 1]):

$$M(S) \subseteq M(S'), \quad \text{for} \;\; S, S' \subseteq \mathbb{E}, \quad \text{with} \;\; S \subseteq S'. \tag{1.5}$$

Moreover, for S, S' subsets of $\mathbb{E}$, one has;

$$M(S \cap S') = M(S) \cap M(S'). \tag{1.6}$$

$$M(S) \cup M(S') \subseteq M(S \cup S'). \tag{1.7}$$

$$M(\mathbb{E}) = A. \tag{1.8}$$

Now, with $A, \mathbb{E}$ topological algebras, we consider the map (cf. (1.1)),

$$\sigma_A^{\mathbb{E}} : A \to \mathfrak{P}(\mathbb{E}) : x \mapsto \sigma_A^{\mathbb{E}}(x). \tag{1.9}$$

Now, we remark that, on the basis of (1.2), we can extend (1.9), even to subsets of A; for convenience we still use the same notation as for (1.9). Thus, one has the relation,

$$(\sigma_A^{\mathbb{E}} \circ M)(S) \subseteq S, \quad \text{for any} \;\; S \subseteq \mathbb{E}. \tag{1.10}$$

2. Principal extensions preserve "openness"

Let $\mathbb{E}$ be a given *topological algebra* and $\mathcal{K}(\mathbb{E})$ the set of *compact* subsets of $\mathbb{E}$. We consider on the latter set the analogue in our case of the classical *Hausdorff-Fréchet* metric, so that $\mathcal{K}(\mathbb{E})$ becomes a (topological) *metric space* (cf. also e.g. [8: p. 205], and [9: p. 108, Definition 2]). So we next define the *generalized Newburg map*, by the relation

$$\nu : A \to \mathcal{K}(\mathbb{E}) : x \mapsto \nu(x) := \sigma_A^{\mathbb{E}}(x) \subseteq \mathbb{E}. \tag{2.1}$$

On the other hand, we also set the following.

Definition 2.1. We say that the generalized Newburg map is *upper semi-continuous at $x \in A$,* if for every S open in $\mathbb{E}$, with $\sigma_A^{\mathbb{E}}(x) \subseteq S$ (hence, $x \in M(S)$, cf. (1.3)), there exists an open neighborhood U of x, such that,

$$\sigma_A^{\mathbb{E}}(U) \subseteq S \tag{2.2}$$

(see also (1.2)). We say that ν is upper semi-continuous if it is such for every $x \in A$.

Now, the following theorem extends an analogous result in [4: Chapt. I; Theorem 1.1] for suitable topological algebras A. That is, one gets.

Theorem 2.1. *Let A, $\mathbb{E}$ be topological algebras with A having, for every element x, $\sigma_A^{\mathbb{E}}(x) \subseteq \mathbb{E}$ compact and the corresponding generalized Newburg map upper semi-continuous. Then, the generalized principal extension of every open subset of $\mathbb{E}$ is an open subset of A.*

Proof. Let $S \subseteq \mathbb{E}$ open, and $a \in M(S)$. Then $\sigma_A^{\mathbb{E}}(a) \subseteq S$, so (Definition 2.1) there exists an open subset U of A, with $a \in U$, such that $\sigma_A^{\mathbb{E}}(U) \subseteq S$. Now, if $x \in U$, then $\sigma_A^{\mathbb{E}}(x) \subseteq \sigma_A^{\mathbb{E}}(U) := \bigcup_{y \in A} \sigma_A^{\mathbb{E}}(y) \subseteq S$ and $\sigma_A^{\mathbb{E}}(x) \subseteq S$. Thus, for every $a \in M(S)$, there is an open neighborhood $U \subseteq A$ of a, such that $U \subseteq M(S)$, that is $M(S)$ is open. ■

3. Closed sets and strong generalized spectral continuity

Let $A, \mathbb{E}$ be topological algebras such that, *every $x \in A$, has $\sigma_A^{\mathbb{E}}(x) \subseteq \mathbb{E}$, compact.* By considering $\mathcal{K}(\mathbb{E})$, as a *metric space*, under the *Hausdorff-Fréchet* metric, we say that the algebra A has the *strong generalized spectral continuity* (sgsc), if the corresponding generalized Newburg map ν is continuous. The following basic result generalizes the case of Banach algebras (Ackermans, cf. Refs), as well as our previous result in [3: p. 217, Theorem 5.1], for $\mathbb{E} = \mathbb{C}$. That is, we have.

Theorem 3.1. *Let* A*,* $\mathbb{E}$ *be topological algebras with* $\sigma_A^{\mathbb{E}}(x) \subseteq \mathbb{E}$ *compact, for every* $x \in A$*. Then, the generalized Newburg map is continuous if, and only if, the generalized principal extension of any closed* $S \subseteq \mathbb{E}$ *is a closed subset of* A*.*

Proof. For the "if" part of the theorem, suppose that A has the sgsc, and $S \subseteq \mathbb{E}$ is closed. Thus, the map ν (cf. (2.1)) being continuous, is upper semi-continuous, hence (cf. Definition 2.1),

$$N(\complement S) := \{x \in A : \sigma_A^{\mathbb{E}}(x) \cap \complement S \neq \emptyset\} \tag{3.1}$$

is an *open subset of* A. On the other hand, since

$$N(\complement S) = \complement M(S), \tag{3.2}$$

the set $M(S)$ is a *closed* subset of A. Now, for the "*only if*" part of the assertion, suppose that $M(S) \subseteq A$ is closed for S closed in $\mathbb{E}$. So let $a \in A$, $\lambda \in \sigma_A^{\mathbb{E}}(a) \subseteq \mathbb{E}$ and Ω an open neighborhood of λ in $\mathbb{E}$. Then, the set (see also (3.2)),

$$U = \complement M(\mathcal{C}\Omega) = N(\Omega) \tag{3.3}$$

is, by hypothesis, an open neighborhood of a such that $\lambda \in \sigma_A^{\mathbb{E}}(a) \cap \Omega \neq \emptyset$, which now completes the proof. ■

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