Principal extensions of complex sets and Riemann surfaces in topological algebras

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Abstract

The principal extension of a complex set in a topological algebra is a subset of the algebra. The principal extension of a Riemann surface in a topological algebra is an infinite-dimensional strongly analytic manifold modelled on the topological algebra considered. The spectrum of an element of the principal extension is a subset of the Riemann surface at issue. This is extended to a subset of the principal extension, and is further applied to define the principal extension of a subset of a Riemann surface. As an application, the principal extension of the complexes in a topological algebra, is the topological algebra itself.

0. Introduction

The principal extension of a complex set (: subset of $\mathbb{C}$) in a topological algebra is a subset of the algebra. The principal extension of a Riemann surface in a topological algebra is an infinite-dimensional strongly analytic manifold modelled on the topological algebra considered (see [2] and [3]). The spectrum of an element of the principal extension is a subset of the Riemann surface at issue. In particular, the principal extension of the Riemann surface $\mathbb{C}$ in a topological algebra is the topological algebra itself.

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1. Extension of complex sets

Let $\mathbb{A}$ be a *topological algebra* with an identity element $\mathbf{1}_{\mathbb{A}}$ and $S \subseteq \mathbb{C}$ (the complexes). The *principal extension of S in* $\mathbb{A}$ is, by definition, the following subset of $\mathbb{A}$,

$$M(S) := \{x \in \mathbb{A} : sp_{\mathbb{A}}(x) \subseteq S\}. \tag{1.1}$$

The following properties of principal extensions are easily verified:

$$M(S) \subseteq M(S'), \quad \text{for} \quad S, S' \subseteq \mathbb{C}, \quad \text{with} \quad S \subseteq S'. \tag{1.2}$$

$$M(S \cap S') = M(S) \cap M(S'), \quad S, S' \subseteq \mathbb{C}. \tag{1.3}$$

$$M(S) \cup M(S') \subseteq M(S \cup S'), \quad S, S' \subseteq \mathbb{C}. \tag{1.4}$$

$$M(\mathbb{C}) = \mathbb{A}. \tag{1.5}$$

If $\mathbb{A}$ is a *unital locally m-convex algebra*, then

$$M(\emptyset) = \emptyset. \tag{1.6}$$

Let $\mathcal{K}(\mathbb{C})$ be the *set of* all *compact subsets of* $\mathbb{C}$, endowed with the *Hausdorff-Fréchet metric*. Then, the (Newburg) map:

$$\mathbb{A} \to \mathcal{K}(\mathbb{A}) : x \mapsto sp_{\mathbb{A}}(x) \tag{1.7}$$

is upper semicontinuous, if and only if, for every $(x, S) \in \mathbb{A} \times \mathcal{T}_{\mathbb{C}}$, with $x \in M(S)$, there is an open neighbourhood U of $x \in \mathbb{A}$, such that $sp_{\mathbb{A}}(U) \subseteq S$ [1].

Theorem 1.1. *Let $\mathbb{A}$ be a unital topological algebra $\mathbb{A}$, having $sp_{\mathbb{A}}(x) \subseteq \mathbb{C}$ compact, for every $x \in \mathbb{A}$, and the corresponding (Newburg) map: $\mathbb{A} \to \mathcal{K}(\mathbb{C}) : x \mapsto sp_{\mathbb{A}}(x)$ upper semicontinuous. Then, for every open $S \subseteq \mathbb{C}$, $M(S) \subseteq \mathbb{A}$ is open.*

Proof. Let $S \subseteq \mathbb{C}$ open, and $a \in M(S)$. Then $sp_{\mathbb{A}}(a) \subseteq S$, so there is an open subset U of $\mathbb{A}$, with $a \in U$, such that $sp_{\mathbb{A}}(U) \subseteq S$. Let $x \in U$, then $sp_{\mathbb{A}}(x) \subseteq sp_{\mathbb{A}}(U) := \bigcup_{y \in U} sp_{\mathbb{A}}(y) \subseteq S$ and $sp_{\mathbb{A}}(x) \subseteq S$. So $x \in M(S)$ and $U \subseteq M(S)$. Thus, for every $a \in M(S)$, there is an open $U \subseteq \mathbb{A}$, with $a \in U$, such that, $a \in U \subseteq M(S)$, that is, $M(S) \subseteq \mathbb{A}$ is open. □

We assume now that our algebra $\mathbb{A}$ has the *strong spectral continuity* (see [1]). Then, we have the next.

Theorem 1.2. *Let $\mathbb{A}$ be a unital topological algebra, having $sp_{\mathbb{A}}(x) \subseteq \mathbb{C}$ compact, for every $x \in \mathbb{A}$, and the corresponding (Newburg) map: $\mathbb{A} \to \mathcal{K}(\mathbb{C}) : x \to sp_{\mathbb{A}}(x)$ continuous ("strong spectral continuity"). Then, $M(S) \subseteq \mathbb{C}$ is closed, for every closed $S \subseteq \mathbb{C}$.*

Proof. Let $S \subseteq \mathbb{C}$ closed. Then, the set

$$N(\complement S) := \{x \in \mathbb{A} : sp_{\mathbb{A}}(x) \cap \complement S \neq \emptyset\} \tag{1.8}$$

is an open subset of $\mathbb{A}$. On the other hand,

$$N(\complement S) = \complement M(S). \tag{1.9}$$

So, the set $M(S)$ is a closed subset of $\mathbb{A}$. □

2. Extension of a Riemann surface and local spectra

Let $\mathbb{A}$ be a *unital commutative complete semisimple locally m-convex algebra*, having $sp_{\mathbb{A}}(x) \subseteq \mathbb{C}$ *compact, for every* $x \in \mathbb{A}$, and the corresponding *(Newburg) map:* $\mathbb{A} \to \mathcal{K}(\mathbb{C}) : x \mapsto sp_{\mathbb{A}}(x)$ *upper semicontinuous.* Let also $(X, \mathcal{A} = \{(U_i, \varphi_i), i \in I\})$ be a *Riemann surface.*
Now, consider the maps:

$$\varphi_{ij} := \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j) \tag{2.1}$$

where $(U_i, \varphi_i), (U_j, \varphi_j) \in \mathcal{A}$ (: atlas of X), (see [6]), and

$$M(\varphi_{ij}) \equiv F_{ij} : M(\varphi_i(U_i \cap U_j)) \to M(\varphi_j(U_i \cap U_j)) \tag{2.2}$$

such that

$$F_{ij}(x) = \frac{1}{2\pi_i} \int_{\Gamma} \varphi_{ij}(z)(z - x)^{-1} dz, \tag{2.3}$$

where Γ is a closed regular curve, with $\Gamma \subseteq \Omega \cap (sp_{\mathbb{A}}(x))^c$ and Ω an open neighbourhood of $sp_{\mathbb{A}}(x)$ in $\mathbb{C}$ (see [5]), such that

$$sp_{\mathbb{A}}(x) \subseteq \varphi_i(U_i \cap U_j) \subseteq \Omega. \tag{2.4}$$

Furthermore, we define the following equivalence relation:

$$(M(\varphi_i(U_i)), a_i) \sim (M(\varphi_j(U_j)), a_j), \tag{2.5}$$

with $a_i \in M(\varphi_i(U_i \cap U_j))$, $a_j \in M(\varphi_j(U_i \cap U_j))$ such that $F_{ij}(a_i) = a_j$.

So we now set

$$M(X) := \{[(M(\varphi_i(U_i)), a_i)] : a_i \in M(\varphi_i(U_i)),\ i \in I\}. \tag{2.6}$$

On the other hand, we also define the maps:

$$g_i : M(\varphi_i(U_i)) \to M(X) : a \mapsto g_i(a) := [(M(\varphi_i(U_i)), a)], \tag{2.7}$$

for every chart $(U_i, \varphi_i) \in \mathcal{A}$.
The map g_i is $1-1$. We next consider the family:

$$M(\mathcal{A}) := \{(g_i(M(\varphi_i(U_i))), g_i^{-1}),\ i \in I\}. \tag{2.8}$$

The family $M(\mathcal{A})$ is a *strongly analytic atlas* of $M(X)$ (see [2]), so that the pair

$$(M(X), M(\mathcal{A})) \tag{2.9}$$

yields a strongly analytic manifold, called the *principal extension of the Reimann surface* $(X, \mathcal{A})$ *in the topological algebra* $\mathbb{A}$.
Now, let $z = [(M(\varphi_i(U_i)), a_i)] \in M(X)$ where $a_i \in M(\varphi_i(U_i))$ and $(U_i, \varphi_i) \in \mathcal{A}$. We consider the set:

$$sp(z) := \varphi_i^{-1}(sp_{\mathbb{A}}(a_i)) \text{ (see [4]).} \tag{2.10}$$

The subset $sp(z)$ of X is called the *spectrum* of z, and it is independent of $i \in I$. It is clear that $sp(z) \neq \emptyset$, for every $z \in M(X)$, and also $sp(z)$ is a compact subset of X. Now, if $Z \subseteq M(X)$, one defines the set

$$sp(Z) := \bigcup_{z \in Z} sp(z), \tag{2.11}$$

called the *spectrum* of Z.
If $U \subseteq X$, then the set

$$M(U) := \{z \in M(X) : sp(z) \subseteq U\} \tag{2.12}$$

is called the *principal extension* of U *in* $M(X)$.

The following properties of principal extensions are easily verified.

$$M(U) \cup M(U') \subseteq M(U \cup U'), \tag{2.13}$$

$$M(U \cap U') = M(U) \cap M(U'), \tag{2.14}$$

for $U, U' \subseteq X$,.

$$M(U) \subseteq M(U'),\ U \subseteq U'. \tag{2.15}$$

$$sp(M(U)) = U. \tag{2.16}$$

Thus, we have now the next.

Theorem 2.1. *Let $\mathbb{A}$ be a unital commutative complete semisimple locally m-convex algebra, having $sp_{\mathbb{A}}(x) \subseteq \mathbb{C}$ compact, for every $x \in \mathbb{A}$, and the corresponding (Newburg) map: $\mathbb{A} \to \mathcal{K}(\mathbb{C}) : x \mapsto sp_{\mathbb{A}}(x)$ upper semicontinuous. Moreover, let $(M(X), M(\mathcal{A}))$ be the principal extension of the Riemann surface $(X, \mathcal{A})$ in the topological algebra $\mathbb{A}$. Then, for every open subset U of X, the principal extension $M(U)$ of U is an open subset of $M(X)$, as well.*

Proof. Let $z \in M(U)$, then $z = [(M(\varphi_i(U_i)), a_i)]$, with $(U_i, \varphi_i) \in \mathcal{A}$, $a_i \in M(\varphi_i(U_i))$. That is, $sp_{\mathbb{A}}(a_i) \subseteq \varphi_i(U_i)$. We have,

$$sp(z) := \varphi_i^{-1}(sp_{\mathbb{A}}(a_i)) \subseteq U_i \text{ and } sp(z) \subseteq U$$

So, $sp(z) \subseteq U \cap U_i$.

On the other hand, $U, U_i, U \cap U_i$ are open subsets of X, so that we have $M(\varphi_i(U_i \cap U_j)) \subseteq \mathbb{A}$ open. Obviously, $M(\varphi_i(U_i)) \subseteq \mathbb{A}$ is open, and $M(\varphi_i(U \cap U_i)) \subseteq M(\varphi_i(U_i))$. Yet, by definition, $sp(z) := \varphi_i^{-1}(sp_{\mathbb{A}}(a_i) \subseteq U \cap U_i$, so that $sp_{\mathbb{A}}(a_i) \subseteq \varphi_i(U \cap U_i)$. Then, $a_i \in M(\varphi_i(U \cap U_i))$, hence, $z = [(M(\varphi_i(U_i)), a_i)]$. Now, one has:

$$g_i(M(\varphi_i(U \cap U_i))) = M(U \cap U_i),$$

where g_i^{-1} is a homeomorphism, so that $M(U \cap U_i) \subseteq M(X)$ is open, with $z \in M(U \cap U_i)$, since $sp(z) \subseteq U \cap U_i$, which thus proves the theorem. □

Now, we assume again that our algebra $\mathbb{A}$ has the "strong spectral continuity". Then, we have the next.

Theorem 2.2. *Let $\mathbb{A}$ be a unital commutative complete semisimple locally m-convex algebra, having $sp_{\mathbb{A}}(x) \subseteq \mathbb{C}$ compact, for every $x \in \mathbb{A}$, and the corresponding (Newburg) map: $\mathbb{A} \to \mathcal{K}(\mathbb{C}) : x \mapsto sp_{\mathbb{A}}(x)$ continuous (: "strong spectral continuity). Moreover, let $(M(X), M(\mathcal{A}))$ be the principal extension of the Riemann surface $(X, \mathcal{A})$ in the topological algebra $\mathbb{A}$. Then, for $K \subseteq X$ closed, $M(K) \subseteq M(X)$ is closed.*

Proof. Let $(z_\delta) \subseteq M(K)$, with $z = \lim_\delta z_\delta$ and $z = [(M(\varphi(U)), a)] \in M(X)$, $a \in M(\varphi(U))$, $(U, \varphi) \in \mathcal{A}$. So, $sp_{\mathbb{A}}(a) \subseteq \varphi(U)$. Then, there exists $a_\delta \in M(\varphi(U))$, with $\lim_\delta a_\delta = a$ and $z_\delta = [(M(\varphi(U)), a_\delta)]$, so that $sp_{\mathbb{A}}(a_\delta) \subseteq \varphi(U)$. Since, $z_\delta \in M(K)$, we have $sp(z_\delta) := \varphi^{-1}(sp_{\mathbb{A}}(a_\delta)) \subseteq K$, hence $sp_{\mathbb{A}}(a_\delta) \subseteq \varphi(U \cap K)$.

On the other hand, $K \subseteq X$ closed, so we have that, $U \cap K$ closed in U and $\varphi(U \cap K) \subseteq \varphi(U)$ closed. By the hypothesis, for the algebra $\mathbb{A}$, we have, $sp_{\mathbb{A}}(a_\delta) \to sp_{\mathbb{A}}(a)$, in the space $\mathcal{K}(\mathbb{C})$ (see Section 1), and $sp_{\mathbb{A}}(a) \subseteq \varphi(U \cap K)$. That is, $z \in M(K)$ and $M(K) \subseteq M(X)$, is closed. □

3. Extension of the Riemann surface $\mathbb{C}$

As an application of the preceding, we consider below the particular classical case of *the complexes*, considered *as a Riemann surface*, so that one can further apply our previous results in Section 1.

Thus, consider the *Riemann surface*

$$(\mathbb{C}, \mathcal{A}_{\mathbb{C}} = \{(\mathbb{C}, id_{\mathbb{C}})\}) \tag{3.1}$$

and $\mathbb{A}$ a *unital commutative complete semisimple locally m-convex algebra, having $sp_{\mathbb{A}}(x) \subseteq \mathbb{C}$ compact, for every $x \in \mathbb{A}$, and the corresponding (Newburg) map: $\mathbb{A} \to \mathcal{K}(\mathbb{C}) : x \mapsto sp_{\mathbb{A}}(x)$ upper semicontinuous.* Then,

$$M(\mathbb{C}) := \{[(M(id_{\mathbb{C}}(\mathbb{C})), a)]\} \tag{3.2}$$

where $a \in M(id_{\mathbb{C}}(\mathbb{C})) = M(\mathbb{C}) = \mathbb{A}$, so that one gets

$$M(\mathbb{C}) = \{[(\mathbb{A}, a)], a \in \mathbb{A}\}. \tag{3.3}$$

Furthermore, $[(\mathbb{A}, a)] = [(\mathbb{A}, b)]$, with $a, b \in \mathbb{A}$, if and only if, $M(id_{\mathbb{C}})(a) = b$, where $M(id_{\mathbb{C}}) = id_{\mathbb{A}}$ (cf. [2]), equivalently, when $a = b$. Accordingly, since $[(A, a)] \cong a$, one has,

$$M(\mathbb{C}) = \mathbb{A}. \tag{3.4}$$

Moreover, one gets:

$$g : M(id_{\mathbb{C}}(\mathbb{C})) = \mathbb{A} \to M(\mathbb{C}) = \mathbb{A} : a \mapsto g(a) := [(\mathbb{A}, a)] \cong a. \tag{3.5}$$

Hence, $g = id_{\mathbb{A}}$, so that one has,

$$M(\mathcal{A}_{\mathbb{C}}) = \{(\mathbb{A}, id_{\mathbb{A}})\},$$

so the pair

$$(M(\mathbb{C}), M(\mathcal{A}_{\mathbb{C}})) = (\mathbb{A}, \{(\mathbb{A}, id_{\mathbb{A}})\}), \tag{3.6}$$

is the *principal extension of the Riemann surface* $(\mathbb{C}, \mathcal{A}_{\mathbb{C}})$ in the topological algebra $\mathbb{A}$. Now, let $a = [(\mathbb{A}, a)] \in M(\mathbb{C}) = \mathbb{A}$. Then, (cf. (2.10)), $sp(a) := id_{\mathbb{C}}^{-1}(sp_{\mathbb{A}}(a)) = sp_{\mathbb{A}}(a)$, therefore, one has

$$M(S) := \{x \in M(\mathbb{C}) = \mathbb{A} : sp(x) \subseteq S\} = \{x \in \mathbb{A} : sp_{\mathbb{A}}(x) \subseteq S\}, \ S \subseteq \mathbb{C},$$

(see also (1.1)). Consequently, by considering $\mathbb{C}$, as a Riemann susrface, we finally conclude the coincidence of the respective notions in Sections 1 and 2.

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