

# Principal extensions of complex sets and Riemann surfaces in topological algebras

Zoi Daoultzi-Malamou

## Abstract

The principal extension of a complex set in a topological algebra is a subset of the algebra. The principal extension of a Riemann surface in a topological algebra is an infinite-dimensional strongly analytic manifold modelled on the topological algebra considered. The spectrum of an element of the principal extension is a subset of the Riemann surface at issue. This is extended to a subset of the principal extension, and is further applied to define the principal extension of a subset of a Riemann surface. As an application, the principal extension of the complexes in a topological algebra, is the topological algebra itself.

## 0. Introduction

The principal extension of a complex set ( $\subset \mathbb{C}$ ) in a topological algebra is a subset of the algebra. The principal extension of a Riemann surface in a topological algebra is an infinite-dimensional strongly analytic manifold modelled on the topological algebra considered (see [2] and [3]). The spectrum of an element of the principal extension is a subset of the Riemann surface at issue. In particular, the principal extension of the Riemann surface  $\mathbb{C}$  in a topological algebra is the topological algebra itself.

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## 1. Extension of complex sets

Let  $\mathbb{A}$  be a *topological algebra* with an identity element  $1_{\mathbb{A}}$  and  $S \subseteq \mathbb{C}$  (the complexes). The *principal extension of  $S$  in  $\mathbb{A}$*  is, by definition, the following subset of  $\mathbb{A}$ ,

$$(1.1) \quad M(S) := \{x \in \mathbb{A} : sp_{\mathbb{A}}(x) \subseteq S\}.$$

The following properties of principal extensions are easily verified:

$$(1.2) \quad M(S) \subseteq M(S'), \text{ for } S, S' \subseteq \mathbb{C}, \text{ with } S \subseteq S'.$$

$$(1.3) \quad M(S \cap S') = M(S) \cap M(S'), \quad S, S' \subseteq \mathbb{C}.$$

$$(1.4) \quad M(S) \cup M(S') \subseteq M(S \cup S'), \quad S, S' \subseteq \mathbb{C}.$$

$$(1.5) \quad M(\mathbb{C}) = \mathbb{A}.$$

If  $\mathbb{A}$  is a *unital locally  $m$ -convex algebra*, then

$$(1.6) \quad M(\emptyset) = \emptyset.$$

Let  $\mathcal{K}(\mathbb{C})$  be the *set of all compact subsets of  $\mathbb{C}$* , endowed with the *Hausdorff-Fréchet metric*. Then, the (Newburg) map:

$$(1.7) \quad \mathbb{A} \rightarrow \mathcal{K}(\mathbb{A}) : x \mapsto sp_{\mathbb{A}}(x)$$

is upper semicontinuous, if and only if, for every  $(x, S) \in \mathbb{A} \times \mathcal{T}_{\mathbb{C}}$ , with  $x \in M(S)$ , there is an open neighbourhood  $U$  of  $x \in \mathbb{A}$ , such that  $sp_{\mathbb{A}}(U) \subseteq S$  [1].

**Theorem 1.1.** *Let  $\mathbb{A}$  be a unital topological algebra  $\mathbb{A}$ , having  $sp_{\mathbb{A}}(x) \subseteq \mathbb{C}$  compact, for every  $x \in \mathbb{A}$ , and the corresponding (Newburg) map:  $\mathbb{A} \rightarrow \mathcal{K}(\mathbb{C}) : x \mapsto sp_{\mathbb{A}}(x)$  upper semicontinuous. Then, for every open  $S \subseteq \mathbb{C}$ ,  $M(S) \subseteq \mathbb{A}$  is open.*

**Proof.** Let  $S \subseteq \mathbb{C}$  open, and  $a \in M(S)$ . Then  $sp_{\mathbb{A}}(a) \subseteq S$ , so there is an open subset  $U$  of  $\mathbb{A}$ , with  $a \in U$ , such that  $sp_{\mathbb{A}}(U) \subseteq S$ . Let  $x \in U$ , then  $sp_{\mathbb{A}}(x) \subseteq sp_{\mathbb{A}}(U) := \bigcup_{y \in U} sp_{\mathbb{A}}(y) \subseteq S$  and  $sp_{\mathbb{A}}(x) \subseteq S$ . So  $x \in M(S)$  and  $U \subseteq M(S)$ . Thus, for every  $a \in M(S)$ , there is an open  $U \subseteq \mathbb{A}$ , with  $a \in U$ , such that,  $a \in U \subseteq M(S)$ , that is,  $M(S) \subseteq \mathbb{A}$  is open.  $\square$

We assume now that our algebra  $\mathbb{A}$  has the *strong spectral continuity* (see [1]). Then, we have the next.

