

# PRINCIPAL EXTENSIONS OF RIEMANN SURFACES IN TOPOLOGICAL ALGEBRAS AND LOCAL SPECTRA

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ABSTRACT. The principal extension of a Riemann surface in a topological algebra, being proved to be an infinite-dimensional complex analytic manifold modelled on the topological algebra, are considered. The spectrum of an element of the principal extension, which is defined as a subset of the Riemann surface, is studied as an application. The notion is extended to a subset of the principal extension, while the principal extension of a subset of a Riemann surface are also considered.

## 1. PRELIMINARIES

Let  $\mathbb{A}$  be a commutative complete semisimple locally  $m$ -convex algebra with a unit element  $e_{\mathbb{A}}$  and

$$\text{sp}_{\mathbb{A}}(x) = \{\lambda \in \mathbb{C} : x - \lambda e_{\mathbb{A}} \text{ is not invertible in } \mathbb{A}\}$$

the spectrum of  $x \in \mathbb{A}$ . We suppose that  $\text{sp}_{\mathbb{A}}(x) \subset \mathbb{C}$  is compact for every  $x \in \mathbb{A}$  and the map  $x \mapsto \text{sp}_{\mathbb{A}}(x)$  is upper semicontinuous. Moreover, let  $(X, \mathcal{A})$  be a *Riemann surface*, where  $\mathcal{A} = \{(U_i, \phi_i)\}$  is an atlas of  $X$  and for every subset  $S \subset \mathbb{C}$  let

$$M(S) = \{x \in \mathbb{A} : \text{sp}_{\mathbb{A}}(x) \subset S\}.$$

For any  $i, j \in I$  and  $(U_i, \phi_i), (U_j, \phi_j) \in \mathcal{A}$  let

$$\phi_{ij} := \phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$$

and let

$$M(\phi_{ij}) \equiv F_{ij} : M(\phi_i(U_i \cap U_j)) \rightarrow M(\phi_j(U_i \cap U_j))$$

such that

$$F_{ij}(x) = \frac{1}{2\pi i} \int_{\Gamma} \phi_{ij}(z)(z - x)^{-1} dz$$

for each  $x \in M(\phi_i(U_i \cap U_j))$ , where  $\Gamma$  is a closed regular curve with

$$\Gamma \subseteq \Omega \subseteq \mathbb{C} \setminus \text{sp}_{\mathbb{A}}(x)$$

and  $\Omega$  is an open neighbourhood of  $\text{sp}_{\mathbb{A}}(x)$  in  $\mathbb{C}$  such, that

$$\text{sp}_{\mathbb{A}}(x) \subset \phi_i(U_i \cap U_j) \subset \Omega.$$

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We will say that  $(M(\phi_i(U_i)), a_i) \sim (M(\phi_j(U_j)), a_j)$ , where  $a_i \in M(\phi_i(U_i \cap U_j))$  and  $a_j \in M(\phi_j(U_i \cap U_j))$  if  $F_{ij}(a_i) = a_j$  and cosets with respect to this equivalence we denote by  $[(M(\phi_i(U_i)), a_i)]$ . Let

$$M(X) = \{[(M(\phi_i(U_i)), a_i)] : a_i \in M(\phi(U_i)) \text{ and } i \in I\}.$$

and  $g_i : M(\phi_i(U_i)) \rightarrow M(X)$  be a map defined by  $g_i(a) = [(M(\phi_i(U_i)), a)]$  for any chart  $(U_i, \phi_i) \in \mathcal{A}$  and any  $a \in M(\phi_i(U_i))$  (then  $g_i$  is a 1 - 1 map) and let

$$M(\mathcal{A}) = \{(g_i(M(\phi_i(U_i))), g_i^{-1}) : (U_i, \phi_i) \in \mathcal{A}\}.$$

Then the family  $M(\mathcal{A})$  is a *strongly analytic atlas* of  $M(X)$  and  $(M(X), M(\mathcal{A}))$  is a *strongly analytic manifold*, which are called the *principal extension of the Riemann surface  $(X, \mathcal{A})$  in the topological algebra  $\mathbb{A}$* .

## 2. LOCAL SPECTRA

Let  $\mathbb{A}$  be a unital commutative complete semisimple locally  $m$ -convex algebra,  $(X, \mathcal{A})$  a Riemann surface and  $(M(X), M(\mathcal{A}))$  the principal extension of  $(X, \mathcal{A})$  in  $\mathbb{A}$ .

**Definition 1.** Let  $\mathbb{A}$  be a unital complete semisimple locally  $m$ -convex algebra, for which  $sp_{\mathbb{A}}(x)$  is compact for each  $x \in \mathbb{A}$  and the map  $x \mapsto sp_{\mathbb{A}}(x)$  is upper semicontinuous,  $(X, \mathcal{A})$  be a Riemann surface and  $(M(X), M(\mathcal{A}))$  the principal extension of  $(X, \mathcal{A})$  in  $\mathbb{A}$ . Let  $z = [(M(\phi_i(U_i)), a_i)] \in M(X)$ , where  $a_i \in M(\phi_i(U_i))$  and  $(U_i, \phi_i) \in \mathcal{A}$ . Then the subset  $sp(z) = \phi_i^{-1}(sp_{\mathbb{A}}(a_i))$  of  $X$  is called the *spectrum of  $z$* .

To verify that  $sp(z)$  is independent of the choice of  $i \in I$ , let  $(U_j, \phi_j) \in \mathcal{A}$  and  $(M(\phi_i(U_i)), a_i), (M(\phi_j(U_j)), a_j) \in z$ . Since  $(M(\phi_i(U_i)), a_i) \sim (M(\phi_j(U_j)), a_j)$ , then  $F_{ij}(a_i) = a_j$  and

$$sp_{\mathbb{A}}(a_j) = sp_{\mathbb{A}}(F_{ij}(a_i)) = \phi_{ij}(sp_{\mathbb{A}}(a_i)) = (\phi_j \circ \phi_i^{-1})(sp_{\mathbb{A}}(a_i)).$$

Therefore  $\phi_j^{-1}(sp_{\mathbb{A}}(a_j)) = \phi_i^{-1}(sp_{\mathbb{A}}(a_i))$ . It means that the spectrum  $sp(z)$  does not depend on  $i \in I$ .

It is easy to see that  $sp(z)$  is not empty for every  $z \in M(X)$ . Hence,  $sp(z)$  is a compact subset of  $X$ .

**Definition 2.** The set

$$sp(Z) = \bigcup_{z \in Z} sp(z)$$

is called the *spectrum of  $Z \subset M(X)$  and, the set*

$$M(U) = \{z \in M(X) : sp(z) \subset U\}$$

is called the *principal extension of  $U \subset X$  in  $M(X)$* .

**Theorem 1.** Let  $\mathbb{A}$  be a unital commutative complete semisimple locally  $m$ -convex algebra, for which the spectrum  $sp_{\mathbb{A}}(x)$  is compact for each  $x \in \mathbb{A}$  and the map  $x \mapsto sp_{\mathbb{A}}(x)$  is upper semicontinuous. Let  $(X, \mathcal{A})$  be a Riemann surface and  $(M(X), M(\mathcal{A}))$  be the principal extension of  $(X, \mathcal{A})$  in  $\mathbb{A}$ . Then for every open subset  $U$  of  $X$  the principal extension  $M(U)$  of  $U$  is an open subset of  $M(X)$ .

*Proof.* Let  $z \in M(U)$ . Then  $z = [(M(\phi_i(U_i)), a_i)]$ , where  $(U_i, \phi_i) \in \mathcal{A}$  and  $a_i \in M(\phi_i(U_i))$ . Since  $sp_{\mathbb{A}}(a_i) \subset \phi_i(U_i)$  then  $sp(z) = \phi_i^{-1}(sp_{\mathbb{A}}(a_i)) \subset U_i$  and  $sp(z) \subset U$ . Thus,  $sp(z) \subset U \cap U_i$  or  $z \in M(U \cap U_i) \subset M(U)$ . On the other hand, since  $U \cap U_i$  is open in  $X$  then  $M(\phi_i(U \cap U_i))$  is open in  $M(X)$ . It is easy

