PRINCIPAL EXTENSIONS OF RIEMANN SURFACES IN TOPOLOGICAL ALGEBRAS AND LOCAL SPECTRA

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ABSTRACT. The principal extension of a Rieman surface in a topological algebra, being proved to be an infinite-dimensional complex analytic manifold modelled on the topological algebra, are considered. The spectrum of an element of the principal extension, which is defined as a subset of the Riemann surface, is studied as an application. The notion is extended to a subset of the principal extension, while the principal extension of a subset of a Riemann surface are also considered.

1. PRELIMINARIES

Let $\mathbb{A}$ be a commutative complete semisimple locally m-convex algebra with a unit element $e_{\mathcal{A}}$ and

$$\mathrm{sp}_{\mathbb{A}}(x) = \{\lambda \in \mathbb{C} : x - \lambda e_{\mathcal{A}} \text{ is not invertile in } \mathbb{A}\}$$

the spectrum of $x \in \mathbb{A}$. We suppose that $\mathrm{sp}_{\mathbb{A}}(x) \subset \mathbb{C}$ is compact for every $x \in \mathbb{A}$ and the map $x \mapsto sp_{\mathbb{A}}(x)$ is upper semicontinuous. Moreover, let $(X, \mathcal{A})$ be a *Riemann surface*, where $\mathcal{A} = \{(U_i, \phi_i)\}$ is an atlas of X and for every subset $S \subset \mathbb{C}$ let

$$M(S) = \{x \in \mathbb{A} : sp_{\mathbb{A}}(x) \subset S\}.$$

For any $i, j \in I$ and $(U_i, \phi_i), (U_j, \phi_j) \in \mathcal{A}$ let

$$\phi_{ij} := \phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

and let

$$M(\phi_{ij}) \equiv F_{ij} : M(\phi_i(U_i \cap U_j)) \to M(\phi_j(U_i \cap U_j)$$

such that

$$F_{ij}(x) = \frac{1}{2\pi i} \int_{\Gamma} \phi_{ij}(z)(z - x)^{-1} dz$$

for each $x \in M(\phi_i(U_i \cap U_j))$, where Γ is a closed regular curve with

$$\Gamma \subseteq \Omega \subseteq \mathbb{C} \setminus sp_{\mathbb{A}}(x)$$

and Ω is an open neighbourhood of $sp_{\mathbb{A}}(x)$ in $\mathbb{C}$ such, that

$$\mathrm{sp}_{\mathbb{A}}(x) \subset \phi_i(U_i \cap U_j) \subset \Omega.$$

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We will say that $(M(\phi_i(U_i)), a_i) \sim (M(\phi_j(U_j)), a_j)$, where $a_i \in M(\phi_i(U_i \cap U_j))$ and $a_j \in M(\phi_j(U_i \cap U_j))$ if $F_{ij}(a_i) = a_j$ and cosets with respect to this equivalence we denote by $[(M(\phi_i(U_i)), a_i)]$. Let

$$M(X) = \{[(M(\phi_i(U_i)), a_i)] : a_i \in M(\phi(U_i)) \text{ and } i \in I\}.$$

and $g_i : M(\phi_i(U_i)) \to M(X)$ be a map defined by $g_i(a) = [(M(\phi_i(U_i)), a)]$ for any chart $(U_i, \phi_i) \in \mathcal{A}$ and any $a \in M(\phi_i(U_i))$ (then g_i is a $1-1$ map) and let

$$M(\mathcal{A}) = \{(g_i(M(\phi_i(U_i))), g_i^{-1}) : (U_i, \phi_i) \in \mathcal{A}\}.$$

Then the family $M(\mathcal{A})$ is a *strongly analytic atlas of* $M(X)$ and $(M(X), M(\mathcal{A}))$ is a *strongly analytic manifold*, which are called the *principal extension of the Riemann surface* $(X, \mathcal{A})$ *in the topological algebra* $\mathbb{A}$.

2. Local spectra

Let $\mathbb{A}$ be a unital commutative complete semisimple locally m-convex algebra, $(X, \mathcal{A})$ a Riemann surface and $(M(X), M(\mathcal{A}))$ the principal extension of $(X, \mathcal{A})$ in $\mathbb{A}$.

Definition 1. *Let $\mathbb{A}$ be a unital complete semisimple locally m-convex algebra, for which $sp_{\mathbb{A}}(x)$ is compact for each $x \in \mathbb{A}$ and the map $x \mapsto sp_{\mathbb{A}}(x)$ is upper semicontinuous, $(X, \mathcal{A})$ be a Riemann surface and $(M(X), M(\mathcal{A}))$ the principal extension of $(X, \mathcal{A})$ in $\mathbb{A}$. Let $z = [(M(\phi_i(U_i)), a_i)] \in M(X)$, where $a_i \in M(\phi_i(U_i))$ and $(U_i, \phi_i) \in \mathcal{A}$. Then the subset $sp(z) = \phi_i^{-1}(sp_{\mathbb{A}}(a_i))$ of X is called the spectrum of z.*

To verify that $sp(z)$ is independent of the choise of $i \in I$, let $(U_j, \phi_j) \in \mathcal{A}$ and $(M(\phi_i(U_i)), a_i), (M(\phi_j(U_j)), a_j) \in z$, Since $(M(\phi_i(U_i)), a_i) \sim (M(\phi_j(U_j)), a_j)$, then $F_{ij}(a_i) = a_j$ and

$$sp_{\mathbb{A}}(a_j) = sp_{\mathbb{A}}(F_{ij}(a_i)) = \phi_{ij}(sp_{\mathbb{A}}(a_i)) = (\phi_j \circ \phi_i^{-1})(sp_{\mathbb{A}}(a_i)).$$

Therefore $\phi_j^{-1}(sp_{\mathbb{A}}(a_j)) = \phi_i^{-1}(sp_{\mathbb{A}}(a_i))$. It means that the spectrum sp(z) does not depend on $i \in I$.

It is easy to see that $sp(z)$ is not empty for every $z \in M(X)$. Hence, $sp(z)$ is a compact subset of X.

Definition 2. *The set*

$$sp(Z) = \bigcup_{z \in Z} sp(z)$$

is called the spectrum of $Z \subset M(X)$ and , the set

$$M(U) = \{z \in M(X) : sp(z) \subset U\}$$

is called the principal extension of $U \subset X$ in $M(X)$.

Theorem 1. *Let $\mathbb{A}$ be a unital commutative complete semisimple locally m-convex algebra, for which the spectrum $sp_{\mathbb{A}}(x)$ is compact for each $x \in \mathbb{A}$ and the map $x \mapsto sp_{\mathbb{A}}(x)$ is upper semicontinuous. Let $(X, \mathcal{A})$ be a Riemann surface and $(M(X), M(\mathcal{A}))$ be the principal extension of $(X, \mathcal{A})$ in $\mathbb{A}$. Then for every open subset U of X the principal extension $M(U)$ of U is an open subset of $M(X)$.*

Proof. Let $z \in M(U)$. Then $z = [(M(\phi_i(U_i)), a_i)]$, where $(U_i, \phi_i) \in \mathcal{A}$ and $a_i \in M(\phi_i(U_i))$. Since $sp_{\mathbb{A}}(a_i) \subset \phi_i(U_i)$ then $sp(z) = \phi_i^{-1}(sp_{\mathbb{A}}(a_i) \subset U_i$ and $sp(z) \subset U$. Thus, $sp(z) \subset U \cap U_i$ or $z \in M(U \cap U_i) \subset M(U)$. On the other hand, since $U \cap U_i$ is open in X then $M(\phi_i(U \cap U_i))$ is open in $M(X)$. It is easy to see that $g_i(M(\phi_i(U \cap U_i))) = M(U \cap U_i)$. Since g_i is a homeomorphism, then $M(U \cap U_i)$ is open in $M(X)$. Hence $M(U)$ is open subset of $M(X)$.

Theorem 2. *Let $\mathbb{A}$ be a unital commutative complete semisimple locally m-convex algebra, the spectrum $sp_{\mathbb{A}}(x)$ of which is compact for every $x \in \mathbb{A}$ and the map $x \mapsto sp_{\mathbb{A}}(x)$ is continuous ("strong spectral continuity"). Let $(M(X), M(\mathcal{A}))$ be the principal extension of the Riemann surface $(X, \mathcal{A})$ in $\mathbb{A}$. Then $M(K) \subset M(X)$ is closed for each closed subset $K \subset X$.*

Proof. Let (z_δ) be a converget net in $M(K)$ and $z = \lim_\delta z_\delta$. Since $z \in M(X)$ then $z = [(M(\phi(U)), a)]$, where $(U, \phi) \in \mathcal{A}$ and $a \in M(\phi(U))$. Now there exists a convergent net (a_δ) in $M(\phi(U))$ such that $\lim_\delta a_\delta = a$ and $z_\delta = [(M(\phi(U)), a_\delta)]$ for each δ. Then $sp(a_\delta) \subset \phi(U)$. Since $z_\delta \in M(K)$ for each δ then

$$sp(z_\delta) = \phi^{-1}(sp_{\mathbb{A}}(a_\delta)) \subset K.$$

Hence $sp_{\mathbb{A}}(a_\delta) \subset \phi(U \cap K)$.

On the other hand, $K \subset X$ is closed. Therefore $U \cap K$ is closed in U and $\phi(U \cap K)$ is closed in $\phi(U)$. As the map $x \mapsto sp_{\mathbb{A}}(x)$ is continuous, then

$$sp_{\mathbb{A}}(a_\delta) \to sp_{\mathbb{A}}(a) \quad \text{and} \quad sp_{\mathbb{A}}(a) \subset \phi(U \cap K) \subset \phi(K).$$

Consequently, $z \in M(K)$ and $M(K) \subset M(X)$ is closed.

Corollary 1. *Suppose that the conditions of Theorem 2 are satisfied. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be Riemann surfaces such that $X \subset Y$ is open and closed. Then $M(X) \subset M(Y)$ is an open and closed subset.*

Theorem 3. *Suppose that the conditions of Theorem 2 hold true. Let $(X, \mathcal{A})$ be a non connected Riemann surface. Then the principal extension $(M(X), M(\mathcal{A}))$ of $(X, \mathcal{A})$ is also a non connected strongly analytic manifold.*

Proof. Let $X_1 \subset X$ be such a nonempty open and closed subset that $X_1 \neq X$. Then $M(X_1) \subset M(X)$ is also nonempty open and closed subset by Theorems 1 and 2. Since, $X \setminus X_1 \subset X$ is open, then there exists a chart (U, ϕ) of the maximal atlas $\mathcal{A}'$ of X, for which $U \subset X \setminus X_1$. Let $z = [(M(\phi(U)), a)] \in M(X)$ with $sp(z) = \phi^{-1}(sp_{\mathbb{A}}(a)) \subset X \setminus X_1$. Then $sp(z) \not\subset X_1$, because of which $z \notin M(X_1)$. Thus, we have $M(X_1) \neq M(X)$, which proves the theorem.

Theorem 4. *Let $\mathbb{A}$ be a unital commutative complete semisimple locally m-convex algebra, spectrum $sp_{\mathbb{A}}(x)$ in which is compact for every $x \in \mathbb{A}$ and the map $x \mapsto sp_{\mathbb{A}}(x)$ is upper semicontinuous. Let $(X, \mathbb{A})$ be a Riemann surface and $(M(X), M(\mathcal{A}))$ the principal extension of $(X, \mathcal{A})$ in $\mathbb{A}$. Then $sp(U) \subset X$ is open for every open $U \subset M(X)$.*

Proof. Let $x \in sp(U)$. Then there is a $z \in U$ such that $z = [(M(\phi_i(U_i)), a_i)]$, where $x \in sp(z) = \phi_i^{-1}(sp_{\mathbb{A}}(a_i)) \subset U_i$, $a_i \in M(\phi_i(U_i)$ and $(U_i, \phi_i) \in \mathcal{A}$. On the other hand,

$$g_i^{-1}(U \cap g_i(M(\phi_i(U_i)))) \subset \mathbb{A} \quad \text{is open}$$

for every chart $(g_i(M(\phi_i(U_i))), g_i^{-1}) \in M(\mathcal{A})$. It means that

$$sp_{\mathbb{A}}(g_i^{-1}(U \cap g_i(M(\phi_i(U_i)))) \subset \mathbb{C} \quad \text{is open.}$$

Since $z \in U$ and $z = [(M(\phi_i(U_i)), a_i)] = g_i(a_i)$ with $a_i \in M(\phi_i(U_i))$, then $a_i \in g_i^{-1}(U \cap g_i(M(\phi_i(U_i)))$ and

$$sp_{\mathbb{A}}(a_i) \subset sp_{\mathbb{A}}(g_i^{-1}(U \cap g_i(M(\phi_i(U_i))))).$$

Now

$$x \in sp(z) = \phi_i^{-1}(sp_{\mathbb{A}}(a_i)) \subset \phi_i^{-1}(sp_{\mathbb{A}}(g_i^{-1}(U \cap g_i(M(\phi_i(U_i)))))),$$

where $\phi_i^{-1}(sp_{\mathbb{A}}(g_i^{-1}(U \cap g_i(M(\phi_i(U_i))))))$ is open in X. Since

$$x \in \phi_i^{-1}(sp_{\mathbb{A}}(g_i^{-1}(U \cap g_i(M(\phi_i(U_i)))))) \subset sp(U)$$

and x is an arbitrary element of sp(U), then $sp(U)$is open in X .

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