Journal of Mathematical Sciences, Vol. 96, No. 6, 1999

EXTENSIONS OF RIEMANN SURFACES IN TOPOLOGICAL ALGEBRAS

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UDC 517.546; 517.986

0. Introduction

Motivated by the previous work of S. T. M. Ackermans [1–3], we consider in this paper (strongly) analytic manifolds, modeled on a topological algebra **A**; in particular, **A** is a unital commutative complete locally m-convex Q-algebra (or else *Waelbroeck algebra* [7]). Our main result here is that such an analytic manifold can be embedded (strongly analytically) in the principal extension of a (suitably defined) Riemann surface (Theorem 4.1). By the latter notion, we still mean a strongly analytic manifold, as before, that can always be suitably defined for any given Riemann surface, provided the corresponding topological algebra (model) is of an appropriate class. In this context, one employs the spectral mapping theorem, within the framework of lmc-algebras, hence partially the type of topological algebras considered, as above [7].

The present paper is based on the author's Doctoral Thesis [6]. On this occasion, I would like to express my gratitude to Prof. Anastasios Mallios for help and encouragement, as well as for several valuable discussions on the subject matter of this work.

1. Preliminaries

We start by first recalling the following algebraic notion; namely, given an algebra **A** and $S \subseteq \mathbf{C}$ (the complexes), the *principal extension* of S in **A** is, by definition, the following subset of **A**:

$$M(S) := \{x \in \mathbf{A} : \mathrm{Sp}_{\mathbf{A}}(x) \subseteq S\}. \tag{1.1}$$

Now, since we are going to employ right away the "spectral mapping theorem" for lmc-algebras (see A. Mallios [7, p. 202, Theorem 3.1]) we assume in the sequel that

$$\mathbf{A} \text{ is a unital commutative complete locally } m\text{-convex (for short, lmc) algebra.} \tag{1.2}$$

Thus, given $S, S' \subseteq \mathbf{C}$, assume that $f : S \to S'$ is holomorphic, so that we consider the following map:

$$F : M(S) \to M(S'), \quad x \mapsto F(x) := \frac{1}{2\pi i} \int\limits_{G(X|Y)} f(z)(z-x)^{-1} dz, \tag{1.3}$$

where $G(X|Y)$ is a "closed regular curve," with $G(X|Y) \subseteq \Omega \cap (\mathrm{Sp}_{\mathbf{A}}(x))^c$ and Ω an open neighborhood of $\mathrm{Sp}_{\mathbf{A}}(x)$ in **C**, such that

$$\mathrm{Sp}_{\mathbf{A}}(x) \subseteq S \subseteq \Omega. \tag{1.3'}$$

(We note that (1.3) is well-defined, in view of (1.2), the result quoted above, and our hypothesis for $G(X|Y)$; note, e.g., that $z - x \in \mathbf{A}^{\bullet}$ (group of units of **A**), for any $z \in G(X|Y)$. See also [7, p. 203, (3.16)].)

Now, the map F, as in (1.3), is called the *principal extension* of (the given holomorphic map) f and is denoted by $M(f)$.

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory. Vol. 49, Functional Analysis–4, 1997.

In particular, assume now that the algebra $\mathbf{A}$ (cf. (1.2)) is, moreover, *semisimple*. (By definition, this means that $\varphi(x) = 0$, $\varphi \in \mathfrak{M}(\mathbf{A})$ (the *spectrum* of $\mathbf{A}$), implies $x = 0$.) Thus, by considering further the holomorphic maps $f : S_1 \to S_2$ and $g : S_2 \to S_2$, with open $S_1, S_2, S_3 \subseteq \mathbf{C}$, one obtains for the respective principal extensions

$$M(g \circ f) = M(g) \circ M(f), \tag{1.4}$$

while for any open $S \subseteq \mathbf{C}$, one has

$$M(\mathrm{id}_S) = \mathrm{id}_{M(S)}. \tag{1.5}$$

Thus, in view of (1.4) and (1.5), one obtains for any biholomorphic map $f : S \to S'$, with open $S \subseteq \mathbf{C}$, the relation

$$M(f^{-1}) = M(f)^{-1}. \tag{1.6}$$

2. Analytic Manifolds

Suppose we have a topological algebra $\mathbf{A}$, as in (1.2), and open $U \subseteq \mathbf{A}$. We say that a map

$$F : U \to \mathbf{A} \tag{2.1}$$

is *strongly analytic* if there exists a (local) holomorphic map

$$f : \mathrm{Sp}_{\mathbf{A}}(U) \to \mathbf{C}, \tag{2.2}$$

such that

$$F(x) = M(f)(x), \quad x \in U \subseteq M(\mathrm{Sp}_{\mathbf{A}}(U)). \tag{2.3}$$

Here we have set $\mathrm{Sp}_{\mathbf{A}}(U) := \bigcup_{x \in U} \mathrm{Sp}_{\mathbf{A}}(x)$. (We note that *for any open* $U \subseteq \mathbf{A}$, $\mathrm{Sp}_{\mathbf{A}}(U) \subseteq \mathbf{C}$ *is open, as well, for any unital topological algebra* $\mathbf{A}$ (A. Mallios).) To repeat it, one thus sets

$$F = M(f)\big|_{U \subseteq M(\mathrm{Sp}_{\mathbf{A}}(U))}. \tag{2.3$'$}$$

It is clear that *the identity map in* $\mathbf{A}$ *is strongly analytic.* On the other hand, if $\mathbf{A}$, as above, is also semisimple, then the composition of two strongly analytic maps is strongly analytic, as well (cf. (1.4)).

Now, given an arbitrary (nonvoid!) set X, a pair (U, φ), with $U \subseteq X$ and $\varphi : U \to \varphi(U) \subseteq \mathbf{A}$ one-to-one with $\varphi(U)$ open, is called a (local) *chart* of X with respect to $\mathbf{A}$. Two local charts (U, φ) and (V, ψ) of X, such that $U \cap V \neq \emptyset$, are said to be (strongly analytically) *compatible* whenever $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbf{A}$, while $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are strongly analytic maps. Thus, following the classical pattern, one further defines a strongly analytic atlas $\mathcal{A}$ of X, as well as a maximal one, modeled on $\mathbf{A}$, as above.

So we are now ready to give the following basic definition.

Definition 2.1. A strongly analytic manifold, with respect to a topological algebra $\mathbf{A}$, as in (1.2), is a pair $(X, \mathcal{A})$, where X is a given (nonempty) set and $\mathcal{A}$ a maximal strongly analytic atlas on X.

Thus, according to the very definitions, an atlas $\mathcal{A}$, as above, is a family

$$\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}, \tag{2.4}$$

where (U_i, φ_i) are charts of X, covering X, which are pairwise strongly analytically compatible. (We also assume that $\mathcal{A}$ is "complete," with respect to the compatibility of a given chart of X with those of $\mathcal{A}$.) An atlas $\mathcal{A}$ on X, as in (2.4), not necessarily complete, provides a topology in X, denoted by $\tau_{\mathcal{A}}$, whose open sets are those $M \subseteq X$ such that $\varphi_i(M \cap U_i) \subseteq \mathbf{A}$ is open, for every $i \in I$, as in (2.4).

For use immediately below, we consider the notion of (strongly analytically) *equivalent* manifolds $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, referring to the existence of a homeomorphism $f : X \to Y$, such that $f(\mathcal{A}) = \mathcal{B}$, where

$$f(\mathcal{A}) = \{(f(U), \varphi \circ f^{-1}) : (U, \varphi) \in \mathcal{A}\}.$$

3. Extension of a Riemann Surface

Consider a *Riemann surface* $(X, \mathcal{A})$, viz., a (usual) complex (analytic) manifold of complex dimension 1. Thus, here one has

$$\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}, \quad \text{with } \varphi_i(U_i) \subseteq \mathbf{C} \text{ open.} \tag{3.1}$$

Furthermore, consider the set

$$\{(\varphi_i(U_i), z_i), \quad \text{with } z_i \in \varphi_i(U_i \cap U_j) \text{ and } (U_j, \varphi_j) \in \mathcal{A}\}. \tag{3.2}$$

In the same set, we now define the following equivalence relation, that is, we say that

$$(\varphi_i(U_i), z_i) \sim (\varphi_j(U_j), z_j) \text{ if and only if } (\varphi_j \circ \varphi_i^{-1})(z_i) = z_j. \tag{3.3}$$

It is readily seen that, indeed, (3.3) is an equivalence relation in the set (3.2).

Now, consider the sets

$$W_{ij} := \varphi_i(U_i \cap U_j) \subseteq \mathbf{C}, \tag{3.4}$$

which are open by the hypothesis for $\mathcal{A}$, as well as the maps

$$\varphi_{ij} := \varphi_j \circ \varphi_i^{-1} : W_{ij} \equiv \varphi_i(U_i \cap U_j) \to W_{ji} \equiv \varphi_j(U_i \cap U_j), \tag{3.5}$$

which are biholomorphic, or else holomorphic isomorphisms, according to the same hypothesis. Thus, based on (3.3), one has

$$(\varphi_i(U_i), z_i) \sim (\varphi_j(U_j), \varphi_{ij}(z_i)) \tag{3.6}$$

for any $z_i \in W_{ij}$, a fact that will be of use in the sequel. Consequently, we get

$$[(\varphi_i(U_i), z_i)] = \{(\varphi_j(U_j), \varphi_{ij}(z_i)) : z_i \in \varphi_i(U_i \cap U_j)\}. \tag{3.7}$$

Now, assume that our topological algebra $\mathbf{A}$, as in (1.2), is also semisimple (i.e., the corresponding Gel'fand map of $\mathbf{A}$ is one-to-one [7]), while we next consider the (principal) extensions (cf. (1.1) and (3.4))

$$M(W_{ij}) := \{a \in \mathbf{A} : \mathrm{Sp}_{\mathbf{A}}(a) \subseteq W_{ij}\}, \tag{3.8}$$

along with those of the maps (3.5) (see also (1.3)):

$$M(\varphi_{ij}) \equiv F_{ij} : M(W_{ij}) \to M(W_{ji}). \tag{3.9}$$

Furthermore, we define the following equivalence relation:

$$\begin{gathered}(M(\varphi_i(U_i)), a_i) \sim (M(\varphi_j(U_j)), a_j), \\ \text{with } a_i \in M(\varphi_i(U_i \cap U_j)), \quad a_j \in M(\varphi_j(U_i \cap U_j)), \quad \text{such that } F_{ij}(a_i) = a_j.\end{gathered} \tag{3.10}$$

So we now set (see also (2.4))

$$M(X) := \{[(M(\varphi_i(U_i)), a_i)] : a_i \in M(\varphi_i(U_i)), \ i \in I\}. \tag{3.11}$$

On the other hand, we also define the maps (see (3.11))

$$g_i : M(\varphi_i(U_i)) \to M(X), \quad a \mapsto g_i(a) := [(M(\varphi_i(U_i)), a)] \tag{3.12}$$

for any $i \in I$. We prove next that g_i is one-to-one. If $a, b \in M(\varphi_i(U)i))$ and $g_i(a) = g_i(b)$, then (cf. (3.12))

$$[(M(\varphi_i(U_i)), a)] = [(M(\varphi_i(U_i)), b)],$$

so that, in view of (3.10), (3.9), and (1.5) (here we employ the semisimplicity of $\mathbf{A}$), one obtains

$$a = F_{ii}(b) \equiv M(\varphi_{ii})(b) = M(\mathrm{id}_{\mathbf{C}}|_{\varphi_i(U_i)})(b) = \mathrm{id}_{M(\varphi_i(U_i))}(b) = b,$$

which proves our claim. □

On the other hand, consider the family

$$M(\mathcal{A}) := \{(g_i(M(\varphi_i(U_i))), g_i^{-1}) : i \in I\}. \tag{3.13}$$

Thus, we prove now that $M(\mathcal{A})$ is a strongly analytic atlas of $M(X)$, modeled on the topological algebra $\mathbf{A}$, as above. First, in view of (3.12), one has

$$\emptyset \neq g_i(M(\varphi_i(U_i))) \subseteq M(X), \tag{3.14}$$

while, due to what we already proved for g_i, g_i^{-1} is one-to-one. Now, to proceed further, we still assume for $\mathbf{A}$ that

$$\begin{gathered}\operatorname{Sp}_{\mathbf{A}}(a) \subseteq \mathbf{C} \textit{ compact for every } a \in \mathbf{A}, \textit{ while the Newburg map} \\ \mathbf{A} \to \mathcal{K}(\mathbf{C}) : a \mapsto \operatorname{Sp}_{\mathbf{A}}(a) \\ \textit{is upper semicontinuous.}\end{gathered} \tag{3.15}$$

(We recall that the range of (3.15) is the space of the compact subsets of $\mathbf{C}$, with the Hausdorff–Fréchet metric.)

Consequently (cf. [6, Theorem 1.1]), given that $\varphi_i(U_i) \subseteq \mathbf{C}$ is open, its (principal) extension $M(\varphi_i(U_i)) \subseteq \mathbf{A}$ is open as well, that is,

$$g_i^{-1}(g_i(M(\varphi_i(U_i)))) = M(\varphi_i(U_i)) \subseteq \mathbf{A}$$

is open, so that the pair

$$(g_i(M(\varphi_i(U_i))), g_i^{-1}) \tag{3.16}$$

is a (local) chart of $M(X)$. Now, it is clear that these charts cover $M(X)$; indeed, if $[(M(\varphi_i(U_i)), a_i)] \in M(X)$ (cf. (3.11)), then (cf. (3.12))

$$[(M(\varphi_i(U_i)), a_i)] = g_i(a_i) \in g_i(M(\varphi_i(U_i))), \tag{3.17}$$

since, in view of (3.11), $a_i \in M(\varphi_i(U_i))$. Furthermore, the same charts are mutually (strongly analytically) compatible; for, given two intersecting charts from $M(\mathcal{A})$ (see (3.13)), one has

$$\begin{aligned} & g_i^{-1}\Big(g_i(M(\varphi_i(U_i))) \cap g_j(M(\varphi_j(U_j)))\Big) \\ &= g_i^{-1}\Big(\Big\{[(M(\varphi_i(U_i)), a_i)] : a_i = M(\varphi_{ij})(a_j),\ a_j \in M(\varphi_j(U_i \cap U_j))\Big\}\Big) \\ &= M(\varphi_{ij})^{-1}(M(\varphi_j(U_i \cap U_j))) \subseteq \mathbf{A} \ \text{ open}, \end{aligned} \tag{3.18}$$

since $M(\varphi_{ij})$ is certainly continuous and $M(\varphi_j(U_i \cap U_j)) \subseteq \mathbf{A}$ is open.

Finally, we prove that the respective coordinate transition functions are strongly analytic maps, that is, one first has (cf. (3.18))

$$\begin{aligned} & g_j^{-1} \circ (g_i^{-1})^{-1} : g_i^{-1}\Big(g_i(M(\varphi_i(U_i))) \cap g_j(M(\varphi_j(U_j)))\Big) \\ & M(\varphi_{ij})^{-1}(M(\varphi_j(U_i \cap U_j))) \to M(\varphi_{ji})^{-1}(M(\varphi_i(U_i \cap U_j))). \end{aligned} \tag{3.19}$$

However, one still obtains

$$(g_j^{-1} \circ (g_i^{-1})^{-1})(a_i) = (g_j^{-1} \circ g_i)(a_i) = g_j^{-1}(g_i(a_i)) = g_j^{-1}([(M(\varphi_i(U_i)), a_i)]) = a_j = M(\varphi_{ij})(a_i) \tag{3.20}$$

(cf. also (3.10), that is, one has (cf. 3.9)),

$$g_j^{-1} \circ (g_i^{-1})^{-1} = M(\varphi_{ij}) = F_{ij}, \tag{3.21}$$

the latter map being, by the very definitions, strongly analytic. An analogous argument is certainly valid for the inverse map of (3.19). Thus, by the preceding, we already proved our claim for the family $M(\mathcal{A})$, as defined by (3.13). □

Indeed, the above provide the proof of the following:

Theorem 3.1. *Let $(X, \mathcal{A})$ be a Riemann surface. Moreover, let $\mathbf{A}$ be a unital commutative complete semisimple locally m-convex algebra, having $\operatorname{Sp}_{\mathbf{A}}(a) \subseteq \mathbf{C}$ compact, for every $a \in \mathbf{A}$, and the corresponding Newburg map $\mathbf{A} \to \mathcal{K}(\mathbf{C})$* (cf. (3.15)) *upper semicontinuous. Then the pair* (cf. (3.11), (3.13))

$$(M(X), M(\mathcal{A})) \tag{3.22}$$

yields a strongly analytic manifold.

Remark 3.1. Concerning the hypothesis of Theorem 3.1, we note that if $\mathbf{A}$ is a locally m-convex Q-algebra, then both conditions in (3.15) are fulfilled (A. Mallios). A recent result of A. Beddaa [4] shows that any given topological algebra is Q if and only if the respective Newburg map is upper-semicontinuous.

So based on the previous theorem, we now set the following:

Definition 3.1. Given a Riemann surface $(X, \mathcal{A})$ and a topological algebra $\mathbf{A}$, as in Theorem 3.1, the strongly analytic manifold $(M(X), M(\mathcal{A}))$, as given by (3.22), is called the principal extension of the Riemann surface $(X, \mathcal{A})$ in the topological algebra $\mathbf{A}$.

4. Embedding of Strongly Analytic Manifolds

In this section, we examine the way that a strongly analytic manifold can be embedded in the principal extension of a Riemann surface.

Henceforth we assume that $(X, \mathcal{A})$ is a strongly analytic manifold modeled on a topological algebra $\mathbf{A}$, for which we still assume that it satisfies the conditions of Theorem 3.1 above.

Now, by considering the previous atlas $\mathcal{A}$, we define an equivalence relation between pairs of the form $(\varphi_i(U_i), a_i)$ with $(U_i, \varphi_i) \in \mathcal{A}$ and $a_i \in \varphi_i(U_i) \subseteq \mathbf{A}$, such that

$$(\varphi_i(U_i), a_i) \sim (\varphi_j(U_j), a_j) \underset{\text{def}}{\Longleftrightarrow} (\varphi_j \circ \varphi_i^{-1})(a_i) = a_j. \tag{4.1}$$

Furthermore, consider the sets

$$W_i = \mathrm{Sp}_{\mathbf{A}}(\varphi_i(U_i)) \quad \text{and} \quad W_{ij} = \mathrm{Sp}_{\mathbf{A}}(\varphi_i(U_i \cap U_j)) \tag{4.2}$$

along with the transition functions

$$\varphi_{ij} \equiv \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \subseteq \mathbf{A} \to \varphi_j(U_i \cap U_j) \subseteq \mathbf{A} \tag{4.3}$$

for any pair of (local) charts (U_i, φ_i) and (U_j, φ_j) in $\mathcal{A}$ (with $U_i \cap U_j \neq \emptyset$), which are, by the hypothesis for $\mathcal{A}$, strongly bianalytic. So there exists, by the very definitions, a local analytic map (see (4.2))

$$f_{ij} : W_{ij} \equiv \mathrm{Sp}_{\mathbf{A}}(\varphi_i(U_i \cap U_j)) \subseteq \mathbf{C} \to \mathbf{C}, \tag{4.4}$$

such that

$$M(f_{ij}) = \varphi_{ij}, \quad \text{restricted on} \quad \varphi_i(U_i \cap U_j) \subseteq M(W_{ij}). \tag{4.5}$$

On the other hand, we next consider the following equivalence relation (see also (4.2) and (4.4)):

$$(W_i, z_i) \sim (W_j, z_j) \quad (\text{with } z_i \in W_i) \underset{\text{def}}{\Longleftrightarrow} f_{ij}(z_i) = z_j. \tag{4.6}$$

Thus, we further define the set

$$Y = \{[(W_i, z_i)] : z_i \in W_i,\ i \in I\}, \tag{4.7}$$

along with the maps

$$\psi_i : W_i \to Y : z \mapsto \psi_i(z) := [(W_i, z)], \tag{4.8}$$

for any $i \in I$, each one being one-to-one. Indeed, if $z, z' \in W_i$ and $[(W_i, z)] = [(W_i, z')]$, then (cf. (4.6)) $f_{ii}(z) = z = z'$ (since $M(1) = 1$, cf. (1.5)).

Now, we further consider the family

$$\mathcal{B} = \{(\psi_i(W_i), \psi_i^{-1}) : i \in I\}, \tag{4.9}$$

which is proved to be an analytic atlas for Y. It is clear that it suffices to prove only the compatibility of the (local) charts in (4.9). Indeed, one gets

$$\psi_i^{-1}(\psi_i(W_i)\cap\psi_j(W_j)) = \psi_i^{-1}\Big(\{[(W_i,z_i)] : z_i\in W_i\}\cap\{[(W_j,z_j)] : z_j\in W_j\}\Big)$$
$$= \psi_i^{-1}\Big(\Big\{[(W_i,f_{ji}(z_j))] : z_j\in W_j\Big\}\Big) \tag{4.10}$$
$$= \{f_{ji}(z_j) : z_j\in W_j\} = f_{ji}(W_j)\subseteq \mathbf{C},$$

an open subset, given that φ_{ij} is strongly bianalytic (cf. also (4.5) and (1.6)). An analogous argument holds, of course, for the set

$$\psi_j^{-1}(\psi_i(W_i)\cap\psi_j(W_j)).$$

Now, concerning the corresponding transition functions between the same (local) charts in (4.9), one obtains (see (4.10))

$$\psi_j^{-1}\circ(\psi_i^{-1})^{-1} : f_{ji}(W_j)\subseteq\mathbf{C}\to f_{ij}(W_i)\subseteq\mathbf{C}. \tag{4.11}$$

Furthermore, one has (cf. (4.8))

$$(\psi_j^{-1}\circ\psi_i)(f_{ji}(z_j)) = \psi_j^{-1}([(W_i,f_{ji}(z_j))]) = \psi_j^{-1}([(W_j,z_j)]) = \psi_j^{-1}(\psi_j(z_j)) = z_j = f_{ij}(f_{ji}(z_j)).$$

Hence, one gets

$$\psi_j^{-1}\circ(\psi_i^{-1})^{-1} = \psi_j^{-1}\circ\psi_i = f_{ij}, \tag{4.12}$$

and analogously $\psi_i^{-1}\circ\psi_j = f_{ji}$, which thus proves our claim for (4.9). □

In other words,

$$(Y,\mathcal{B})\text{ provides a Riemann surface.} \tag{4.13}$$

Accordingly, based on Theorem 3.1, let now

$$(M(Y),M(\mathcal{B})) \tag{4.14}$$

be the principal extension of $(Y,\mathcal{B})$ with respect to (the given topological algebra) $\mathbf{A}$. Thus, our next task is to find an appropriate embedding

$$(X,\mathcal{A}) \underset{\rightarrow}{\subset} (M(Y),M(\mathcal{B})). \tag{4.15}$$

First, in view of (3.11), (4.7), and (4.9), one obtains

$$M(Y) = \{[(M(\psi_i^{-1}(\psi_i(W_i))),a_i)] : a_i\in M(W_i)\} = \{[(M(W_i),a_i)] : a_i\in M(W_i)\}, \tag{4.16}$$

while, based on (3.13) and (3.12), we also set

$$M(\mathcal{B}) = \{(g_i(M(W_i)),g_i^{-1}) : i\in I\}, \tag{4.17}$$

such that

$$g_i : M(W_i)\to M(Y) : a\mapsto g_i(a) := [(M(W_i),a)]. \tag{4.18}$$

Thus, we next consider the following map:

$$h : X\to M(Y), \tag{4.19}$$

such that

$$h(x) := [(M(W_i),\varphi_i(x))], \tag{4.19'}$$

where $(U_i,\varphi_i)\in\mathcal{A}$, with $x\in U_i$, so that $\varphi_i(x)\in\varphi_i(U_i)$ and $\mathrm{Sp}_{\mathbf{A}}(\varphi_i(x))\subseteq\mathrm{Sp}_{\mathbf{A}}(\varphi_i(U_i))\equiv W_i$, hence, $\varphi(x)\in M(W_i)$. Indeed, (4.19′) is independent of the above chart (U_i,φ_i), for, if $x\in(U_j,\varphi_j)\in\mathcal{A}$, since $(\varphi_j\circ\varphi_i^{-1})(\varphi_i(x)) = \varphi_j(x)$ one gets (see also (4.1) and (3.10))

$$(M(W_i),\varphi_i(x))\sim(M(W_j),\varphi_j(x)), \tag{4.20}$$

which proves the assertion, by virtue of (4.19′). □

$$(M(W_i), \varphi_i(x)) \sim (M(W_j), \varphi_j(y)),$$

hence, $(\varphi_j \circ \varphi_i^{-1})(\varphi_i(x)) = \varphi_j(y)$, that is, $\varphi_i(x) = \varphi_j(y)$; thus $x = y$. □

Furthermore, h is a homeomorphism; indeed, one proves that

$$h = g_i \circ \varphi_i|_{U_i} \tag{4.21}$$

(cf. (4.18), (2.4)), where $\varphi_i(U_i) \subseteq M(W_i) \equiv M(\mathrm{Sp}_{\mathbf{A}}(\varphi_i(U_i)))$. So, for every $x \in U_i$, one gets (cf. also (4.19′))

$$(g_i \circ \varphi_i)(x) = g_i(\varphi_i(x)) = [(M(W_i), \varphi_i(x))] = h(x), \tag{4.22}$$

viz. (4.21), which thus proves our assertion for h, due to our hypothesis for $g_i; \varphi_i$. □

Now, consider the pair

$$(h(X), h(\mathcal{A})) \equiv h((X, \mathcal{A})), \tag{4.23}$$

such that one has

$$h(\mathcal{A}) = \{(h(U_i), \varphi_i \circ h^{-1}) : (U_i, \varphi_i) \in \mathcal{A}\}. \tag{4.24}$$

So, by standard argument, one concludes that $(X, \mathcal{A})$ and $(h(X), h(\mathcal{A}))$ are strongly analytically isomorphic.

Furthermore, we note that

$$h(\mathcal{A}) \subseteq M(\mathcal{B})^* \tag{4.25}$$

(see (4.24), (4.17); here $*$ indicates the maximal (strongly analytic) atlas containing $M(\mathcal{B})$); that is, if

$$(h(U_i), \varphi_i \circ h^{-1}) \in h(\mathcal{A}),$$

then (cf. (4.19′), (4.21))

$$h(U_i) = \{[(M(W_i), \varphi_i(x))] : x \in U_i\} = g_i(\varphi_i(U_i) \subseteq g_i(M(W_i)), \tag{4.26}$$

that is, $h(U_i)$ is open in $M(Y)$. Moreover, based on (4.21), one obtains

$$\varphi_i \circ h^{-1} = g_i^{-1}|_{h(U_i)}, \tag{4.27}$$

which thus, together with (4.26), proves (4.25). □

Thus, (4.23) is, in effect, an open (strongly analytic) submanifold of (4.14).

We can now summarize the preceding into the following:

Theorem 4.1. *Let* $\mathbf{A}$ *be a topological algebra satisfying the conditions of Theorem 3.1. Then every strongly analytic manifold modeled on* $\mathbf{A}$ *is embedded, as an open strongly analytic submanifold, into a principal extension of a Riemann surface.*

REFERENCES

1. S. T. M. Ackermans, "On the principal extension of complex sets in a Banach algebra," *Indag. Math.*, **29**, 146–150 (1967).
2. S. T. M. Ackermans, "A case of strong spectral continuity," *Indag. Math.*, **29**, 455–459 (1967).
3. S. T. M. Ackermans, *The Principal Extension of Riemann Surfaces to a Banach Algebra*, T.H. Report 68. WSK–04 (1968).
4. A. Beddaa, "Caractérisations des algèbres localement multiplicativement convexes," to appear.
5. Z. Daoultzi-Malamou, "Strong spectral continuity in topological matrix algebras," *Boll. U.M.I.*, (7) 2-A, 213–219 (1988).
6. Z. Daoultzi-Malamou, *Infinite-dimensional holomorphy: Analytic manifolds modeled on topological algebras and extensions of Riemann surfaces* [Doctoral Dissertation], Univ. of Athens, to be submitted.
7. A. Mallios, *Topological Algebras. Selected Topics*, North-Holland, Amsterdam (1986).