

Strong Spectral Continuity in Topological Matrix Algebras.

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Sunto. - *Data una Q -algebra A commutativa localmente m -convessa con un elemento identità, si mostra che l'algebra $M_n(A)$ delle matrici $n \times n$ con elementi da A , ha la continuità spettrale forte (s.s.c.).*

1. - The vector spaces and algebras considered are over the field \mathbf{C} of complex numbers. We denote by $\mathcal{K}(\mathbf{C})$ the set of all compact subsets of \mathbf{C} , endowed with the *Hausdorff-Fréchet metric* [7: p. 63 ff.].

By a *locally m -convex algebra* we mean an algebra A which is a topological vector space with a local basis consisting of m -convex (i.e., multiplicative (viz. idempotent) and convex) sets (see [5] and/or [3]).

If A is a locally m -convex algebra, $\text{Sp}_A(x)$ denotes the spectrum of x in A ; this is a compact subset of \mathbf{C} if A is a Q -algebra (i.e., the set of quasi-regular elements is open [3]).

2. - Let A be a topological algebra. We denote by $M_n(A)$ the set of $n \times n$ matrices with entries in A which by the usual algebraic operations of matrices becomes a (complex) algebra. Thus, one gets the identification

$$(2.1) \quad M_n(A) = A^{n^2},$$

valid within a bijection; so one considers $M_n(A)$ topologized by (2.1), where A^{n^2} carries the cartesian product topology. Now since A^{n^2} is a topological vector space, the same holds for $M_n(A)$. Moreover, this topology makes the respective ring multiplication (composition of matrices) separately, resp. jointly, continuous if (and only if) this is the case for A .

Thus one has the following. For the notation applied see [4].

THEOREM 2.1. - *If A is a topological algebra consider the map*

$$(2.2) \quad D: A \rightarrow M_n(A): a \mapsto D(a) = (a_{ij}),$$

where $a_{ij} = a$, for any $i = j$, and $a_{ij} = 0$, for $i \neq j$, $1 \leq i, j \leq n$. Then, D is an (into) isomorphism of topological algebras. In particular, if A is a commutative locally m -convex algebra with an identity element, the map D preserves the corresponding semi-norms; i.e., one has

$$(2.3) \quad N_{\bar{p}}(D(a)) = p(a),$$

for every $a \in A$.

3. - If A is a topological algebra, we denote by $\mathcal{M}(A)$ the spectrum of A , i.e., the set of non-zero continuous complex morphisms of A with the relative topology from the weak dual A'_s of A . The Gel'fand map of A is given by $\mathfrak{G}: A \rightarrow C_c(\mathcal{M}(A)): x \mapsto \mathfrak{G}(x) \equiv \hat{x}$, such that $\hat{x}(f) = f(x)$, $f \in \mathcal{M}(A)$. Furthermore, one defines the radical of A by

$$(3.1) \quad \mathcal{R}(A) := \bigcap_{f \in \mathcal{M}(A)} \ker(f).$$

A is semi-simple if $\mathcal{R}(A) = (0)$ or, equivalently, whenever \mathfrak{G} is 1-1.

THEOREM 3.1. - Let A be a topological algebra. Then, one gets an algebra morphism

$$(3.2) \quad h: M_n(A) \rightarrow M_n(C(\mathcal{M}(A))),$$

defined by $h(\alpha) := (\hat{\alpha}_{ij})$, for every $\alpha = (a_{ij}) \in M_n(A)$. Besides, one has

$$(3.3) \quad \ker(h) = M_n(\mathcal{R}(A)).$$

Thus h is 1-1 if, and only if, A is semi-simple. In particular, h is continuous if, and only if, \mathfrak{G} is.

Now given an algebra A with an identity element, we denote by A^* the group of its invertible elements. Moreover, we denote by $\rho(x)$ the resolvent set of $x \in A$. On the other hand, if A is a topological algebra, one gets, for every $f \in \mathcal{M}(A)$, the following continuous algebra morphism:

$$(3.4) \quad \check{f}: M_n(A) \rightarrow M_n(C): \alpha = (a_{ij}) \mapsto \check{f}(\alpha) = \hat{\alpha}(f) := (f(a_{ij})).$$

We come next to the following.

THEOREM 3.2. - Let A be a commutative advertibly complete locally m -convex algebra with an identity element. Moreover, let $M_n(A)$ be

