Bollettino U.M.I.
(7) 2-A (1988), 213-219

Strong Spectral Continuity in Topological Matrix Algebras.

Z. Daoultzi - Malamou

Sunto. – *Data una Q-algebra A commutativa localmente m-convessa con un elemento identità, si mostra che l'algebra $M_n(A)$ delle matrici $n \times n$ con elementi da A, ha la continuità spettrale forte (s.s.c.).*

1. – The vector spaces and algebras considered are over the field $\boldsymbol{C}$ of complex numbers. We denote by $\mathcal{K}(\boldsymbol{C})$ the set of all compact subsets of $\boldsymbol{C}$, endowed with the *Hausdorff-Fréchet metric* [7: p. 63 ff.].

By a *locally m-convex algebra* we mean an algebra A which is a topological vector space with a local basis consisting of m-convex (i.e., multiplicative (viz. idempotent) and convex) sets (see [5] and/or [3]).

If A is a locally m-convex algebra, $\mathrm{Sp}_A(x)$ denotes the spectrum of x in A; this is a compact subset of $\boldsymbol{C}$ if A is a Q-algebra (i.e., the set of quasi-regular elements is open [3]).

2. – Let A be a topological algebra. We denote by $M_n(A)$ the set of $n \times n$ matrices with entries in A which by the usual algebraic operations of matrices becomes a (complex) algebra. Thus, one gets the identification

$$M_n(A) = A^{n^2}, \tag{2.1}$$

valid within a bijection; so one considers $M_n(A)$ topologized by (2.1), where A^{n^2} carries the cartesian product topology. Now since A^{n^2} is a topological vector space, the same holds for $M_n(A)$. Moreover, this topology makes the respective ring multiplication (composition of matrices) separately, resp. jointly, continuous if (and only if) this is the case for A.

Thus one has the following. For the notation applied see [4].

Theorem 2.1. – *If A is a topological algebra consider the map*

$$D: A \to M_n(A): a \mapsto D(a) = (a_{ij}), \tag{2.2}$$

where $a_{ij} = a$, *for any* $i = j$, *and* $a_{ij} = 0$, *for* $i \neq j$, $1 \leqslant i, j \leqslant n$. *Then, D is an (into) isomorphism of topological algebras. In particular, if A is a commutative locally m-convex algebra with an identity element, the map D preserves the corresponding semi-norms; i.e., one has*

$$N_{\bar{p}}(D(a)) = p(a)\,, \tag{2.3}$$

for every $a \in A$.

3. – If A is a topological algebra, we denote by $\mathcal{M}(A)$ the *spectrum* of A, i.e., the set of non-zero continuous complex morphisms of A with the relative topology from the weak dual A'_s of A. The *Gel'fand map* of A is given by $\mathcal{G}: A \to \mathcal{C}_c(\mathcal{M}(A)): x \mapsto \mathcal{G}(x) \equiv \hat{x}$, such that $\hat{x}(f) = f(x)$, $f \in \mathcal{M}(A)$. Furthermore, one defines the *radical* of A by

$$\mathcal{R}(A) := \bigcap_{f \in \mathcal{M}(A)} \ker(f)\,. \tag{3.1}$$

A is *semi-simple* if $\mathcal{R}(A) = (0)$ or, equivalently, whenever $\mathcal{G}$ is 1-1.

THEOREM 3.1. – *Let A be a topological algebra. Then, one gets an algebra morphism*

$$h: M_n(A) \to M_n\big(\mathcal{C}(\mathcal{M}(A))\big)\,, \tag{3.2}$$

defined by $h(\alpha) := (\hat{a}_{ij})$, *for every* $\alpha = (a_{ij}) \in M_n(A)$. *Besides, one has*

$$\ker(h) = M_n(\mathcal{R}(A))\,. \tag{3.3}$$

Thus h is 1-1 if, and only if, A is semi-simple. In particular, h is continuous if, and only if, $\mathcal{G}$ is.

Now given an algebra A with an identity element, we denote by $A^{\bullet}$ the *group of its invertible elements*. Moreover, we denote by $\varrho(x)$ the *resolvent set* of $x \in A$. On the other hand, if A is a topological algebra, one gets, for every $f \in \mathcal{M}(A)$, the following continuous algebra morphism:

$$\check{f}: M_n(A) \to M_n(\mathbf{C}): \alpha = (a_{ij}) \mapsto \check{f}(\alpha) = \hat{\alpha}(f) := \big(f(a_{ij})\big)\,. \tag{3.4}$$

We come next to the following.

THEOREM 3.2. – *Let A be a commutative advertibly complete locally m-convex algebra with an identity element. Moreover, let $M_n(A)$ be* *the respective algebra of $n \times n$ matrices with entries in A. Then, one has*

$$\mathrm{Sp}_{M_n(A)}(\alpha) = \bigcup_{f \in \mathcal{M}(A)} \mathrm{Sp}_{M_n(\boldsymbol{C})}(\hat{\alpha}(f)), \tag{3.5}$$

for every $\alpha \in M_n(A)$.

PROOF. – It is enough to show that

$$\varrho(\alpha) = \bigcap_{f \in \mathcal{M}(A)} \varrho(\hat{\alpha}(f)). \tag{3.6}$$

Indeed, one has

$$\varrho(\alpha) = \{\lambda \in \boldsymbol{C} : \alpha - \lambda I_n \in M_n(A)^{\bullet}\} = \{\lambda \in \boldsymbol{C} : \det(\alpha - \lambda I_n) \in A^{\bullet}\} =$$
$$= \{\lambda \in \boldsymbol{C} : f(\det(\alpha - \lambda I_n)) \neq 0,\ f \in \mathcal{M}(A)\}$$

(see [3: p. 98, Corollary 5.2]). On the other hand, one has

$$f(\det(\alpha - \lambda I_n)) = \det(\check{f}(\alpha - \lambda I_n)) = \det(\check{f}(\alpha) - \lambda I).$$

Therefore, we have

$$\varrho(\alpha) = \{\lambda \in \boldsymbol{C} : \det(\check{f}(\alpha) - \lambda I) \neq 0,\ f \in \mathcal{M}(A)\} = \bigcap_{f \in \mathcal{M}(A)} \varrho(\hat{\alpha}(f)). \quad \blacksquare$$

COROLLARY 3.1. – *Suppose that the conditions of Theorem 3.2 are satisfied, and let $\alpha = (a_{ij}) \in M_n(A)$ be a triagonal matrix. Then,*

$$\mathrm{Sp}_{M_n(A)}(\alpha) = \bigcup_{i=1}^{n} \mathrm{Sp}_A(a_{ii}). \tag{3.7}$$

PROOF. – Since $f(\det(\alpha - \lambda I_n)) = \prod_{i=1}^{n} (f(a_{ii} - \lambda))$, $f \in \mathcal{M}(A)$, the proof of the previous theorem yields $\lambda \in \varrho(\alpha)$ if, and only if, $\lambda \in \bigcap_{i=1}^{n} \varrho(a_{ii})$, which proves (3.7). $\blacksquare$

4. – To state the next result we need some more terminology. Thus, given an algebra A with an identity element, we denote by

$$R(A) \tag{4.1}$$

the set of (*properly*) *nilpotent elements* of A (« *radical* » of A; cf. [6: p. 161]). This coincides with the intersection of all maximal left ideals of A (ibid. p. 161 I). Thus, it is clear from (3.1) that

$$R(A) \subseteq \mathfrak{R}(A)\,. \tag{4.2}$$

In particular, *if A is a commutative Gel'fand-Mazur Q-algebra with an identity element one has*

$$R(A) = \mathfrak{R}(A) \tag{4.3}$$

(cf. [3: Chapt. II; Corollary 7.3]). We note that *every commutative locally m-convex algebra is a Gel'fand-Mazur algebra* (something more general is actually true; cf. [3: Chapt. VIII; 9.(5)]).

We are now in the position to state the main result of this section.

THEOREM 4.1. – *Let A be a commutative locally m-convex Q-algebra with an identity element. Then, one has*

$$R(M_n(A)) = M_n(R(A)) = M_n(\mathfrak{R}(A))\,, \qquad n \in N\,. \tag{4.4}$$

PROOF. – We know that $\alpha = (a_{ij}) \in R(M_n(A))$ if, and only if, $I_n - \beta\cdot\alpha \in M_n(A)^{\bullet}$, for every $\beta \in M_n(A)$ [6: p. 162, II]. So let $\beta = (b_{ij}) \in M_n(A)$, with $b_{lk} = b \neq 0$ and $b_{ij} = 0$, for any $(i, j) \neq (l, k)$; then $I_n - \beta\cdot\alpha \in M_n(A)^{\bullet}$ if, and only if, $\det(I_n - \beta\cdot\alpha) \in A^{\bullet}$. But, $\det(I_n - \beta\alpha) = 1_A - b\cdot a_{kl}$; so the existence of $(1_A - ba_{kl})^{-1}$, for every $b \in A$, is equivalent with $a_{kl} \in R(A)$. Thus $\alpha = (a_{kl}) \in M_n(R(A))$, i.e.,

$$R(M_n(A)) \subseteq M_n(R(A))\,. \tag{4.5}$$

Now, if $\alpha = (a_{ij}) \in M_n(R(A))$, then (cf. (4.3)) $f(a_{ij}) = 0$, for every $f \in \mathcal{M}(A)$; hence (Theorem 3.2), since every Q-algebra is advertibly complete [9], $\mathrm{Sp}_{M_n(A)}(\alpha) = 0$. Therefore $r(\alpha) = 0$, so that $\alpha \in M_n(R(A))$ is a quasi-regular element; that is the (2-sided) ideal $M_n(R(A))$ is a regular ideal of $M_n(A)$. Consequently, $\alpha \in M_n(R(A)) \subseteq R(M_n(A))$ (cf. [8: p. 55, Theorem (2.3.2)]), which together with (4.5) proves (4.4). ∎

5. – We come now to our main result, concerning the s.s.c. of $M_n(A)$ for suitable topological algebras A. Thus, we first have the following criterion.

THEOREM 5.1. – *Let A be a topological algebra, such that* $\mathrm{Sp}_A(x) \subseteq \boldsymbol{C}$ *is compact, for every $x \in A$. Then, the following assertions are equivalent:*

1) *The algebra A has the s.s.c.*
2) *For every closed $S \subseteq \boldsymbol{C}$, the set*

$$M(S, A) = \{x \in A : \mathrm{Sp}_A(x) \subseteq S\} \tag{5.1}$$

is a closed subset of A.

PROOF. – The map

$$a \mapsto \mathrm{Sp}_A(a) : A \to \mathcal{K}(\boldsymbol{C}) \tag{5.2}$$

is continuous if, and only if, for any $a \in A$, $\lambda \in \mathrm{Sp}_A(a)$, and $\Omega \subseteq \boldsymbol{C}$ an (open) neighborhood of λ, there exists an open neighborhood U of α in A such that

$$\mathrm{Sp}_A(x) \cap \Omega \neq \emptyset \tag{5.3}$$

for every $x \in U$. Now, if 1) is valid and $S \subseteq \boldsymbol{C}$ is closed, consider the set

$$N(\complement S, A) = \{x \in A : \mathrm{Sp}_A(x) \cap \complement S \neq \emptyset\}, \tag{5.4}$$

so that (cf. (5.1)) one has $N(\complement S, A) = \complement M(S, A)$.

Now we prove that (5.4) is open, hence the assertion 1) $\Rightarrow$ 2): Indeed, the assertion is a direct consequence of the hypothesis and (5.3).

On the other hand, 2) $\Rightarrow$ 1): That is, if $a \in A$, $\lambda \in \mathrm{Sp}_A(a)$, and Ω is an open neighborhood of λ, the set $\complement M(\complement \Omega, A) = N(\Omega, A) = \{x \in A : \mathrm{Sp}_A(x) \cap \Omega \neq \emptyset\}$ is by hypothesis an open neighborhood of α satisfying, of course, (5.3) and this completes the proof. ■

So we finally have the next.

THEOREM 5.2. – *Let A be a commutative locally m-convex Q-algebra with an identity element. Then, the algebra $M_n(A)$ satisfies s.s.c.*

PROOF. – We first remark that the algebra $M_n(\boldsymbol{C})$ fulfils s.s.c. [2]. Thus (Theorem 5.1), for every closed $S \subseteq \boldsymbol{C}$, $M(S, M_n(\boldsymbol{C}))$ (cf. (5.1)) is a closed subset of $M_n(\boldsymbol{C})$.

Now, by considering the continuous map (3.4), one obtains that, for any $f \in \mathcal{M}(A)$ and closed $S \subseteq \boldsymbol{C}$,

$$M_f(S) = \check{f}^{-1}\big(M(S, M_n(\boldsymbol{C}))\big) \tag{5.5}$$

is a closed subset of $M_n(A)$. On the other hand, we still have

$$M(S, M_n(A)) = \bigcap_{f \in \mathcal{M}(A)} M_f(S) \tag{5.6}$$

(see also (3.4), (3.5)). Thus, (5.6) is a closed subset of $M_n(A)$, for every closed $S \subseteq \boldsymbol{C}$, which yields the assertion, according to Theorem 5.1. ∎

The next result asserts that s.s.c. in an algebra A as above is « hereditary » with respect to subalgebras of A, this being also a characteristic property.

However, we first comment a bit more on the terminology. Thus, for any algebra A and any subalgebra B of A one has

$$\mathrm{Sp}_A(x) \subseteq \mathrm{Sp}_B(x)\,, \tag{5.7}$$

with $x \in B \subseteq A$. On the other hand, we get.

LEMMA 5.1. – *Let A be a topological Q-algebra with an identity element and B a closed subalgebra of A with the same identity element which is also a topological Q-algebra in the relative topology. Then,*

$$\mathrm{bd}\, \complement B^{\bullet} \subseteq B \cap \mathrm{bd}\, \complement A^{\bullet}\,. \tag{5.8}$$

The proof follows standard patterns. See, for instance, [8: p. 22, Theorem (1.5.7)] and [3].

Thus, we come next to the following.

THEOREM 5.3. – *Let A be a topological Q-algebra with an identity element. Then, A satisfies s.s.c. if, and only if, this is the case for every closed subalgebra B of A with the same identity element, which is also a topological Q-algebra in the relative topology.*

PROOF. – It suffices, of course, to prove the « only if » part of the assertion. Thus, suppose that B is a subalgebra of A satisfying the hypothesis. We shall prove that cond. 2) of Theorem 5.1, holds: So suppose that $S \subseteq \boldsymbol{C}$ is a closed set, and let

$$M(S, B) = \{x \in B : \mathrm{Sp}_B(x) \subseteq S\} \tag{5.9}$$

Now, let $(x_\delta) \subseteq M(S, B)$, with $x = \lim_\delta x_\delta$, and assume that $x \notin M(S, B)$, i.e., $\mathrm{Sp}_B(x) \cap \complement S \neq \emptyset$. Thus, if $\lambda \in \mathrm{Sp}_B(x) \cap \complement S$, then $x - \lambda \cdot e = \lim_\delta (x_\delta - \lambda e) \in \complement B^\bullet$, while $x_\delta - \lambda e \in B^\bullet$, by (5.9). Therefore (Lemma 5.1), $x - \lambda e \in \complement A^\bullet$, that is, $\lambda \in \mathrm{Sp}_A(x)$. On the other hand, by hypothesis, $M(S, A) \subseteq A$ is closed, with $x_\delta \in M(S, A)$ (cf. (5.7)). Hence, $x \in M(S, A)$, so that $\lambda \in \mathrm{Sp}_A(x) \subseteq S$, which is a contradiction, and the proof is complete. ∎

Acknowledgment. – The author expresses her gratitude to Professor Anastasios Mallios for introducing her into this area of study, and for several helpful and stimulating discussions during the preparation of this paper.

REFERENCES

[1] S. T. M. Ackermans, *On the principal extension of complex sets in a Banach algebra*, Indag. Math., **29** (1967), 146-150.

[2] S. T. M. Ackermans, *A case of strong spectral continuity*, Indag. Math., **29** (1967), 455-459.

[3] A. Mallios, *Topological algebras. Selected topics*, North-Holland, Amsterdam, 1986.

[4] A. Mallios, *Hermitian K-theory over topological *-algebras*, J. Math. Anal. Appl., **106** (1985), 454-539.

[5] E. A. Michael, *Locally multiplicativelly-convex topological algebras*, Mem. Amer. Math. Soc., **11** (1952).

[6] M. A. Naimark, *Normed algebras*, Groningen, 1972.

[7] O. A. Nielsen, *Direct integral theory*, Marcel Decker, New York, 1980.

[8] C. E. Rickart, *General theory of Banach algebras*, Princeton Univ. Press, Princeton, New Jersey, 1960.

[9] S. Warner, *Polynomial completeness in locally multiplicatively-convex algebras*, Duke Math. J., **23** (1956), 1-11.

Mathematical Institute, University of Athens, Greece

Pervenuta in Redazione
il 9 gennaio 1987