Magnetic Driving of Gamma-Ray Burst Outflows

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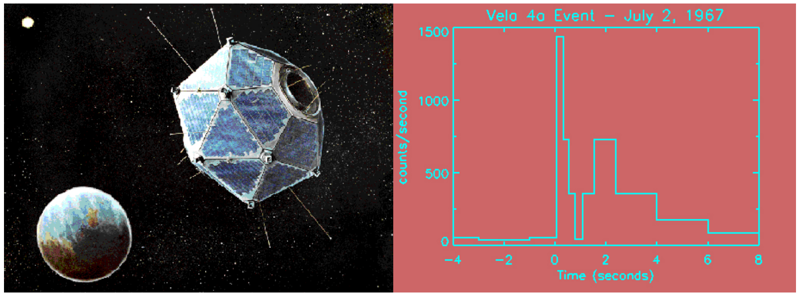
Outline

- GRBs and their afterglows
  - observations
  - our understanding
- the MHD description
  - general theory
  - the model
  - results

Observations

- 1967: the first GRB
  Vela satellites
  (first publication on 1973)



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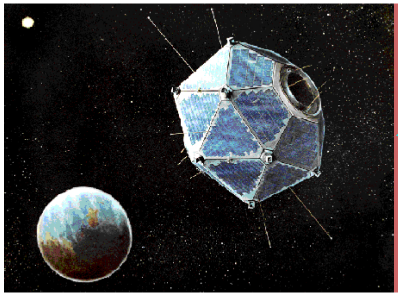

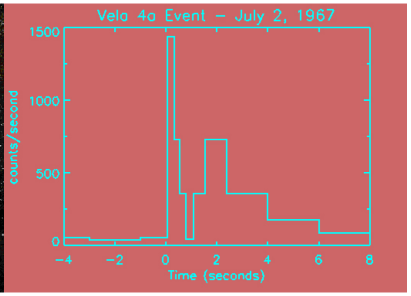

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  2704 GRBs (until May 2000)
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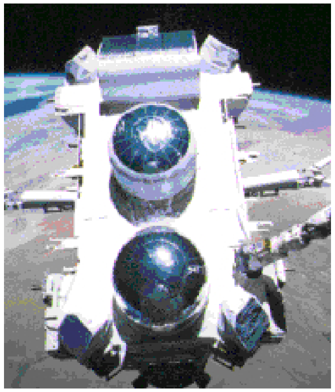

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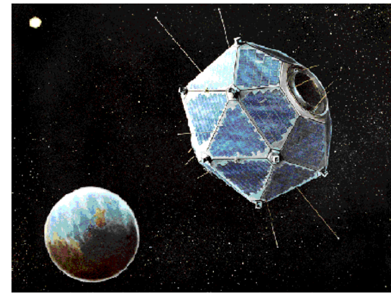
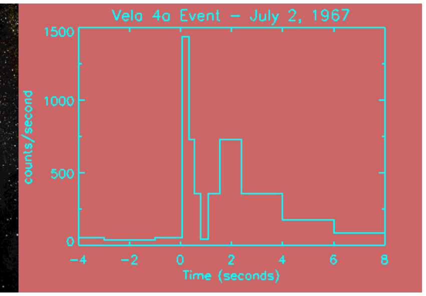

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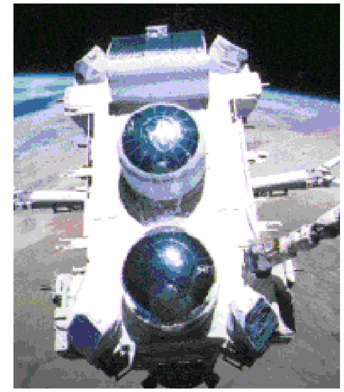

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  X-ray afterglow
  arc-min accuracy positions
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  GRB afterglow at longer wavelengths
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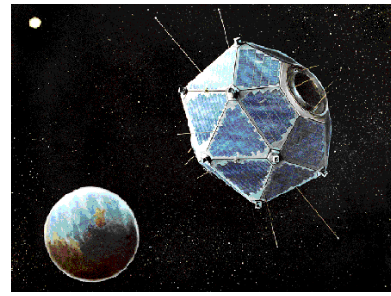

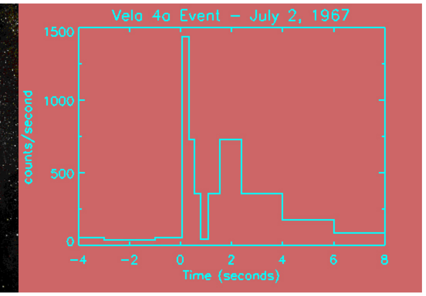

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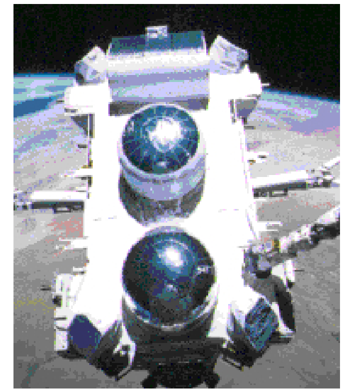

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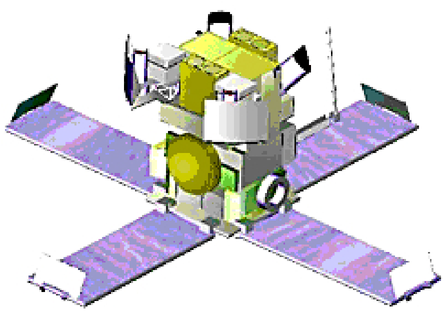

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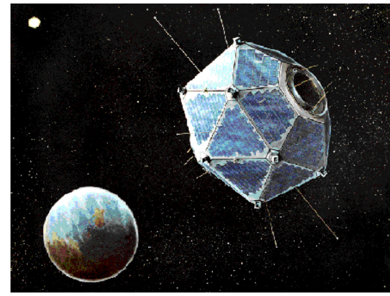

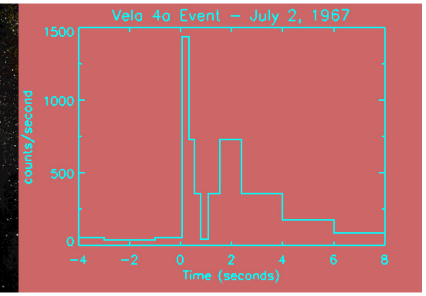

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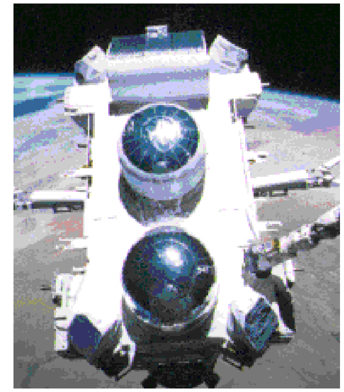

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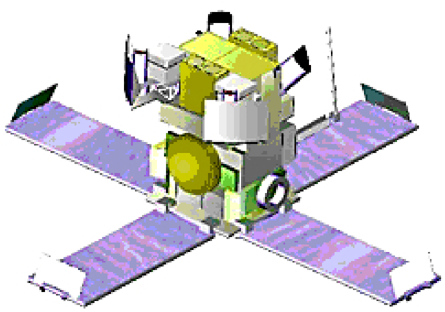
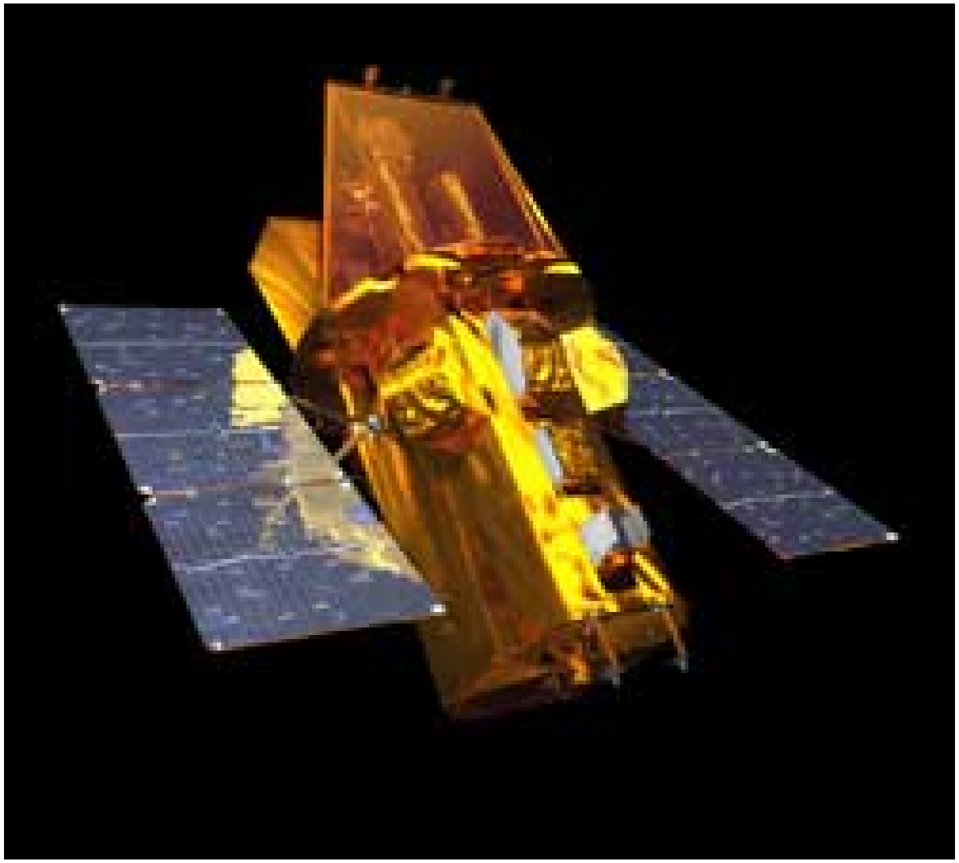


- INTErnationalGamma-RayAstrophysicsLaboratory

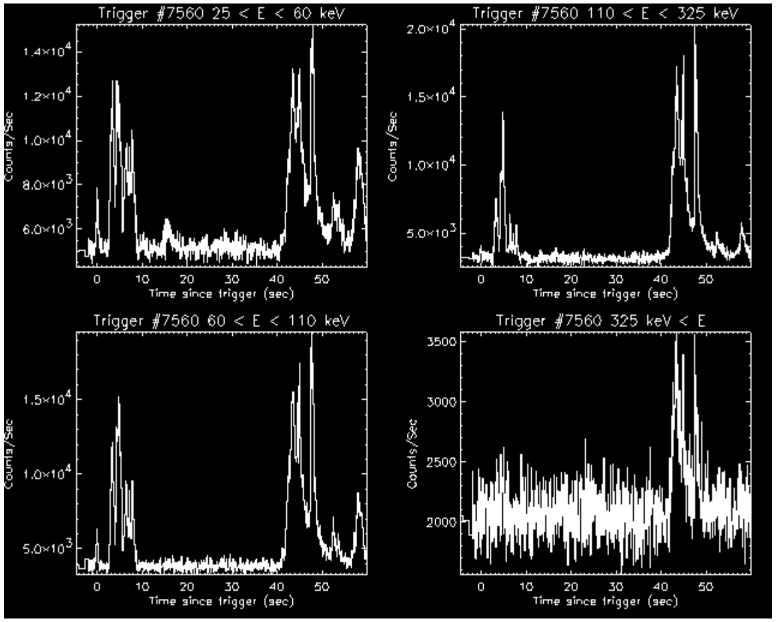
Swift

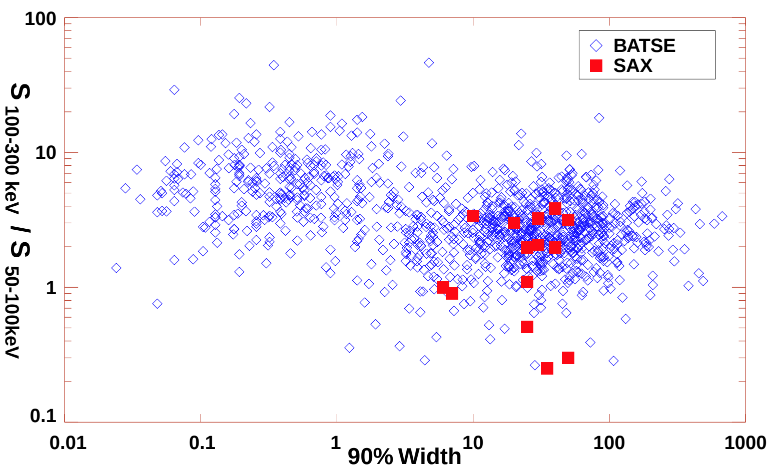


http://swift.gsfc.nasa.gov

Gamma-ray Burst Real-time Sky Map @ http://grb.sonoma.edu/

GRB prompt emission



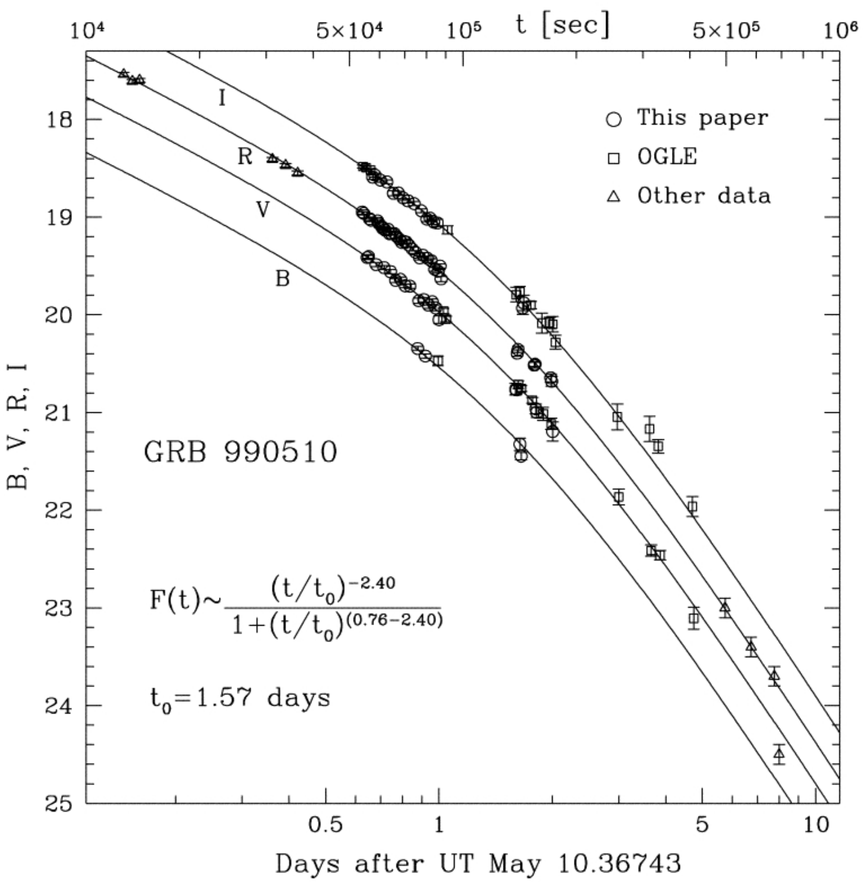


(from Djorgovski et al. 2001)

- Fluence $F_\gamma = 10^{-8} - 10^{-3}$ergs/cm^2
  energy
  $$E_\gamma = 10^{53}\left(\frac{D}{3\ \mathrm{Gpc}}\right)^2\left(\frac{F_\gamma}{10^{-4}\,\frac{\mathrm{ergs}}{\mathrm{cm}^2}}\right)\left(\frac{\Delta\omega}{4\pi}\right)\ \text{ergs}$$
  collimation $\begin{cases}\text{reduces } E_\gamma \\ \text{increases the rate of events}\end{cases}$
- non-thermal spectrum
- Duration $\Delta t = 10^{-3} - 10^3$s
  long bursts > 2 s, short bursts < 2 s
- Variability $\delta t = \Delta t / N\,,\, N = 1 - 1000$
  compact source $R < c\,\delta t \sim 1000$ km
  not a single explosion
  huge optical depth for $\gamma\gamma \rightarrow e^+e^-$
  compactness problem: how the photons escape?

relativistic motion $\gamma \gtrsim 100$ $\begin{cases} R < \gamma^2 c\,\delta t \\ \text{blueshifted photon energy} \\ \text{beaming} \\ \text{optically thin}\end{cases}$

Afterglow



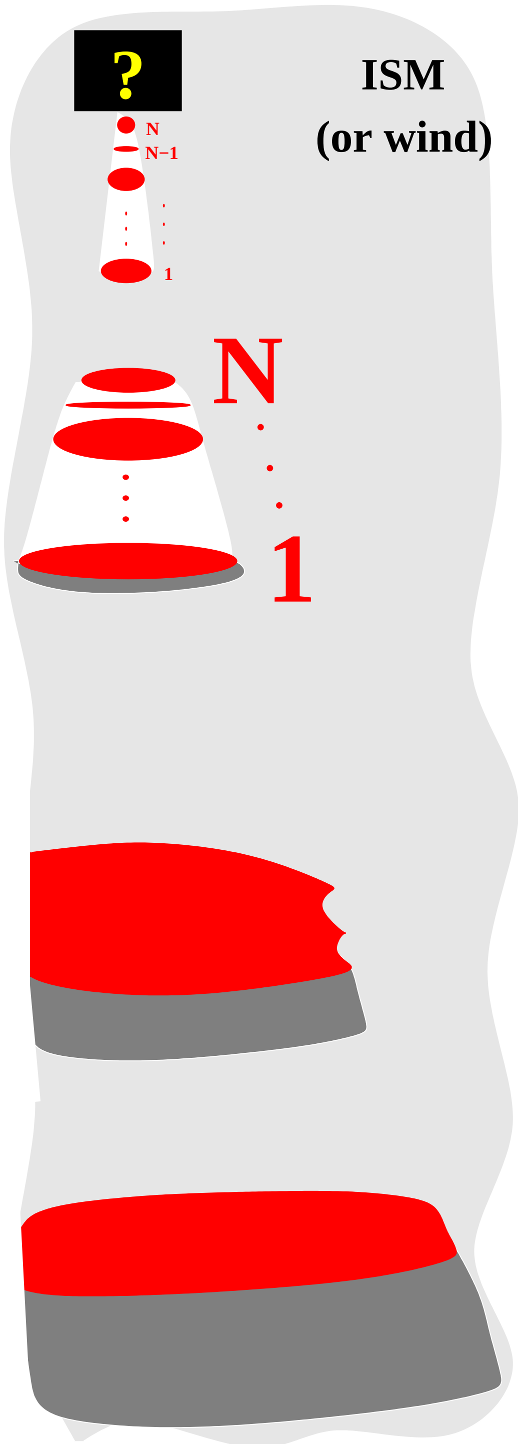
(from Stanek et al. 1999)

- from X-rays to radio
- fading – broken power law

  panchromatic break $F_\nu \propto \begin{cases} t^{-a_1}, t < t_o \\ t^{-a_2}, t > t_o \end{cases}$
- non-thermal spectrum

  (synchrotron + inverse Compton
  with power law electron energy distribution)

The internal–external shocks model



mass outflow (pancake)

N shells (moving with different $\gamma \gg 1$)
Frozen pulse
(if ℓ the path's arclength,
$s \equiv ct - \ell = const$ for each shell,
$\delta s = const$ for two shells)

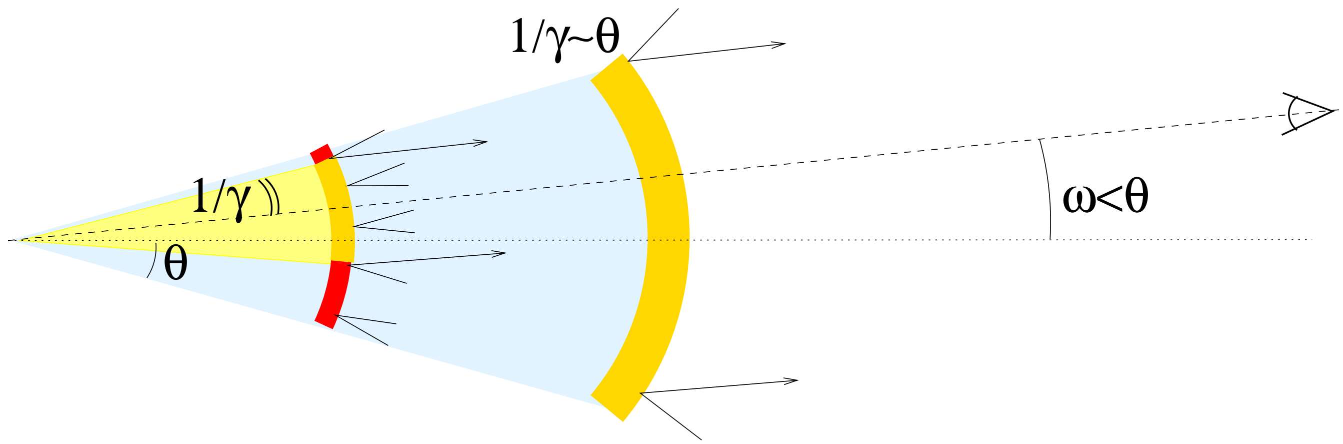
internal shocks

(a few tens of kinetic energy $\longrightarrow$ **GRB**)

external shock

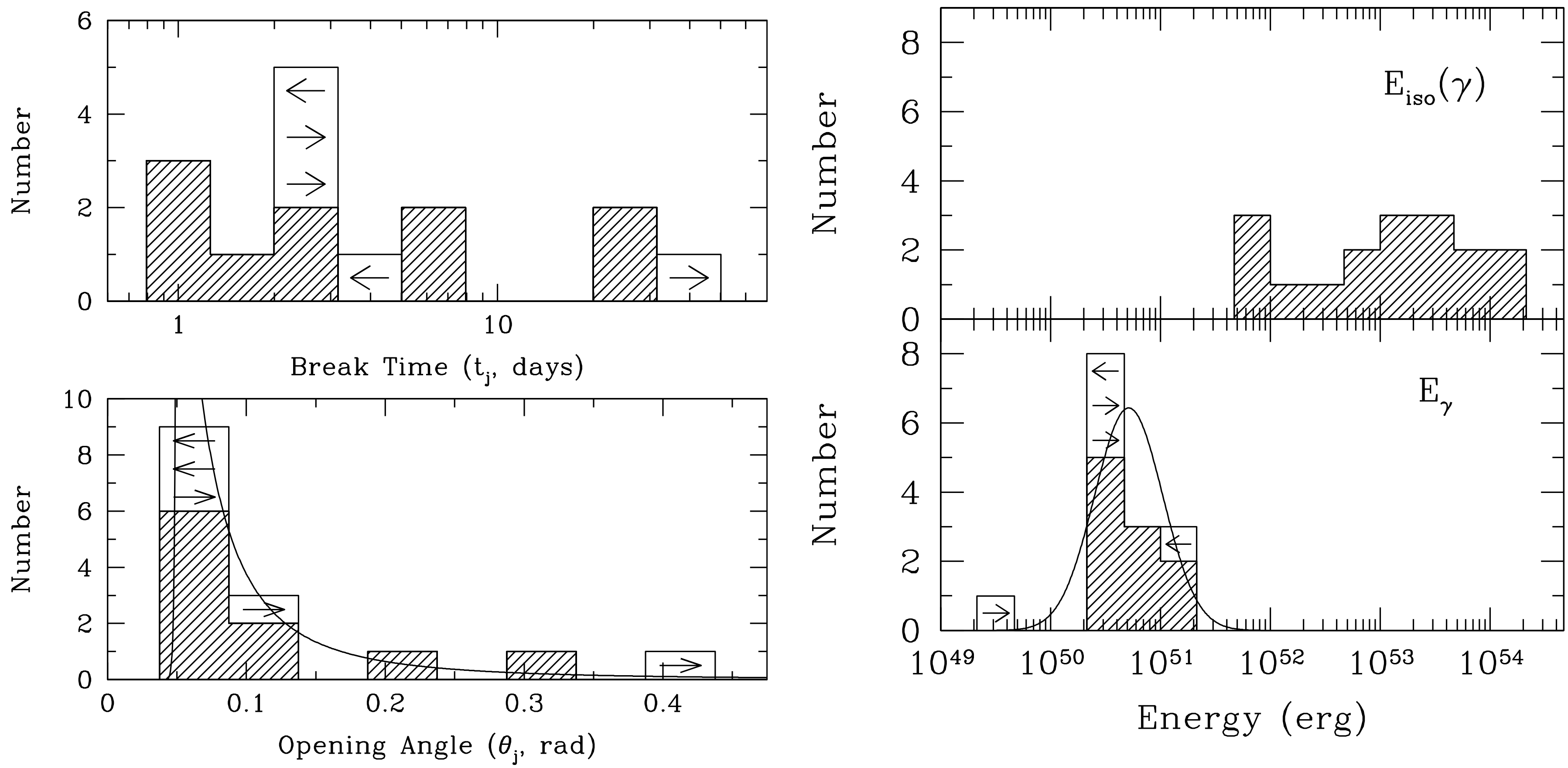
interaction with ISM (or wind)
(when the flow accumulates $M_{ISM} = M/\gamma$)
As γ decreases with time, kinetic energy $\longrightarrow$ X-rays . . . radio
$\longrightarrow$ **Afterglow**

Beaming – Collimation



- During the afterglow γ decreases
  When $1/\gamma > \vartheta$ the $F(t)$ decreases faster
  The broken power-law justifies collimation

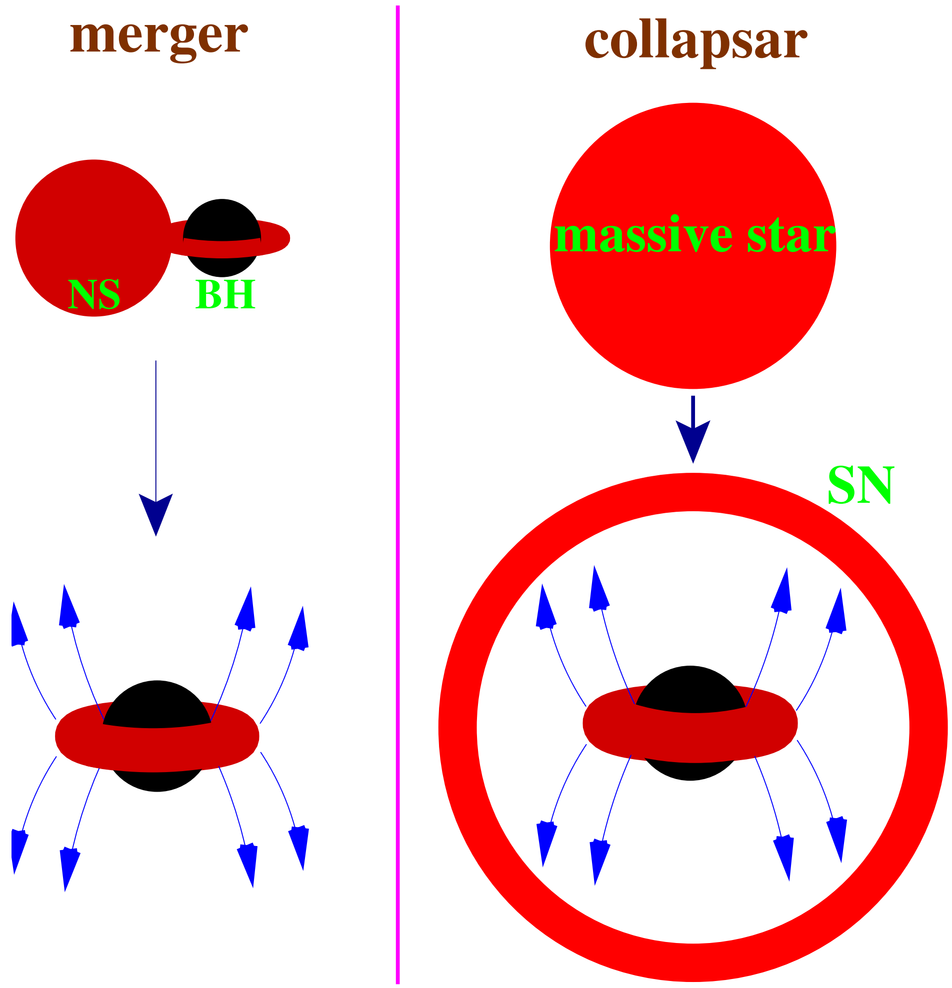
- orphan afterglows ?
  (for $\omega > \vartheta$)



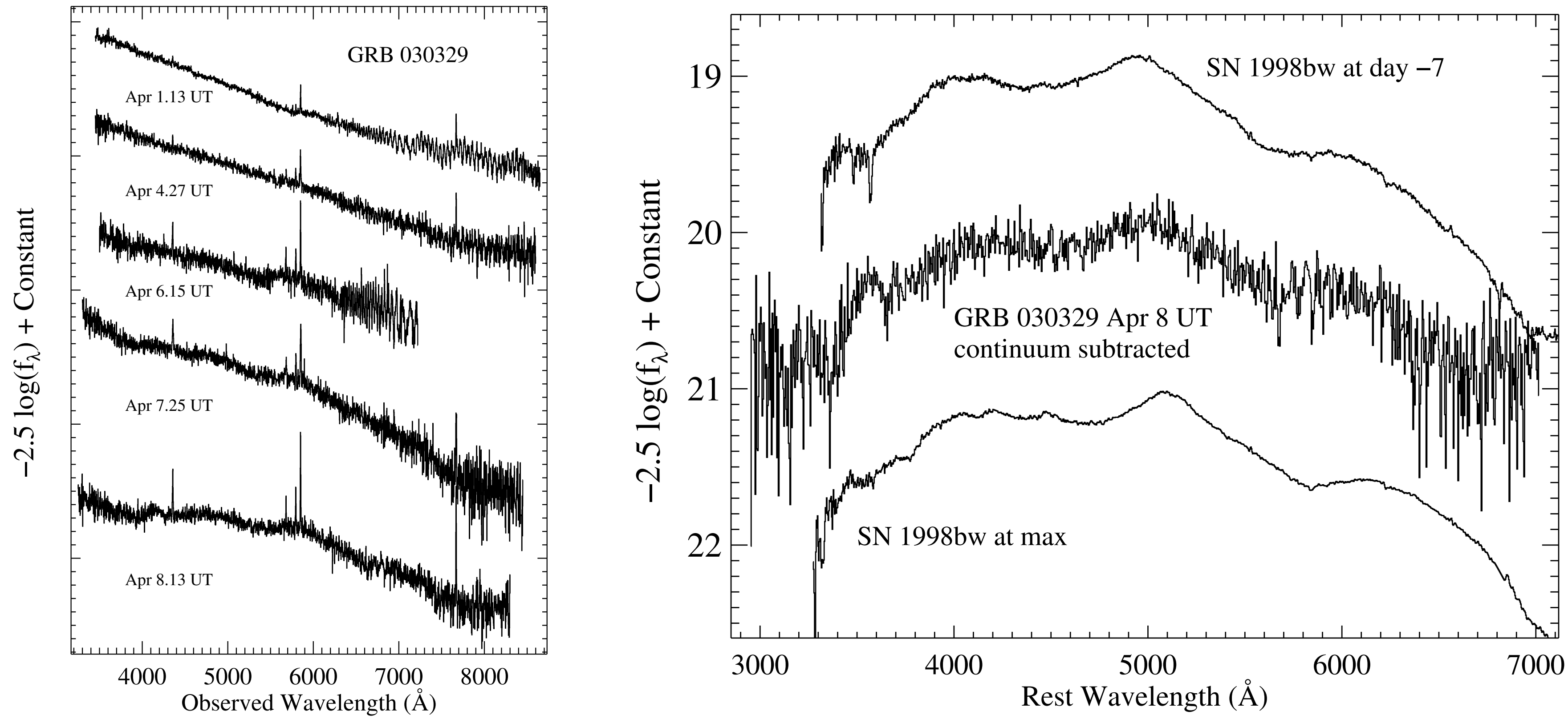
- afterglow fits $\rightarrow$
  - opening half-angle $\vartheta = 1^o - 10^o$
  - energy $E_\gamma = 10^{50} - 10^{51}$ergs (Frail et al. 2001)
  - $E_{\rm afterglow} = 10^{50} - 10^{51}$ergs (Panaitescu & Kumar 2002)

Imagine a Progenitor . . .

- **acceleration** and **collimation** of matter ejecta
- $E \sim 1\%$ of the binding energy of a solar-mass compact object
- small $\delta t \rightarrow$ compact object
- highly relativistic $\rightarrow$ compact object
- two time scales $(\delta t\,, \Delta t)$ + energetics suggest accretion



The supernova connection



Left: Spectrum evolution, from 2.64 to 9.64 days after the burst.
Right: Spectrum of April 8 with the smoothed spectrum of April 1 scaled and subtracted. (From Stanek et al. 2003)

The SN exploded within a few days of the GRB (Hjorth et al. Nature 2003).

The BH – debris-disk system

- **Energy reservoirs:**

① binding energy of the orbiting debris
② spin energy of the newly formed BH

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☞ viscous dissipation $\Rightarrow$ thermal energy $\Rightarrow \nu\bar{\nu} \longrightarrow e^+e^- \Rightarrow e^\pm$/photon/baryon **fireball**

– unlikely that the disk is optically thin to neutrinos (Di Matteo, Perna, & Narayan 2002)
– strong photospheric emission would have been detectable (Daigne & Mochkovitch 2002)
– difficult to explain the collimation
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☞ dissipation of magnetic fields
generated by the differential rotation in the torus $\Rightarrow e^\pm$/photon/baryon “magnetic” **fireball**
– collimation
– strong photospheric emission $\Rightarrow$ detectable thermal emission

☞ MHD extraction (**Poynting** jet)

- $\mathcal{E} = \frac{c}{4\pi} \underbrace{\frac{\varpi\Omega}{c} B_p}_{E} \, B_\phi \times$ area $\times$ duration $\Rightarrow$

$$\frac{B_p B_\phi}{(2 \times 10^{14}\text{G})^2} = \left[\frac{\mathcal{E}}{5 \times 10^{51}\text{ergs}}\right] \left[\frac{\text{area}}{4\pi \times 10^{12}\text{cm}^2}\right]^{-1} \left[\frac{\varpi\Omega}{10^{10}\text{cm s}^{-1}}\right]^{-1} \left[\frac{\text{duration}}{10\text{s}}\right]^{-1}$$

  - from the BH: $B_p \gtrsim 10^{15}$G (small B_ϕ, small area)
  - from the disk: smaller magnetic field required $\sim 10^{14}$G
  - If initially $B_p/B_\phi > 1$, a **trans-Alfvénic** outflow is produced.
  - If initially $B_p/B_\phi < 1$, the outflow is **super-Alfvénic** from the start.

- Is it possible to “use” this energy and accelerate the matter ejecta?

Ideal Magneto-Hydro-Dynamics

in collaboration with Arieh Königl (U of Chicago)

- Outflowing matter:
  - baryons (rest density ρ_0)
  - ambient electrons (neutralize the protons)
  - $e^{\pm}$ pairs (Maxwellian distribution)

- photons (blackbody distribution)

- large scale electromagnetic field $\mathbf{E}\,,\mathbf{B}$

$\tau \gg 1$ ensure local thermodynamic equilibrium

$$\left.\begin{array}{l} \text{charge density } \frac{J^0}{c} \ll \frac{\rho_0}{m_p} e \\ \text{current density } J \ll \frac{\rho_0}{m_p} ec \end{array}\right\} \text{one fluid approximation}$$

$\mathbf{V}$ bulk velocity

$P =$ total pressure (matter + radiation)

$\xi c^2 =$ specific enthalpy (matter + radiation)

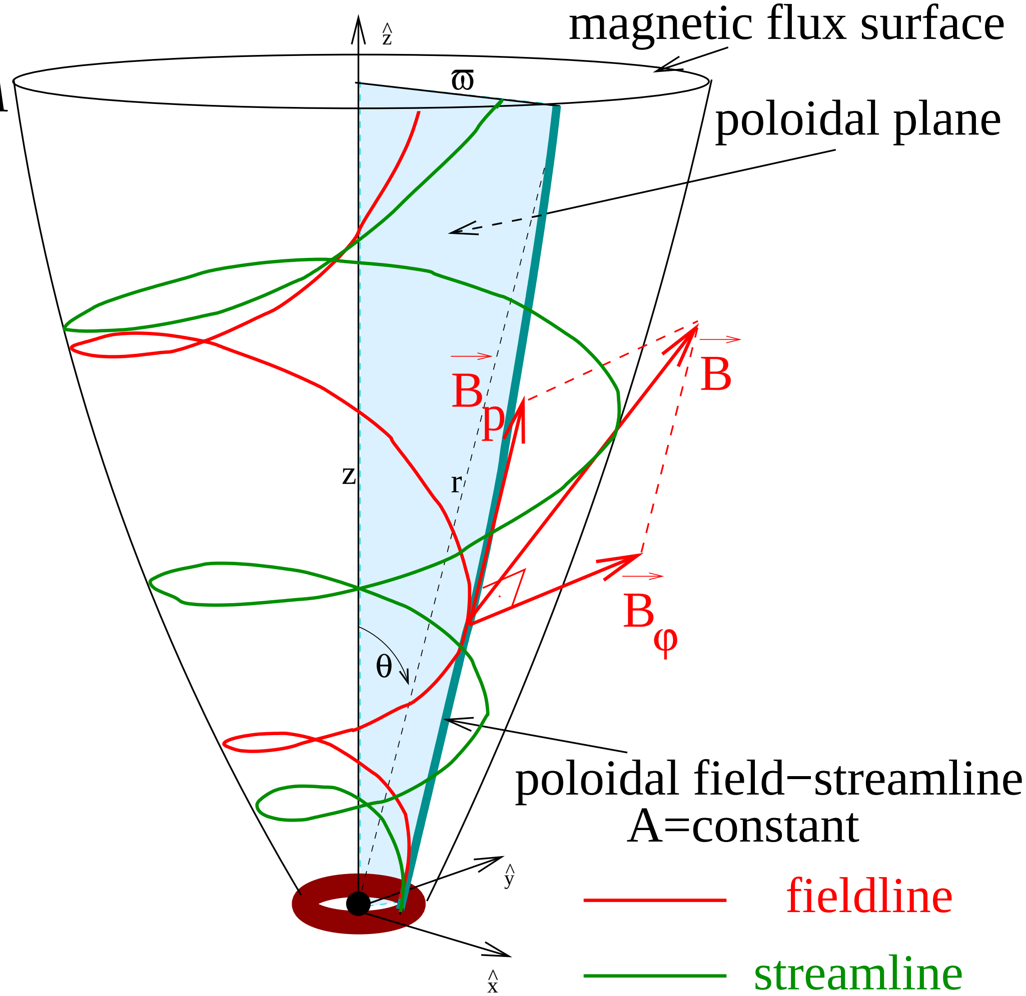
Assumptions

❶ axisymmetry
❷ highly relativistic poloidal motion
❸ quasi-steady poloidal magnetic field $\Leftrightarrow E_\phi = 0 \Leftrightarrow \mathbf{B}_p \parallel \mathbf{V}_p$

Introduce the magnetic flux function A

$\mathbf{B} = \mathbf{B}_p + \mathbf{B}_\phi\,, \quad \mathbf{B}_p = \nabla \times \left(A \frac{\hat{\phi}}{\varpi} \right)$

Faraday + Ohm $\longrightarrow \mathbf{V}_p \parallel \mathbf{B}_p$



The frozen-pulse approximation

- The arclength along a poloidal fieldline

$$\ell = \int_{\frac{s}{c}}^{t} V_p dt \approx ct - s \Rightarrow s = ct - \ell$$

- s is constant for each ejected shell. Moreover, the distance between two different shells $\ell_2 - \ell_1 = s_1 - s_2$ remains the same (even if they move with $\gamma_1 \neq \gamma_2$).

- Eliminating t in terms of s, we show that all terms with $\partial/\partial s$ are $\mathcal{O}(1/\gamma) \times$ remaining terms (generalizing the HD case examined by Piran, Shemi, & Narayan 1993). Thus we may examine the motion of <u>each shell</u> using steady-state equations.
$\left(\text{e.g., } \dfrac{d}{dt} = (c - V_p)\dfrac{\partial}{\partial s} + \mathbf{V} \cdot \nabla_s \approx \mathbf{V} \cdot \nabla_s\right)$
(also $E = |\mathbf{B} \times \mathbf{V}/c| \approx |\mathbf{B} \times \mathbf{V}_p/c| \approx |B_\phi|$)

Integration

The full set of ideal MHD equations can be partially integrated to yield five fieldline constants (functions of A and $s = ct - \ell$):

① the mass-to-magnetic flux ratio

② the field angular velocity

③ the specific angular momentum

④ the total energy-to-mass flux ratio μc^2

⑤ the adiabat $P/\rho_0^{4/3}$

Two integrals remain to be performed, involving the

Bernoulli and transfield force-balance equations.

Known solutions of the ideal MHD equations

Michel's solution gives $\gamma_\infty = \mu^{1/3}$ and $\sigma_\infty = \mu^{2/3} >> 1$
($\mu c^2 = \frac{\text{total energy flux}}{\text{mass flux}}$; μ is the maximum possible γ).

BUT, it does not satisfy the transfield equation.

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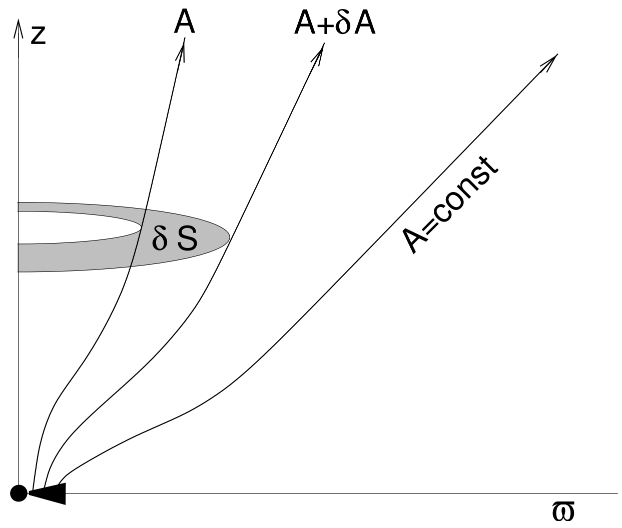
Necessary to solve the transfield because the line shape controls the acceleration:



Poynting-to-mass flux ratio $\propto \varpi |B_\phi| \rightarrow$
$\gamma \uparrow$ when $\varpi |B_\phi| \downarrow$
$E = |\mathbf{V}/c \times \mathbf{B}| \approx |B_\phi|$, $E = (\varpi \Omega / c) B_p$
So, $\varpi |B_\phi| \propto \varpi^2 B_p = (\varpi^2 / \delta S) \delta A$
($A =$ magnetic flux function).

Trying to solve the transfield

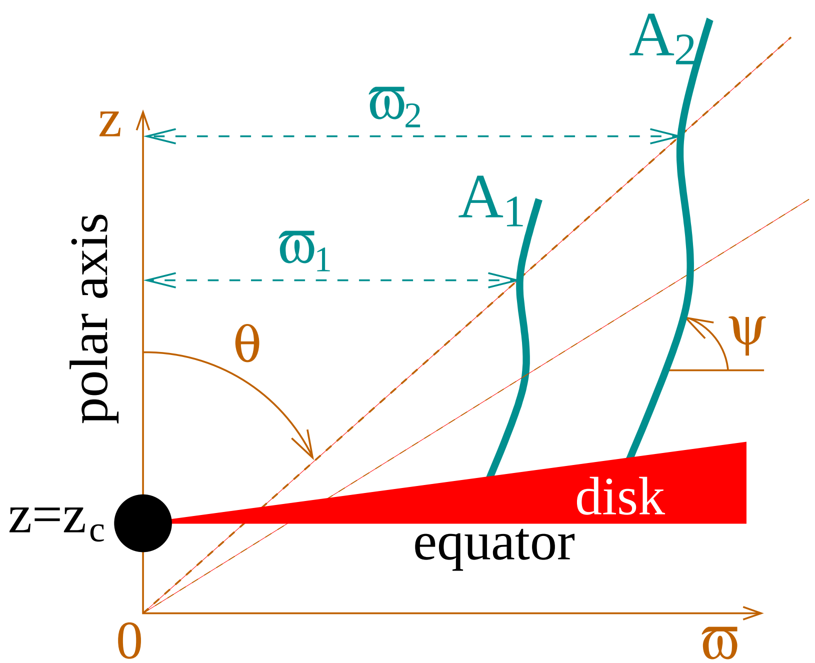
Bernoulli is algebraic

The transfield is a 2nd order PDE of the form

$$a\frac{\partial^2 A}{\partial \varpi^2} + 2b\frac{\partial^2 A}{\partial \varpi \partial z} + c\frac{\partial^2 A}{\partial z^2} = d\,, \quad a, b, c, d = \text{functions of } \frac{\partial A}{\partial \varpi}, \frac{\partial A}{\partial z}, A\,, \varpi$$

- mixed type $\rightarrow$ extremelly difficult numerical work (no solution at present)
- easier to solve numerically the time-dependent flow (only a few rotational periods)
- mixed (Bogovalov)
- analytical solutions: Only one exact solution known: the steady-state, cold, r self-similar model found by Li, Chiueh, & Begelman (1992) and Contopoulos (1994).
  Generalization for non-steady GRB outflows, including radiation and thermal effects.

r self-similarity



self-similar ansatz $r = \mathcal{F}_1(A)\,\mathcal{F}_2(\theta)$

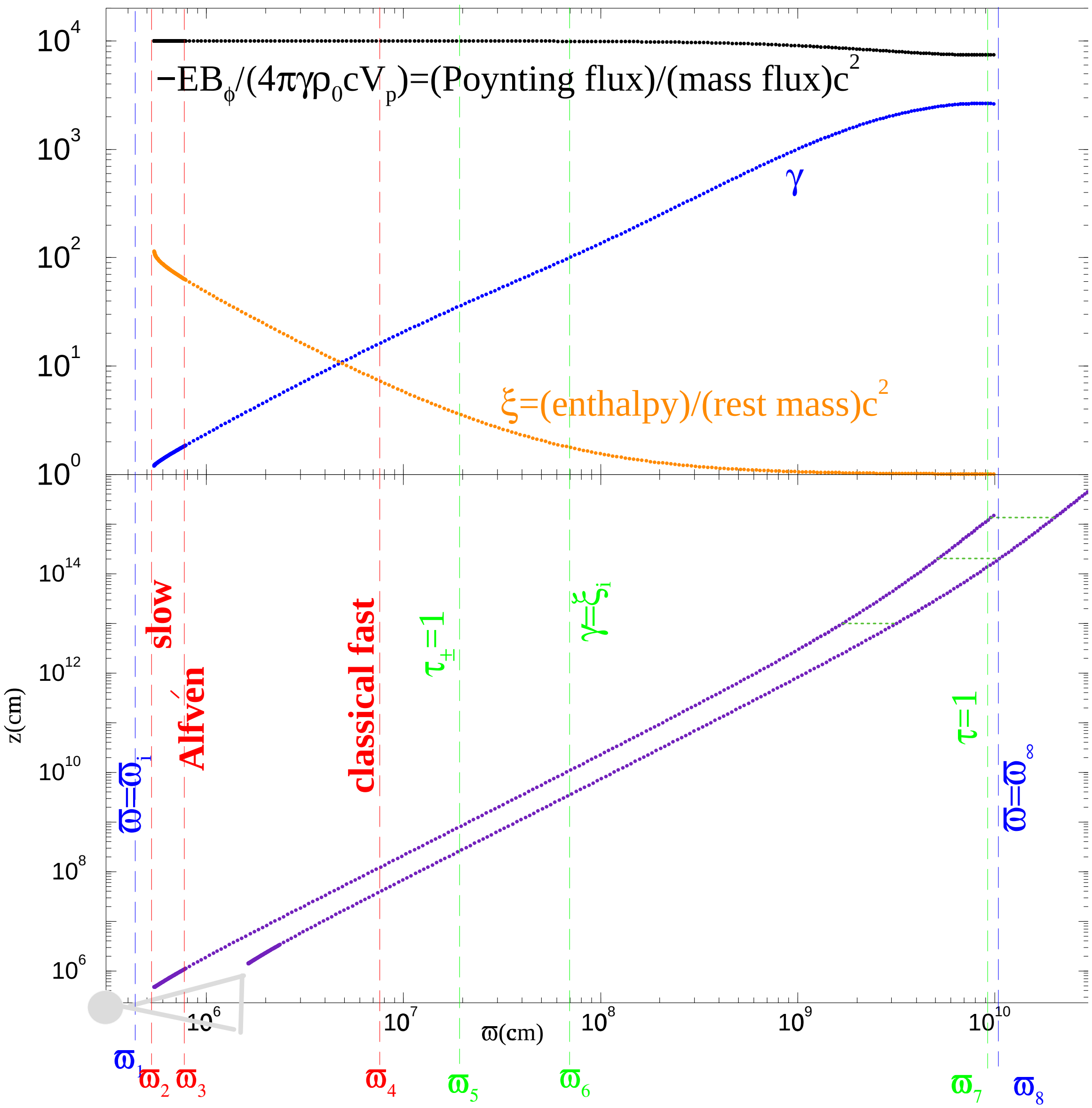
For points on the same cone $\theta = const$,
$$\frac{\varpi_1}{\varpi_2} = \frac{r_1}{r_2} = \frac{\mathcal{F}_1(A_1)}{\mathcal{F}_1(A_2)}.$$

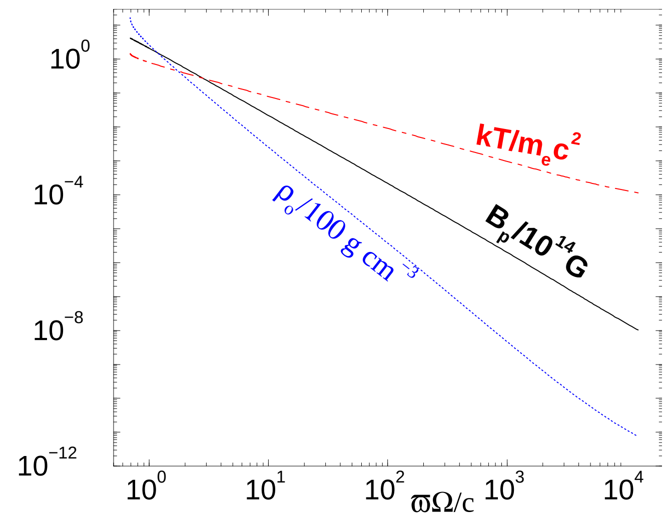
ODEs
$$\left\{ \begin{array}{l} \psi = \psi(x\,, M\,, \theta)\,, \text{(Bernoulli)} \\ \dfrac{dx}{d\theta} = \mathcal{N}_0(x\,, M\,, \psi\,, \theta)\,, \text{(definition of } \psi) \\ \dfrac{dM}{d\theta} = \dfrac{\mathcal{N}(x\,, M\,, \psi\,, \theta)}{\mathcal{D}(x\,, M\,, \psi\,, \theta)}\,, \text{(transfield)} \end{array} \right\}$$

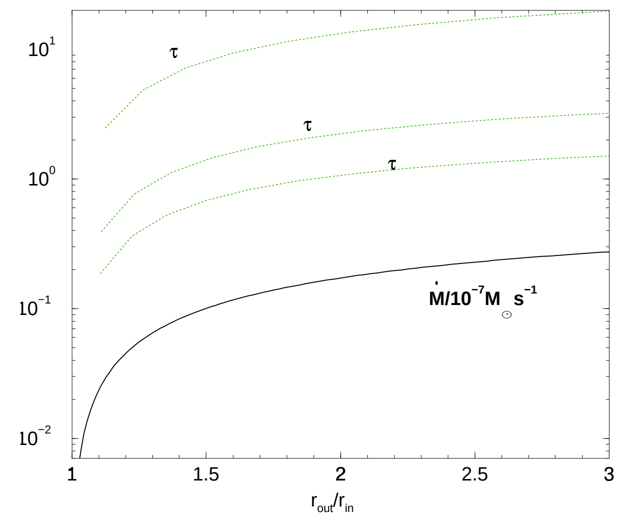
$\mathcal{D} = 0$: singular points
(Alfvén, modified slow - fast)

(start the integration from a cone $\theta = \theta_i$ and give the boundary conditions $B_\theta = -\mathcal{C}_1 r^{F-2}$, $B_\phi = -\mathcal{C}_2 r^{F-2}$, $V_r/c = \mathcal{C}_3$, $V_\theta/c = -\mathcal{C}_4$, $V_\phi/c = \mathcal{C}_5$, $\rho_0 = \mathcal{C}_6 r^{2(F-2)}$, and $P = \mathcal{C}_7 r^{2(F-2)}$, where F = parameter).

Trans-Alfvénic Jets (NV & Königl 2001, 2003a)

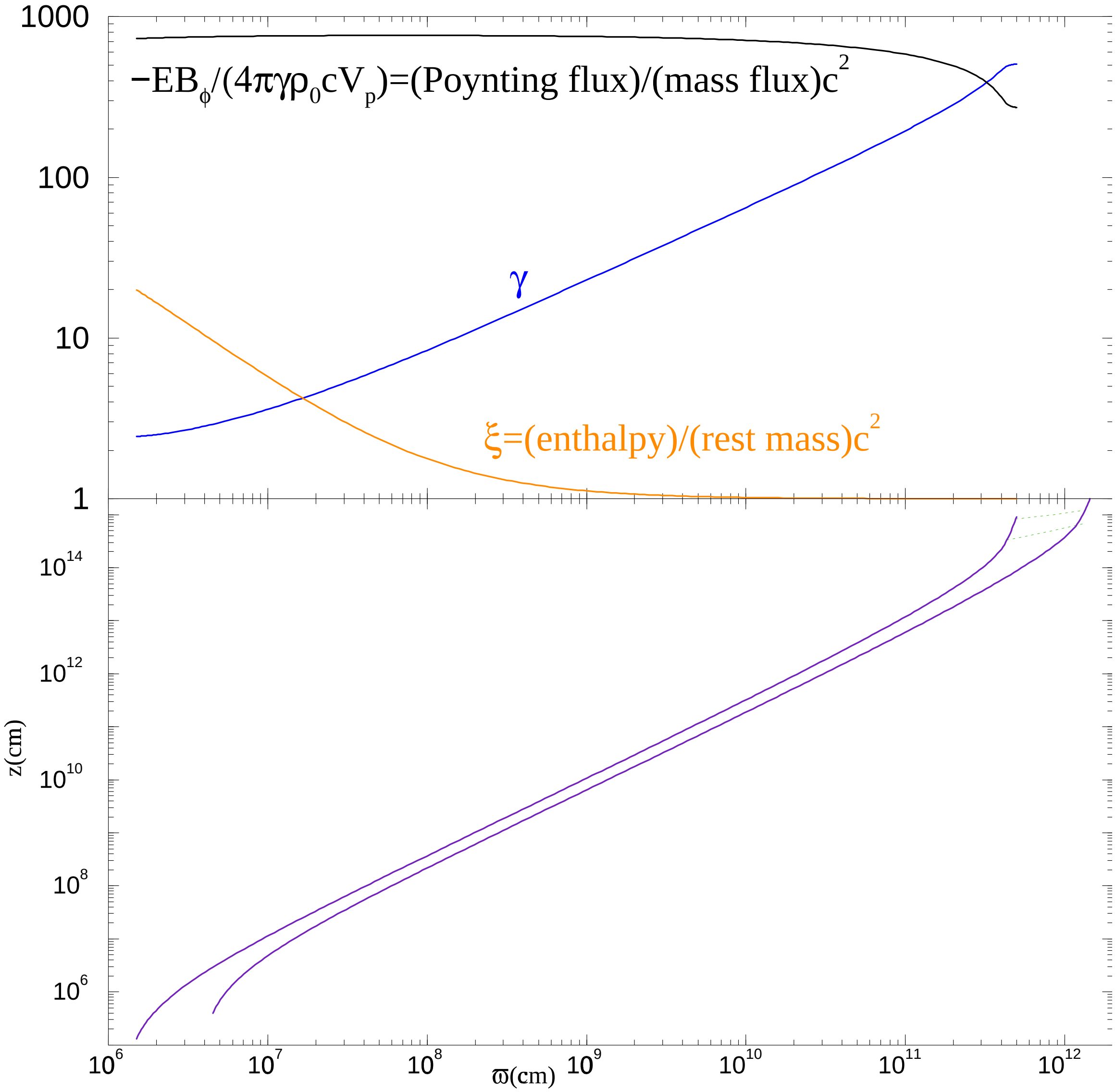


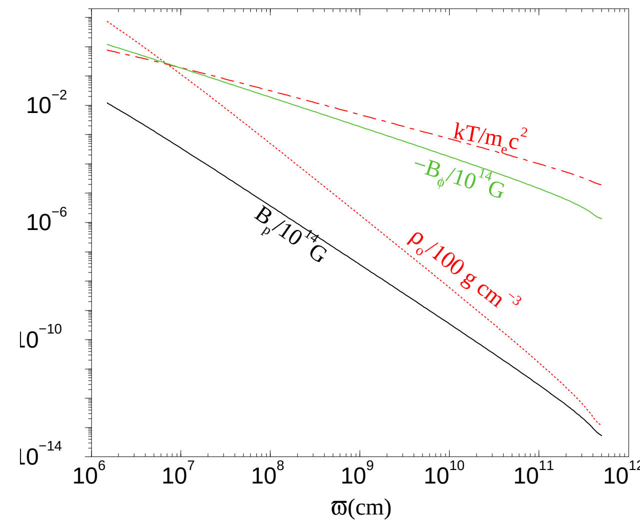


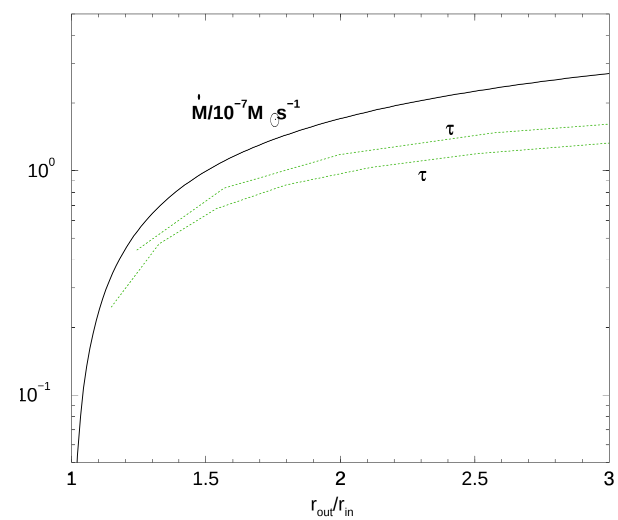


- $\varpi_1 < \varpi < \varpi_6$: Thermal acceleration - force free magnetic field ($\gamma \propto \varpi$, $\rho_0 \propto \varpi^{-3}$, $T \propto \varpi^{-1}$, $\varpi B_\phi = const$, parabolic shape of fieldlines: $z \propto \varpi^2$)
- $\varpi_6 < \varpi < \varpi_8$: Magnetic acceleration ($\gamma \propto \varpi$, $\rho_0 \propto \varpi^{-3}$)
- $\varpi = \varpi_8$: cylindrical regime - equipartition $\gamma_\infty \approx (-EB_\phi/4\pi\gamma\rho_0 V_p)_\infty$

Super-Alfvénic Jets (NV & Königl 2003b)

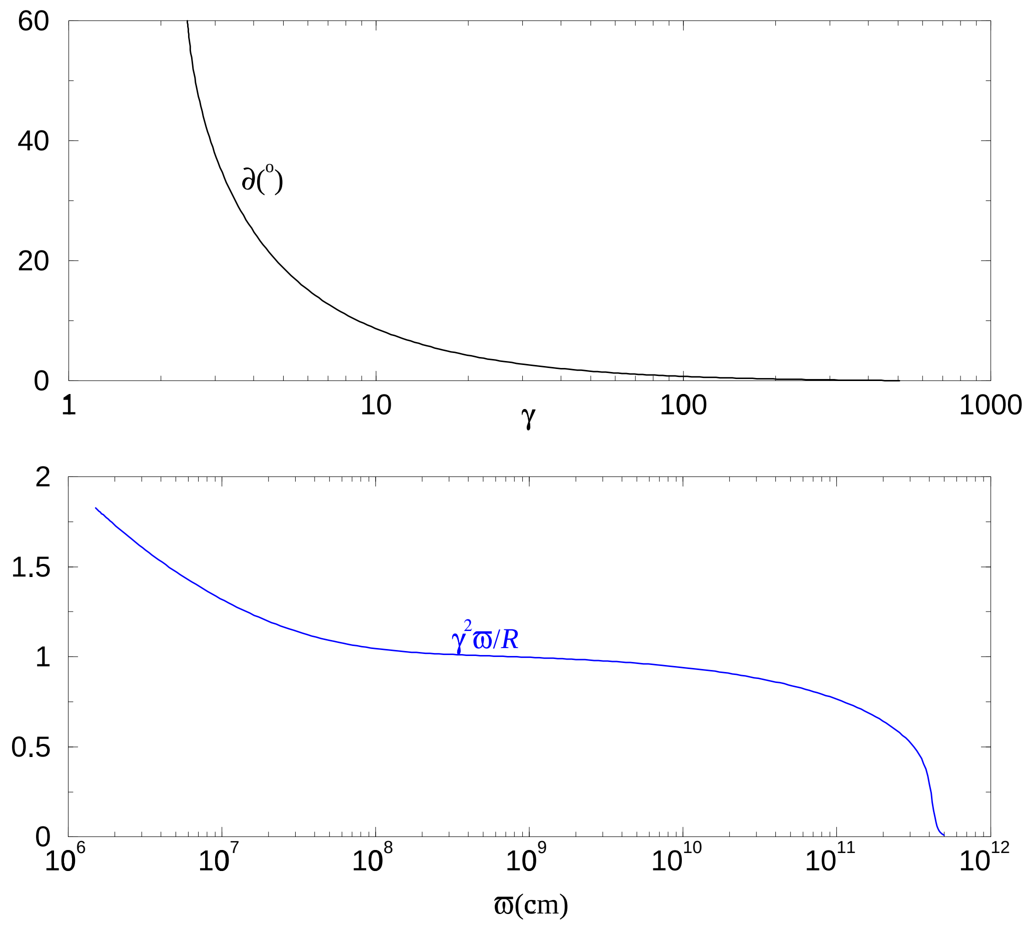






- Thermal acceleration ($\gamma \propto \varpi^{0.44}$, $\rho_0 \propto \varpi^{-2.4}$, $T \propto \varpi^{-0.8}$, $B_\phi \propto \varpi^{-1}$, $z \propto \varpi^{1.5}$)
- Magnetic acceleration ($\gamma \propto \varpi^{0.44}$, $\rho_0 \propto \varpi^{-2.4}$)
- cylindrical regime - equipartition $\gamma_\infty \approx (-EB_\phi/4\pi\gamma\rho_0 V_p)_\infty$

Collimation

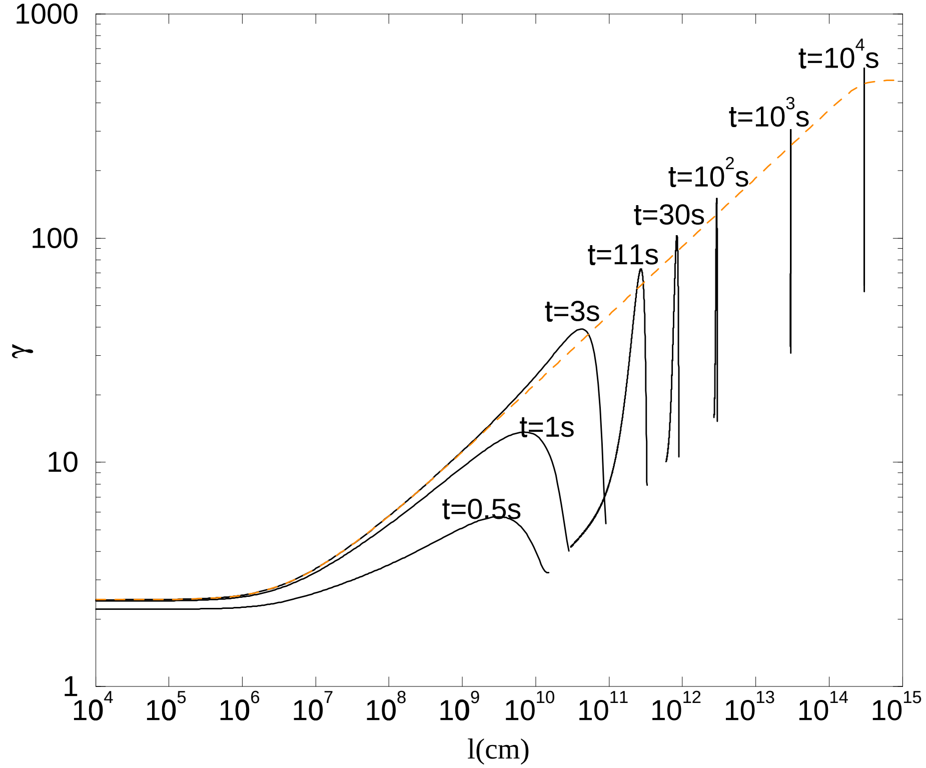


⋆ At $\varpi = 10^8$cm – where $\gamma = 10$ – the opening half-angle is already $\vartheta = 10^o$

⋆ For $\varpi > 10^8$cm, collimation continues slowly ($\mathcal{R} \sim \gamma^2\varpi$)

Time-Dependent Effects

⋆ recovering the time-dependence:



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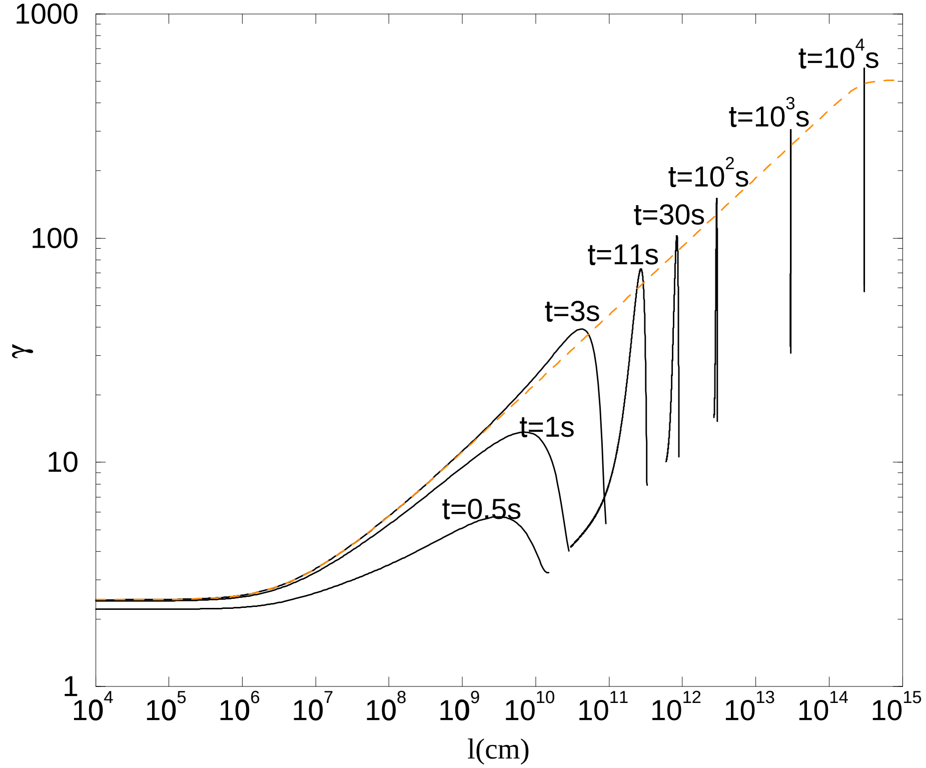


⋆ internal shocks:

The distance between two neighboring shells $s_1\,, s_2 = s_1 + \delta s$

$$\delta\ell = \delta\left(\int_{\frac{s}{c}}^{t} V_p dt\right) = -\delta s - \delta\left(\int_{\frac{s}{c}}^{t}(c - V_p)dt\right) \approx -\delta s - \int_0^t \delta\left(\frac{c}{2\gamma^2}\right) dt$$

Different $V_p \Rightarrow$ collision (at $ct \approx \gamma^2 \delta s$ – inside the cylindrical regime)

The baryon loading problem

- Proton mass in jet: $M_{\rm proton} = 3 \times 10^{-6}\,(\mathcal{E}/10^{51}\text{ergs})\,(\gamma_\infty/200)^{-1} M_\odot$.

- The disk would be $\sim 10^4$ times more massive even if 10% of its gravitational potentional energy could be converted into outflow kinetic energy (baryon loading problem).

A possible resolution (Fuller et al. 2000):

- If the source is neutron-rich, then the neutrons could decouple from the flow before the protons attain their terminal Lorentz factor.

- Disk-fed GRB outflows are expected to be neutron-rich, with n/p as high as $\sim 20 - 30$ (Pruet et al. 2003; Beloborodov 2003; Vlahakis et al. 2003).

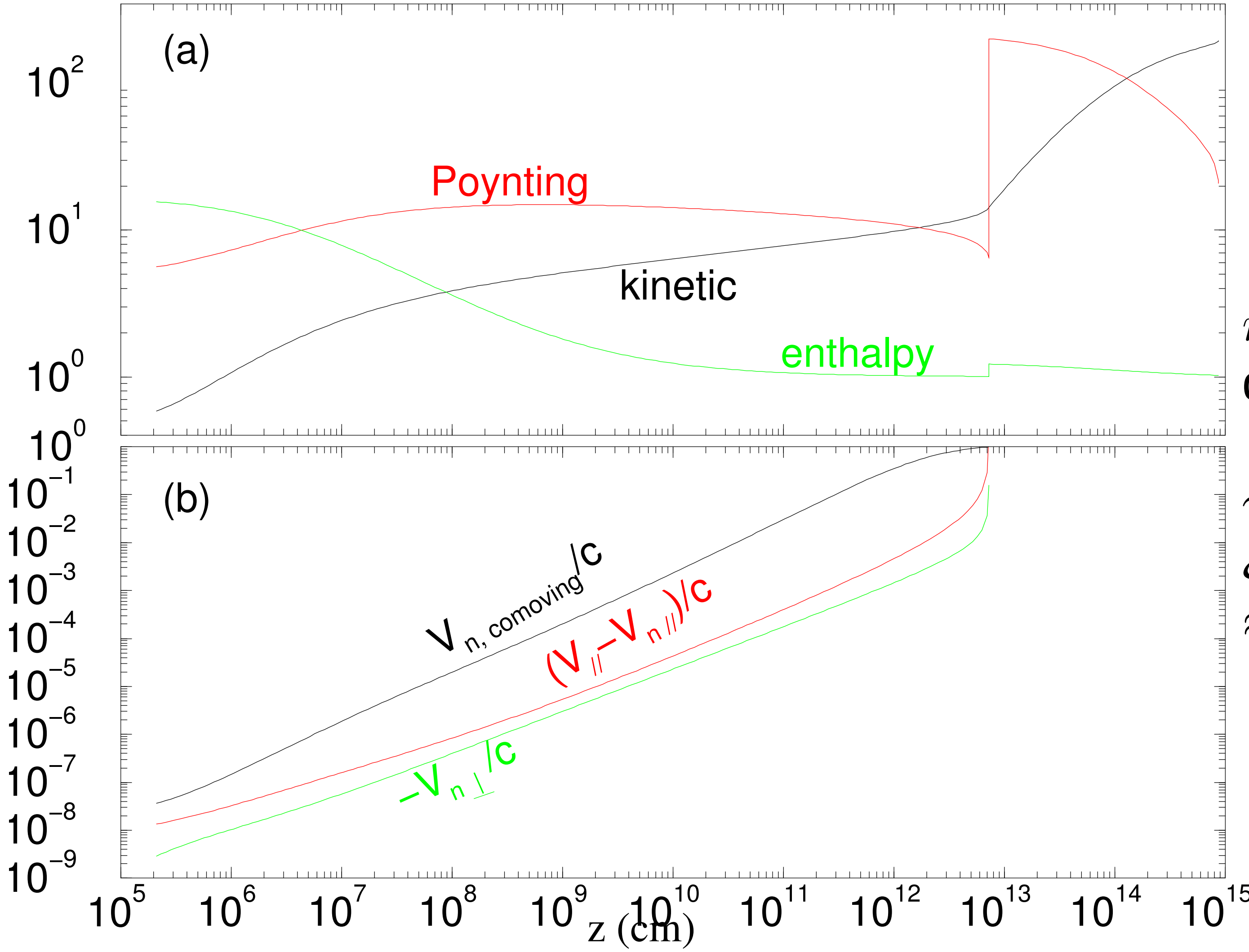
However, it turns out that the decoupling Lorentz factor $\gamma_{\rm d}$ in a thermally driven, purely hydrodynamic outflow is of the order of the inferred value of γ_∞ (e.g., Derishev et al. 1999; Beloborodov 2003), which has so far limited the practical implications of the Fuller at al. (2000) proposal.

Neutron-rich hydromagnetic flows

(Vlahakis, Peng, & Königl 2003 ApJL)

- Part of the thermal energy could be converted to electromagnetic (with the remainder transfered to baryon kinetic).
- The Lorentz factor increases with lower rate compared to the hydrodynamic case. This makes it possible to attain $\gamma_{\rm d} \ll \gamma_\infty$, as it is shown in the following solution.
- The energy deposited into the Poynting flux is returned to the matter beyond the decoupling point.

- Pre-decoupling phase:
  - The momentum equation for the whole system (protons/neutrons/$e^\pm$/photons/electromagnetic field) yields the flow velocity.
  - The momentum equation for the neutrons alone yields the neutron-proton collisional drag-force, and the drift velocity.
  - When $V_{\rm proton} - V_{\rm neutron} \sim c$ the neutrons decouple.
- Post-decoupling phase:
  - We solve for the protons alone ($+$ electromagnetic field).



$n/p = 30$
decoupling at $\gamma_d = 15$

$\gamma_\infty = 200$
$\mathcal{E}_{\rm proton} \approx 10^{51}$ergs
$\approx 0.5\ \mathcal{E}_{\rm neutron}$

(a) The three components of the total energy flux, normalized by the mass flux $\times\ c^2$.
(b) Proton–neutron drift velocity.

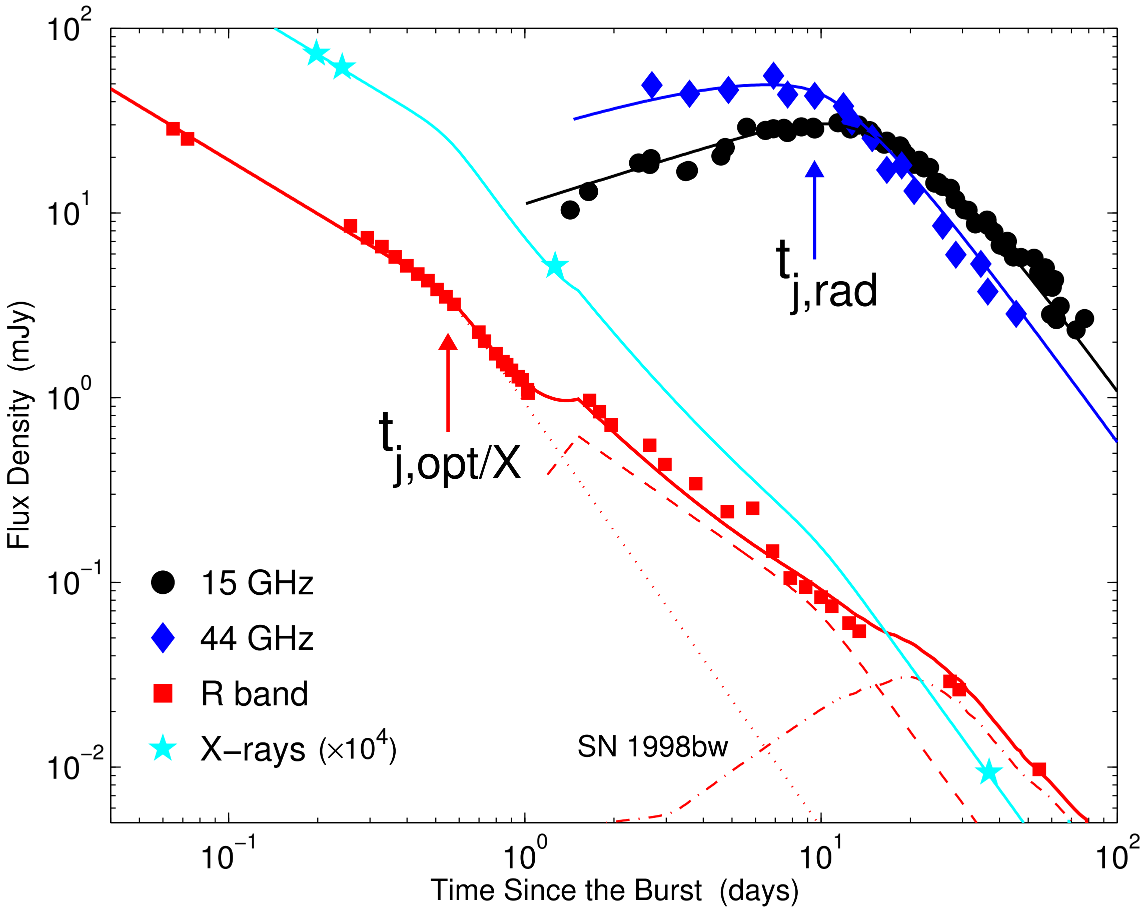
Due to the magnetic collimation $V_{\mathrm{neutron},\perp} \sim 0.1c$ at decoupling.

Thus, a two component outflow is naturally created:

- An inner jet consisting of the protons (with $\gamma = 200$ and $\mathcal{E}_p = 10^{51}$ ergs).
- The decoupled neutrons, after undergoing β decay at a distance $\sim 4 \times 10^{14}(\gamma_d/15)$cm, form a wider proton component (with $\gamma = 15$ and $\mathcal{E}_p = 2 \times 10^{51}$ ergs).

(See Peng, Königl, & Granot, ApJ (2005) for implications.)

Evidence for two jets!



Radio to X-ray lightcurves of the afterglow of GRB 030329 (Berger et al. 2003).
A two-component jet model provides a reasonable fit to the data.

Conclusion

- Trans-Alfvénic flow:
  - ⋆ The flow is initially thermally accelerated ($\xi\gamma = const.$; the magnetic field only guides the flow), and subsequently magnetically accelerated up to Lorentz factors corresponding to equipartition between kinetic and Poynting fluxes, i.e., $\sim 50\%$ of the initial total energy is extracted to baryonic kinetic. $\gamma \propto \varpi$ in both regimes.
  - ⋆ The fieldline shape is parabolic, $z \propto \varpi^2$ and becomes asymptotically cylindrical.

- Super-Alfvénic flow:
  - ⋆ Similar results, except that the Lorentz factor increases with lower rate: $\gamma \propto \varpi^{\beta}\,, \beta < 1$. Also $z \propto \varpi^{\beta+1}$.

- Neutron decoupling:
  - ⋆ In pure-hydro case $\gamma_{\rm d} \sim \gamma_\infty$.
  - ⋆ Magnetic fields make possible $\gamma_{\rm d} \ll \gamma_\infty$.
  - ⋆ The decoupled neutrons decay into protons at a distance $\sim 4 \times 10^{14} (\gamma_{\rm d}/15)$cm. In contrast with the situation in the pure-hydro case, these two components are unlikely to interact with each other in the hydromagnetic case since their motions are not collinear.
  - ⋆ Observational signatures of the neutron component?

The ideal MHD equations

Maxwell:
$$\nabla \cdot \mathbf{B} = 0 = \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{c\partial t}\,, \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{c\partial t} + \frac{4\pi}{c}\mathbf{J}\,, \nabla \cdot \mathbf{E} = \frac{4\pi}{c}J^0$$

Ohm: $\mathbf{E} = \mathbf{B} \times \mathbf{V}/c$

baryon mass conservation (continuity):
$$\frac{d(\gamma\rho_0)}{dt} + \gamma\rho_0\nabla \cdot \mathbf{V} = 0\,, \quad \text{where} \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla$$

energy $U_\mu T^{\mu\nu}_{\ ,\nu} = 0$ (or specific entropy conservation, or first law for thermodynamics):
$$\frac{d\left(P/\rho_0^{4/3}\right)}{dt} = 0$$

momentum $T^{\nu i}_{\ ,\nu} = 0$: $\gamma\rho_o\dfrac{d\left(\xi\gamma\mathbf{V}\right)}{dt} = -\nabla P + \dfrac{J^0\mathbf{E} + \mathbf{J} \times \mathbf{B}}{c}$

Eliminating t in terms of s: $\left(\mathbf{V} \cdot \nabla_s\right)\left(\xi\gamma\mathbf{V}\right) - \dfrac{\left(\nabla_s \cdot \mathbf{E}\right)\mathbf{E} + \left(\nabla_s \times \mathbf{B}\right) \times \mathbf{B}}{4\pi\gamma\rho_0} + \dfrac{\nabla P}{\gamma\rho_0} =$
$$(V_p - c)\frac{\partial\left(\xi\gamma\mathbf{V}\right)}{\partial s} + \frac{\partial(E + B_\phi)}{4\pi\gamma\rho_0\partial s}\frac{\nabla_s A}{|\nabla_s A\,|} \times \mathbf{B} - \nabla_s A\frac{\nabla_s\ell \cdot \nabla_s A}{|\nabla_s A\,|^2}\frac{\partial(E^2 - B_\phi^2)}{8\pi\gamma\rho_0\partial s}$$