ON BAIRE-1/4 FUNCTIONS AND SPREADING MODELS

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Abstract. We prove a characterization of functions in $B_{1/4}(K)\backslash C(K)$, where K is a compact metric space in terms of c_0-spreading models, answering a Problem of R. Haydon, E. Odell and H. Rosenthal. Beginning with $B_{1/4}(K)$ we define a decreasing family $(V_\xi(K), \|\cdot\|_\xi)_{1\leqslant\xi<\omega_1}$ of Banach spaces whose intersection is DBSC(K) and we prove an analogous stronger property for the functions in $V_\xi(K)\backslash C(K)$. Defining the s-spreading model-index, we classify $B_{1/4}(K)$ and we prove that s-SM$[F]>\xi$ for every $F\in V_\xi(K)$. Also we classify the separable Banach spaces by defining the c_0-SM-index which measures the degree to which they have sequences with extending spreading models equivalent to the usual basis of c_0. We give examples of Baire-1 functions and reflexive spaces with arbitrary large indices.

Introduction. The class $B_1(K)$ of bounded Baire-1 functions on a compact metric space K were classified by A. Kechris and A. Louveau in **[10]** into a transfinite increasing hierarchy $B_1^\xi(K)$ for $1\leqslant\xi<\omega_1$. The smallest subclass $B_1^1(K)$, as they proved, consists of all uniform limits of sequences of differences of bounded semicontinuous functions on K (DBSC(K)). Earlier, R. Haydon, E. Odell and H. Rosenthal in **[8]** had characterized the functions in $B_1(K)\backslash B_1^1(K)$ in terms of l^1-spreading models. In the same paper they defined the subclass $B_{1/4}(K)$ of $B_1^1(K)$ and they proved that for every $F\in B_{1/4}(K)\backslash C(K)$ there exists a sequence $(f_n)\subset C(K)$ converging pointwise to F and generating a spreading model equivalent to the summing basis of c_0. They ask in **[8]** where the converse holds.

In Theorem 1.1 we show that the answer to this problem is affirmative and thus we obtain a characterization of the functions in $B_{1/4}(K)\backslash C(K)$.

The class $B_{1/4}(K)$ is a Banach space with respect to a natural norm $\|\cdot\|_{1/4}$ **[8]**. Beginning with $(B_{1/4}(K), \|\cdot\|_{1/4})$ we define a decreasing family $\{V_\xi(K), \|\cdot\|_\xi\}_{1\leqslant\xi<\omega_1}$ of Banach spaces whose intersection coincides with DBSC(K). Extending the notion of spreading models we prove in Theorem 1.5 that the functions in $V_\xi(K)\backslash C(K)$ have a stronger property than the functions in $B_{1/4}(K)\backslash C(K)$.

Furthermore, we define an index s-SM$[F]$ for every $F\in B_1(K)$ and we use it to classify $B_{1/4}(K)$. In Proposition 3.5 we prove that every function F in $V_\xi(K)\backslash C(K)$ has s-SM-index greater than ξ and the converse holds for $\xi=1$. However, we do not know if this condition characterizes the Banach spaces $V_\xi(K)\backslash C(K)$ for every $1\leqslant\xi<\omega_1$ (Problem 3.7).

Finally we define an index c_0-SM$[X]$ for an arbitrary separable Banach space X which is used to classify this class. The first level of this classification

[MATHEMATIKA, **41** (1994), 251–265]

consists of the spaces with sequences generating spreading models equivalent to the usual basis of c_0 and the smallest consists of the spaces which contain c_0. In Example 2.5 we construct reflexive Banach spaces R_ξ for every $\xi<\omega_1$, such that $c_0\text{-SM}[R_\xi]\geqslant\xi$. This index is much smaller than the c_0-index of X (if $c_0\text{-SM}[X]\geqslant\xi$ then $c_0[X]\geqslant\omega^\xi$) which is analogous to the Bourgain index $l^1[X]$. But the two indices are not equivalent; we give examples where $c_0\text{-SM}[X]=0$ and $c_0[X]$ can be made larger than any countable ordinal (Example 2.6).

§1. We will use standard terminology and notation as may be found in [11]. For completeness we will give some basic definitions and notations which we will use in the following.

Let (x_n) be a seminormalized basic sequence in a Banach space X. A basic sequence (e_n) is said to be a spreading model of (x_n) if for every $k\in\mathbb{N}$ and $\varepsilon>0$ there exists $m\in\mathbb{N}$ so that if $m<n_1<n_2<\ldots<n_k$ then

$$\left|\left\|\sum_{i=1}^{k} a_i x_{n_i}\right\|-\left\|\sum_{i=1}^{k} a_i e_i\right\|\right|<\varepsilon$$

for all scalars $a_1,\ldots,a_k$ with $\max_{1\leqslant i<k}|a_i|\leqslant 1$. Every seminormalized basic sequence has a subsequence generating a spreading model. For further information on spreading models see [3].

The summing basis (s_n) of c_0 is characterized by

$$\left\|\sum_{n=1}^{\infty} a_n s_n\right\|=\sup_k\left|\sum_{n=1}^{k} a_n\right|.$$

Let K be a compact metric space. $B_1(K)$ denotes the class of Baire-1 functions on K, *i.e.*, the pointwise limits of (uniformly bounded) sequences of continuous functions on K. The class of continuous functions on K is denoted by $C(K)$ and the class of differences of bounded semicontinuous functions on K by DBSC(K). It is easy to see that

$$\mathrm{DBSC}(K)=\Big\{F\colon K\to\mathbb{R}\colon \text{there exists } (f_n)\subset C(K) \text{ and } C<\infty$$
$$\text{such that } f_0=0,\ (f_n) \text{ converges pointwise to } F$$
$$\text{and } \sum_{n=0}^{\infty}|f_{n+1}(x)-f_n(x)|\leqslant C \text{ for all } x\in K\Big\}.$$

The class DBSC(K) is a Banach space with respect to the norm $|F|_D=\inf\{C\colon$ there exists $(f_n)\subset C(K)$ converging pointwise to F with $f_0=0$ and $\sum_{n=0}^{\infty}|f_{n+1}(x)-f_n(x)|\leqslant C$ for all $x\in K\}$.

Haydon, Odell and Rosenthal in [8] defined the class

$$B_{1/4}(K)=\Big\{F\in B_1(K)\colon \text{there exists a sequence } (F_n)\subset\mathrm{DBSC}(K)$$
$$\text{such that } F_n\to F \text{ uniformly and } \sup_n|F_n|_D<\infty\Big\}.$$

Also they proved that if $F\in B_{1/4}(K)$ then there exists a bounded basic sequence $(f_n)\subseteq C(K)$ converging pointwise to F and $0<C<\infty$ such that every convex block subsequence of (f_n) has a subsequence generating a spreading model C-equivalent to the summing basis of c_0. In [8] it was asked whether the converse holds (Problem 3.9). We prove that it does.

THEOREM 1.1. *Let K be a compact metric space and $F\in B_1(K)$. The following are equivalent.*

(a) $F\in B_{1/4}(K)\setminus C(K)$.

(b) *Every bounded sequence $(f_n)\subseteq C(K)$ converging pointwise to F has a convex block subsequence generating a spreading model equivalent to the summing basis of c_0.*

(c) *There exists a bounded sequence $(f_n)\subseteq C(K)$ converging pointwise to F and generating a spreading model equivalent to the summing basis of c_0.*

Proof. That (a) implies (b) is proved in [8] and (b) implies (c) is trivial.

(c) *implies* (a). If (c) holds, then there exists a bounded sequence $(g_n)\subseteq C(K)$ converging pointwise to F and $0<C<\infty$ such that

$$\left\|\sum_{i=1}^{k}\lambda_i g_{n_i}\right\|_\infty\leqslant C\left\|\sum_{i=1}^{k}\lambda_i s_i\right\|_\infty, \tag{1}$$

for every $k, n_1,\ldots,n_k\in\mathbb{N}$ with $k<n_1<\ldots<n_k$ and scalars $\lambda_1,\ldots,\lambda_k$.

According to [8] (Theorem 6.1) it is sufficient to prove that there exists $0<M<\infty$ such that for every $\varepsilon>0$ there exists $m\in\mathbb{N}$ satisfying

$$\sum_{i\in B((n_i),x)}|g_{n_{i+1}}(x)-g_{n_i}(x)|\leqslant M,$$

for every $x\in K$ and every strictly increasing sequence $(n_i)\subset\mathbb{N}$ with $n_i>m$ for every $i\in\mathbb{N}$.

Here

$$B((n_i),x)=\{i\in\mathbb{N}: |g_{n_{i+1}}(x)-g_{n_i}(x)|\geqslant\varepsilon\}.$$

For $\varepsilon>0$, let m be an even integer with $m>2C/\varepsilon$ and $(n_i)\subset\mathbb{N}$ with $m<n_1<n_2<\ldots$. We observe that $|B((n_i),x)|<m$ for every $x\in K$. Indeed, if $|B((n_i),x)|\geqslant m$ for a $x\in K$ then we can choose $i_1,\ldots,i_m\in B((n_i),x)$. Since

$$m\varepsilon\leqslant\sum_{j=1}^{m}|g_{n_{i(j)+1}}(x)-g_{n_{i(j)}}(x)|\leqslant 2C,$$

with $i(j)=i_j$, we have a contradiction. Hence $|B((n_i),x)|<m$ for every $x\in K$ and from (1)

$$\sum_{i\in B((n_i),x)}|g_{n_{i+1}}(x)-g_{n_i}(x)|\leqslant 2C$$

for every $x\in K$. Thus $F\in B_{1/4}(K)$.

Beginning with $B_{1/4}(K)$ we will define for every $1\leqslant\xi<\omega_1$ a linear subspace $V_\xi(K)$ of $B_1(K)$ and a norm $\|\cdot\|_\xi$ in $V_\xi(K)$ such that $(V_\xi(K),\|\cdot\|_\xi)$ is a Banach space. It is easy to see that $V_\xi(K)\subseteq V_\beta(K)$ for every $\beta<\xi<\omega_1$ and

$\|F\|_\infty \leqslant \|F\|_\beta \leqslant \|F\|_\xi$ for every $F \in V_\xi(K)$ and $\beta < \xi < \omega_1$. Also, as we will prove in the third section, the intersection of all $V_\xi(K)$ $(1 \leqslant \xi < \omega_1)$ coincides with DBSC(K). The spaces $V_n(K)$ for $n \in \mathbb{N}$ were first defined in [8].

Definition 1.2. Let K be a compact metric space. We set $V_1(K) = B_{1/4}(K)$ with the norm

$$\|F\|_1 = \inf \left\{ C > 0 : \text{there exists } (F_n) \subset \mathrm{DBSC}(K) \right.$$
$$\left. \text{with } \|F_n - F\|_\infty \to 0 \text{ and } \sup_n |F_n|_D \leqslant C \right\}$$

and

$$V_{\xi+1}(K) = \left\{ F \in B_1(K) : \text{there exists } (F_n) \subset \mathrm{DBSC}(K) \right.$$
$$\left. \text{with } \|F_n - F\|_\xi \to 0 \text{ and } \sup_n |F_n|_D < \infty \right\}$$

with the norm

$$\|F\|_{\xi+1} = \inf \{ C > 0 : \text{there exists } (F_n) \subset \mathrm{DBSC}(K)$$
$$\text{with } \|F_n - F\|_\xi \to 0 \text{ and } \sup |F_n|_D \leqslant C \}.$$

If ξ is a limit ordinal then we define

$$\|F\|_\xi = \sup \{ \|F\|_\beta : 1 \leqslant \beta < \xi \} \qquad \text{for every} \quad F \in \bigcap_{\beta < \xi} V_\beta(K)$$

and $V_\xi(K) = \{ F \in B_1(K) : \|F\|_\xi < \infty \}$.

PROPOSITION 1.3. *For every* $1 \leqslant \xi < \omega_1$ *the spaces* $(V_\xi(K), \|\ \|_\xi)$ *are complete.*

Proof. The space $(V_1(K), \|\cdot\|_1)$ is complete as proved in [8].

Suppose that $(V_\beta(K), \|\cdot\|_\beta)$ are Banach spaces for every ordinal β with $\beta < \xi < \omega_1$.

If $\xi = \beta + 1$ then $\|F\|_\beta \leqslant \|F\|_\xi$ for every $F \in V_\xi(K)$ and also every $\|\cdot\|_\xi$-closed ball in $V_\xi(K)$ is $\|\cdot\|_\beta$-closed subset of $V_\beta(K)$. Hence $(V_\xi(K), \|\cdot\|_\xi)$ is a Banach space, since $(V_\beta(K), \|\cdot\|_\beta)$ is.

If ξ is a limit ordinal, then $(V_\xi(K), \|\cdot\|_\xi)$ is a Banach space, since it is isometric to the closed linear subspace

$$\left\{ (x_\beta)_{1 \leqslant \beta < \xi} \in \prod_{1 \leqslant \beta < \xi} V_\beta(K) : x_\beta = x_1 \text{ for all } 1 \leqslant \beta < \xi \right\}$$

of the Banach space $(\prod_{1 \leqslant \beta < \xi} V_\beta(K), \|\cdot\|_\infty)$, where $\|(x_\beta)_{1 \leqslant \beta < \xi}\|_\infty = \sup_{1 \leqslant \beta < \xi} \|x_\beta\|_\beta$.

It is easy to prove that a bounded sequence $(f_n)\subseteq C(K)$ converging pointwise to a function F has a subsequence (g_n) generating a spreading model C-equivalent to the summing basis of c_0, if, and only if, it has a subsequence (d_n) such that $(d_{n_1},\dots,d_{n_k})$ is C-equivalent to the summing basis of $c_0(k)$ for every $(n_1,\dots,n_k)$ in the Schreier family $\mathscr{F}_1$ [12]. Alspach and Argyros in [1] generalized the family $\mathscr{F}_1$ by defining, for each ordinal $\xi<\omega_1$, a family $\mathscr{F}_\xi$ consisting of finite subsets of $\mathbb{N}$. We recall this definition since these families are useful below in order to extent results relating to spreading models.

Definition 1.4. [1]. For every limit ordinal number ξ, let (ξ_n) be a sequence of ordinal numbers strictly increasing to ξ. The families $\mathscr{F}_\xi$, $\xi<\omega_1$ are defined inductively as follows:

$$\mathscr{F}_0=\{\{n\}:n\in\mathbb{N}\};$$

$$\mathscr{F}_{\xi+1}=\left\{A\subset\mathbb{N}:A=\bigcup_{i=1}^{n}A_i \text{ with } \{n\}\leqslant A_1<\dots<A_n \text{ and } A_i\in\mathscr{F}_\xi \text{ for all } i=1,\dots,n\right\};$$

and if ξ is a limit ordinal

$$\mathscr{F}_\xi=\{A\subset\mathbb{N}:A\in\mathscr{F}_{\xi_n}\text{ and } n\leqslant\min A\}.$$

It is easy to see that if $A\in\mathscr{F}_\xi$ and $B\subset A$ then $B\in\mathscr{F}_\xi$. Moreover $\mathscr{F}_\xi$ is a well founded closed tree on $\mathbb{N}$ and $O[\mathscr{F}_\xi]=\omega^\xi$ for every $1\leqslant\xi<\omega_1$ (Definition 2.1).

In the following theorem we give a property of the functions in $V_\xi(K)$ extending the analogous property of functions in $B_{1/4}(K)$.

THEOREM 1.5. *Let K be a compact metric space, ξ a countable ordinal and let $F\in V_\xi(K)\setminus C(K)$. Then there exists a uniformly bounded basic sequence (f_n) in $C(K)$ converging pointwise to F such that every convex block subsequence (g_n) of (f_n) has a subsequence (d_n) which satisfies:*

$$C\left\|\sum_{i=1}^{k}\lambda_i s_i\right\|_\infty\leqslant\left\|\sum_{i=1}^{k}\lambda_i d_{n_i}\right\|_\infty\leqslant 2\|F\|_\xi\left\|\sum_{i=1}^{k}\lambda_i s_i\right\|_\infty \tag{2}$$

for $C=C(F)>0$, every $(n_1,\dots,n_k)\in\mathscr{F}_\xi$ and all scalars $\lambda_1,\dots,\lambda_k$.

Proof. For $\xi=1$ the theorem is essentially proved in [8].

Let ξ be a countable ordinal and $F\in V_\xi(K)\setminus C(K)$. We assume that the theorem is proved for every ordinal $\beta<\xi$.

Case 1. Let $\xi=\beta+1$. By Definition 1.2 there exists a sequence $(F_m)\subset\mathrm{DBSC}(K)$ with $\|F_m-F\|_\beta\to 0$ and $\sup_n|F_n|_D\leqslant M=\|F\|_\xi$. Let (ε_m) be a decreasing sequence of positive numbers such that $\varepsilon_m\to 0$ and $\sum_{i=m+1}^{\infty}\varepsilon_i<\varepsilon_m$ for every $m\in\mathbb{N}$. We can assume that $\|F_{m+1}-F_m\|_\beta<\varepsilon_m$ for every $m\in\mathbb{N}$. Hence from the inductive hypothesis for every $m\in\mathbb{N}$ there exists a bounded sequence $(g_n^m)_{n=1}^\infty\subset C(K)$ converging pointwise to $F_{m+1}-F_m$ such that every convex block subsequence of $(g_n^m)_{n=1}^\infty$ has a subsequence $(d_n^m)_{n=1}^\infty$ with

$$\left\|\sum_{i=1}^k \lambda_i d_{n_i}^m\right\|_\infty \leqslant 2\varepsilon_m \left\|\sum_{i=1}^k \lambda_i s_i\right\|_\infty ,$$

for every $(n_1,\ldots,n_k)\in\mathscr{F}_\beta$ and scalars $\lambda_1,\ldots,\lambda_k$.

Since $F\notin C(K)$ we may assume that there exists $\delta>0$ and $x\in K$ so that $\operatorname{osc}_K(F_m,x)>\delta$ for all $m\in\mathbb{N}$.

We will construct for every $m\in\mathbb{N}$ a bounded sequence $(h_n^m)_{n=m}^\infty\subset C(K)$ converging pointwise to F_m and $C(M,\delta)$-equivalent to the summing basis of c_0 and also a convex block subsequence $(d_n^m)_{n=m}^\infty$ of $(g_n^m)_{n=1}^\infty$ such that $h_n^m=h_n^1+d_n^1+\ldots+d_n^{m-1}$ for every $m,n\in\mathbb{N}$ with $m\leqslant n$.

Let $(f_n^1)\subset C(K)$ be a bounded sequence converging pointwise to F_1 and $C(M,\delta)$-equivalent to the summing basis of c_0. The sequence $(f_n^1+g_n^1)$ converges pointwise to $F_2\in\mathrm{DBSC}(K)$, hence there exist convex block subsequences $(f_n^{1,2})$, $(g_n^{1,2})$ of (f_n^1), (g_n^1) respectively such that the sequence (f_n^2), where $f_n^2=f_n^{1,2}+g_n^{1,2}$ for every $n\in\mathbb{N}$, is $C(M,\delta)$-equivalent to the summing basis of c_0. Now, since $(f_n^2+g_n^2)$ converges pointwise to F_3, there exist convex block subsequences $(f_n^{1,2,3})$, $(g_n^{1,2,3})$ and $(g_n^{2,3})$ of $(f_n^{1,2})$, $(g_n^{1,2})$ and (g_n^2) respectively such that the sequence (f_n^3), where $f_n^3=f_n^{2,3}+g_n^{2,3}$ and $f_n^{2,3}=f_n^{1,2,3}+g_n^{1,2,3}$ for every $n\in\mathbb{N}$, is $C(M,\delta)$ equivalent to the summing basis of c_0. We continue in the obvious way. Finally, we set $h_n^m=f_n^{m,\ldots,n}$ and $d_n^m=g_n^{m,\ldots,n}$ for every $n,m\in\mathbb{N}$ with $n\geqslant m$. The sequences $(h_n^m)_{n=m}^\infty$ and $(d_n^m)_{n=m}^\infty$ have the desired properties.

Since the sequence $(d_n^m)_{n=m}^\infty$ is a convex block subsequence of $(g_n^m)_{n=1}^\infty$ for every $m\in\mathbb{N}$ we can assume that

$$\left\|\sum_{i=1}^k \lambda_i d_{n_i}^m\right\| \leqslant 2\varepsilon_m \left\|\sum_{i=1}^k \lambda_i s_i\right\|_\infty$$

for every $m\in\mathbb{N}$, $(n_1,\ldots,n_k)\in\mathscr{F}_\beta$ with $m\leqslant n_1$ and scalars $\lambda_1,\ldots,\lambda_k$. (Otherwise, we replace the sequences $(d_n^m)_{n=m}^\infty$ by subsequences $(d_{n_i}^m)_{i=m}^\infty$ having this property using a diagonal argument. Of course in this case we will replace $(h_n^m)_{n=m}^\infty$ by $(h_{n_i}^m)_{i=m}^\infty$).

We set $h_n=h_n^n$ for every $n\in\mathbb{N}$. Hence we have that $h_n=h_n^m+d_n^m+\ldots+d_n^{n-1}$ for every $n,m\in\mathbb{N}$ with $n>m$ and (h_n) converges pointwise to F. Also we choose an increasing sequence $(n(i))_{i=1}^\infty$ of natural numbers such that $n(i)>i$ and $\varepsilon_{n(i)}<\frac{1}{4}M/i$ for every $i\in\mathbb{N}$.

Let $A=(n_1,\ldots,n_k)$ be an element of $\mathscr{F}_\xi$ and $A=A_1\cup\ldots\cup A_\mu$ for $\mu\in\mathbb{N}$ and $A_1,\ldots,A_\mu\in\mathscr{F}_\beta$ with $\{n(\mu)\}<A_1<\ldots<A_\mu$. Then for arbitrary scalars $\lambda_1,\ldots,\lambda_k$ we have:

$$\left\|\sum_{i=1}^k \lambda_i h_{n_i}\right\|_\infty \leqslant \left\|\sum_{i=1}^k \lambda_i h_{n_i}^{n_1}\right\|_\infty + \left\|\sum_{i=2}^k \lambda_i (d_{n_i}^{n_1}+\ldots+d_{n_i}^{n_i-1})\right\|_\infty .$$

At first

$$\left\| \sum_{i=1}^{k} \lambda_i h_{n_i}^{n_1} \right\|_\infty \leq M \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|_\infty .$$

Secondly

$$\left\| \sum_{i=2}^{k} \lambda_i (d_{n_i}^{n_1} + \ldots + d_{n_i}^{n_i - 1}) \right\|_\infty$$

$$\leq \sum_{\zeta=1}^{k} \sum_{p=n_\zeta}^{n_{\zeta+1}-1} \left\| \sum_{i=\zeta+1}^{k} \lambda_i d_{n_i}^{p} \right\|_\infty$$

$$\leq \sum_{\zeta=1}^{k} \sum_{p=n_\zeta}^{n_{\zeta+1}-1} \sum_{q=1}^{\mu} \left\| \sum_{i\in\Gamma} \lambda_i d_{n_i}^{p} \right\|_\infty , \quad (\text{where } \Gamma = \{\zeta+1 \leq i < k : n_i \in A_q\})$$

$$\leq \sum_{\zeta=1}^{k} \sum_{p=n_\zeta}^{n_{\zeta+1}-1} 2\varepsilon_p \sum_{q=1}^{\mu} \left\| \sum_{i\in\Gamma} \lambda_i s_i \right\|_\infty$$

$$\leq \sum_{\zeta=1}^{k} \sum_{p=n_\zeta}^{n_{\zeta+1}-1} 4\mu\, \varepsilon_p \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|_\infty$$

$$\leq 4\mu \left(\sum_{\zeta=1}^{k} \sum_{p=n_\zeta}^{n_{\zeta+1}-1} \varepsilon_p \right) \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|_\infty$$

$$\leq 4\mu\, \varepsilon_{n_1 - 1} \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|_\infty$$

$$\leq M \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|_\infty .$$

We set $f_i = h_{n(i)}$ for every $i \in \mathbb{N}$. The sequence (f_n) satisfies:

$$\left\| \sum_{i=1}^{k} \lambda_i f_{n_i} \right\|_\infty \leq 2\|F\|_\xi \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|_\infty \tag{3}$$

for every $(n_1, \ldots, n_k) \in \mathscr{F}_\xi$ and scalars $\lambda_1, \ldots, \lambda_k$. Using analogous arguments we can prove that every convex block subsequence (g_n) of (f_n) has a subsequence which satisfies (3).

Since $F \notin C(K)$, every bounded sequence of continuous functions converging pointwise to F has a subsequence (d_n) which is a basic sequence and satisfies

$$\left\| \sum_{i=1}^{k} \lambda_i d_i \right\|_\infty \geq C \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|_\infty$$

for $0 < C = C(F) < \infty$, every $k \in \mathbb{N}$ and scalars $\lambda_1, \ldots, \lambda_k$. Hence every convex block subsequence (g_n) of (f_n) has a subsequence (d_n) which satisfies the inequalities:

$$C \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|_\infty \leq \left\| \sum_{i=1}^{k} \lambda_i d_{n_i} \right\|_\infty \leq 2\|F\|_\xi \left\| \sum_{i=1}^{k} \lambda_i s_{n_i} \right\|_\infty$$

for every $(n_1, \ldots, n_k) \in \mathscr{F}_\xi$ and scalars $\lambda_1, \ldots, \lambda_k$.

Case 2. Let ξ be a limit ordinal, (ξ_m) a sequence of ordinal numbers strictly increasing to ξ and $F\in V_\xi(K)\setminus C(K)$. According to Definition 1.2 there exists $M=\sup_{\beta<\xi}\|F\|_\beta<\infty$. Hence $\|F\|_{\xi_m}\leqslant M$ for every $m\in\mathbb{N}$.

Since $F\in V_{\xi_m}(K)\setminus C(K)$ for every $m\in\mathbb{N}$, using the inductive hypothesis we can find for every $m\in\mathbb{N}$ a bounded basic sequence $(f_n^m)_{n=1}^\infty\subset C(K)$ converging pointwise to F such that every convex block subsequence $(g_n^m)_{n=1}^\infty$ of $(f_n^m)_{n=1}^\infty$ has a subsequence $(d_n)_{n=1}^\infty$ satisfying

$$C\left\|\sum_{i=1}^k \lambda_i s_i\right\|_\infty\leqslant\left\|\sum_{i=1}^k \lambda_i d_{n_i}^m\right\|_\infty\leqslant 2M\left\|\sum_{i=1}^k \lambda_i s_i\right\|_\infty \tag{4}$$

for every $(n_1,\ldots,n_k)\in\mathscr{F}_{\xi_m}$ and scalars $\lambda_1,\ldots,\lambda_k$.

We set $h_n^1=f_n^1$ for every $n\in\mathbb{N}$. Since the minimum distance of the convex hulls $\mathrm{co}\{h_n^1: n\in\mathbb{N}\}$ and $\mathrm{co}\{f_n^2: n\in\mathbb{N}\}$ is equal to zero, there are convex block subsequences (h_n^2), (g_n^2) of (h_n^1) and (f_n^2) respectively with $\|h_n^2-g_n^2\|_\infty\to 0$. Hence, every convex block subsequence of (h_n^2) has a subsequence which satisfies (4) for every $(n_1,\ldots,n_k)\in\mathscr{F}_{\xi_2}$ and scalars $\lambda_1,\ldots,\lambda_k$. Consequently, we can find convex block subsequences (h_n^3), (g_n^3) of (h_n^2) and (f_n^3) respectively with $\|h_n^3-g_n^3\|_\infty\to 0$. We continue in an obvious way to define $(h_n^m)_{n=1}^\infty$ for every $m\in\mathbb{N}$.

We set $f_n=h_n^n$ for every $n\in\mathbb{N}$. Obviously, (f_n) converges pointwise to F and $(f_n)_{n=m}^\infty$ is a convex block subsequence of $(h_n^m)_{n=m}^\infty$ for every $m\in\mathbb{N}$.

Let (g_n) be a convex block subsequence of (f_n). Then for every $m\in\mathbb{N}$ there exists a subsequence of $(g_n)_{n=m}^\infty$ which satisfies (4) for every $(n_1,\ldots,n_k)\in\mathscr{F}_{\xi_m}$ and scalars $\lambda_1,\ldots,\lambda_k$. Using a diagonal argument we can find a subsequence (d_n) of (g_n) such that the sequence $(d_n)_{n=m}^\infty$ satisfies (4) for every $(n_1,\ldots,n_k)\in\mathscr{F}_{\xi_m}$ with $m\leqslant n_1$ and scalars $\lambda_1,\ldots,\lambda_k$.

As we proved in Theorem 1.1 the converse of Theorem 1.5 is true for $\xi=1$. We do not know if it is also true for every $1<\xi<\omega_1$.

COROLLARY 1.5. *Let K be a compact metric space, ξ a countable ordinal and $F\in V_\xi(K)\setminus C(K)$. Then every bounded sequence $(f_n)\subseteq C(K)$ converging pointwise to F has a convex block subsequence (d_n) which satisfies* (2).

§2. In this section we will define an ordinal index for every seminormalized, weakly null sequence in a Banach space, which measures the degree to which it has extending spreading models equivalent to the usual basis of c_0. (An analogous index can be defined for l_1.) For the definition we will need the notion of a tree as defined by Bourgain in [**6**].

Definition 2.1. [**6**]. Let X be an arbitrary set. A *tree* on X is a subset T of $\bigcup_{n=1}^\infty X^n$ with the property that $(x_1,\ldots,x_n)\in T$ whenever $(x_1,\ldots,x_n, x_{n+1})\in T$. A tree is *well founded* if there is no sequence (x_n) in X satisfying $(x_1,\ldots,x_n)\in T$ for every $n\in\mathbb{N}$.

Given a well founded tree T in X define inductively the trees T^ξ for every ordinal ξ in the following way:

$$T^0 = T;$$

$$T^{\xi+1} = \bigcup_{n=1}^{\infty} \{(x_1, \dots, x_n) \in T^\xi : (x_1, \dots, x_n, x) \in T^\xi \text{ for some } x \in X\};$$

and for a limit ordinal ξ

$$T^\xi = \bigcap_{\beta < \xi} T^\beta.$$

The *ordinal index* of the tree T (it is denoted by $O[T]$) is the smallest ordinal ξ so that $T^\xi = \varnothing$.

A particular version of the Kunen–Martin theorem (see **[6]**) gives that every well founded closed tree T on a Polish space has countable ordinal index $(O[T] < \omega_1)$.

Definition 2.2. Let X be a separable Banach space and $\delta > 0$. We set

$$T(X, \delta) = \bigcup_{n=1}^{\infty} \left\{ (x_1, \dots, x_n) \in X^n : \text{for all } (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \right.$$
$$\left. (1/1+\delta) \max_{i \leqslant n} |\lambda_i| \leqslant \left\| \sum_{i=1}^{n} \lambda_i x_i \right\| \leqslant (1+\delta) \max_{i \leqslant n} |\lambda_i| \right\}.$$

Clearly, the tree $T(X, \delta)$ is closed and well founded, if, and only if, c_0 is not δ-isomorphically embeddable into X. Hence if c_0 is not embeddable in X then $O[T(X, \delta)] < \omega_1$ for every $\delta > 0$.

We define the $\delta - c_0$*-index* of X as follows.

$c_0[X, \delta] = O[T(X, \delta)] < \omega_1$, if c_0 is not embeddable in X; and

$c_0[X, \delta] = \omega_1$ otherwise.

Finally we define the c_0*-index* of X as

$$c_0[X] = \sup_{\delta > 0} c_0[X, \delta].$$

Remarks. (1) If A is a bounded subset of X we can define the c_0-index $c_0[A]$ of A in a similar way by taking trees on A instead of X.

(2) Using a result of Giesy–James **[7]**, we see that a separable Banach space X has c_0-index greater that ω, if, and only if, c_0 is finitely representable in X.

Definition 2.3. Let X be a separable Banach space and (x_n) a seminormalized, weakly null sequence in X. For some $\delta > 0$ let

$$C[(x_n), \delta] = \{1 \leqslant \xi < \omega_1 : \text{there exists } (n_1, \dots, n_k) \in \mathscr{F}_\xi \text{ such that}$$
$$(x_{n_1}, \dots, x_{n_k}) \notin T((x_n), \delta)\}.$$

We define the *$\delta - c_0$-spreading model index* of (x_n) as follows.

$$c_0\text{-SM}[(x_n), \delta] = \inf C[(x_n), \delta], \text{ if } C[(x_n), \delta) \neq \emptyset;$$

$$c_0\text{-SM}[(x_n), \delta] = \omega_1, \text{ otherwise.}$$

The *c_0-spreading model index of* (x_n) is

$$c_0\text{-SM}[(x_n)] = \sup_{\delta > 0} c_0\text{-SM}[(x_n), \delta],$$

and the *c_0-spreading model index* of X is

$$c_0\text{-SM}[X] = \sup \{c_0\text{-SM}[(x_n)]: (x_n) \subset X \text{ with } x_n \xrightarrow{w} 0 \text{ and } \|x_n\| = 1 \text{ for all } n \in \mathbb{N}\}$$

Since $O[\mathscr{F}_\xi] = \omega^\xi$ for every $1 \leqslant \xi < \omega_1$ we have that if $c_0\text{-SM}[(x_n), \delta] > \xi$ then $c_0[(x_n), \delta] \geqslant \omega^\xi$. Hence if (x_n) has no subsequence equivalent to the usual basis of c_0 then $c_0[(x_n)] < \omega_1$ and also $c_0\text{-SM}[(x_n)] < \omega_1$. Similarly it can be proved that $c_0[X] \geqslant \omega^\xi$ if $c_0\text{-SM}[X] > \xi$ and that $c_0\text{-SM}[X] = \omega_1$, if, and only if, c_0 embeds in X.

Using the following proposition we can characterize the separable Banach spaces X with $c_0\text{-SM}[X] > 1$.

PROPOSITION 2.4. *Let X be a separable Banach space and (x_n) a seminormalized weakly null sequence in X. The following are equivalent.*

(i) *(x_n) has a basic subsequence with spreading model equivalent to the usual basis of c_0.*

(ii) *(x_n) has a subsequence (y_n) with $c_0\text{-SM}[(y_n)] > 1$.*

Proof. (i)$\Rightarrow$(ii). Let (z_n) be a subsequence of (x_n) with spreading model equivalent to the usual basis of c_0. This means that there exists $\infty > C > 0$ and $n(k) \in \mathbb{N}$ with $n(k) < n(k+1)$ for every $k \in \mathbb{N}$ such that if $n(k) \leqslant p_1 < p_2 < \ldots < p_k$ then $(z_{p_1}, \ldots, z_{p_k})$ is C-equivalent to the usual basis of $c_0(k)$. Set $y_k = z_{n(k)}$ for every $k \in \mathbb{N}$. The subsequence $(y_n)_{n=1}^{\infty}$ of (x_n) has $c_0\text{-SM}[(y_n)] > 1$ since $(y_{p_1}, \ldots, y_{p_k})$ is C-equivalent to the usual basis of $c_0(k)$ if $(p_1, \ldots, p_k) \in \mathscr{F}_1$.

(ii)$\Rightarrow$(i). Let (y_n) be a subsequence of (x_n) with $c_0\text{-SM}[(y_n)] > 1$. The sequence (y_n) has a basic subsequence (z_n) which generates a spreading model (e_n). Since $c_0\text{-SM}[(z_n)] > 1$ there exists $\delta > 0$ so that $(z_{p_1}, \ldots, z_{p_k}) \in T((z_n), \delta)$ for every $(p_1, \ldots, p_k) \in \mathscr{F}_1$. Hence (e_n) is equivalent to the usual basis of c_0.

From the previous proposition we have that X has a sequence with spreading model equivalent to the usual basis of c_0, if, and only if, $c_0\text{-SM}[X] > 1$.

We will give examples of reflexive Banach spaces with arbitrary large c_0-spreading model index.

Example 2.5. For every countable ordinal ξ, let T_ξ be the Tsirelson-like space which has been defined by Argyros in [2]. We will prove that $c_0\text{-SM}[T_\xi^*] > \xi$.

For completeness we recall the definition of T_ξ. Let $1 \leqslant \xi < \omega_1$ and $x: \mathbb{N} \to \mathbb{R}$ be a finitely supported function. For every $m \in \mathbb{N}$ set

$\|x\|_0^\xi = \sup \{|x(p)| : p \in \mathbb{N}\}$, and

$$\|x\|_{m+1}^\xi = \max \left\{ \|x\|_m^\xi, \tfrac{1}{2} \sup \sum_{i=1}^{k-1} \|x|p_i, p_{i+1}-1|\|_m^\xi \text{ for all } (p_1, \ldots, p_k) \in \mathscr{A}_\xi \right\},$$

where $x|p, q|$ $(p \leqslant q)$ denotes the restriction of x on the set $\{p, p+1, \ldots, q\}$ and $\mathscr{A}_\xi = \mathscr{F}_\xi \bigcup \{(n, p) : 2 \leqslant n < p\} \cup (\varnothing\}$.

Finally define

$$\|x\|^\xi = \lim_{m \to \infty} \|x\|_m^\xi$$

$$= \max \left\{ \|x\|_0^\xi, \sup \tfrac{1}{2} \sum_{i=1}^{k-1} \|x|p_i, p_{i+1}-1|\|^\xi \text{ for } \{p_1, \ldots, p_k\} \in \mathscr{A}_\xi \right\}.$$

The space T_ξ is the completion of the linear space of all finitely supported functions with the norm $\|\cdot\|^\xi$.

The usual basis (e_n) is an unconditional basis of T_ξ and as proved in **[2]**, T_ξ is reflexive. Let (e_n^*) be the sequence of biorthogonal functionals of (e_n) and let $R_\xi = [e_n^*] = T_\xi^*$. Hence R_ξ is also a reflexive space with unconditional basis.

Let $F = (p_1, \ldots, p_k) \in \mathscr{F}_\xi$. We will prove that $(e_{p_1}^*, \ldots, e_{p_k}^*) \in T(R_\xi, 2)$. Indeed, for every $x \in T_\xi$ with $\|x\|^\xi \leqslant 1$ we have

$$\sum_{i=1}^{k-1} |e_{p_i}^*(x)| \leqslant \sum_{i=1}^{k-1} \|x|p_i, p_{i+1}-1|\|^\xi \leqslant 2\|x\|^\xi \leqslant 2.$$

Hence it is easy to see that

$$\tfrac{1}{3} \max_{i \leqslant k} |\lambda_i| \leqslant \left\| \sum_{i=1}^{k} \lambda_i e_{p_i}^* \right\|_\infty \leqslant 3 \max_{i \leqslant k} |\lambda_i|.$$

Thus $c_0\text{-SM}[(e_n^*)] \geqslant c_0\text{-SM}[(e_n^*), 2] > \xi$ and consequently $c_0\text{-SM}[T_\xi^*] > \xi$.

As we will see in the following example there exist Banach spaces with arbitrary large c_0-index and c_0-SM-index equal to zero. Hence the c_0-SM-index measures something stronger than the c_0-index.

Example 2.6. We will define inductively a family $(X_\xi)_{1 \leqslant \xi < \omega_1}$ of reflexive Banach spaces as follows:

$$X_1 = l^2;$$

$$X_{\xi+1} = (X_\xi \oplus l^2)_\infty;$$

and for every limit ordinal ξ,

$$X_\xi = \left(\sum_{1 \leqslant \beta < \xi} X_\beta \right)_2.$$

It is easy to prove that $c_0[X_\xi] \geqslant \xi$ for every $1 \leqslant \xi < \omega_1$. We will prove that $c_0\text{-SM}[X_\xi] = 0$ for every $1 \leqslant \xi < \omega_1$ using some techniques of **[1]**.

Let ξ be a countable limit ordinal and (x_n) a seminormalized, weakly null sequence in X_ξ. Then by passing to a subsequence we may assume that $x_n = y_n + z_n$ for every $n \in \mathbb{N}$, where $y_n \in \sum_{\beta \leqslant \lambda} X_\beta$ for $\lambda < \xi$ and $z_n \in \sum_{\beta \in B_n} X_\beta$ where $B_n = \{\beta : \xi_n < \beta \leqslant \xi_{n+1}\}$ for all $n \in \mathbb{N}$ and $\xi_1 > \lambda$. Since $\overline{\text{span}}\,(z_n) = l^2$ we have that $c_0\text{-SM}[(x_n)] = c_0\text{-SM}[(y_n)]$. Hence

$$c_0\text{-SM}[X_\xi] = \sup\{c_0\text{-SM}[X_\beta] : 1 \leqslant \beta < \xi\}.$$

Now if $\xi = \beta + \kappa$ for some $\kappa \in \mathbb{N}$ then also $x_n = y_n + z_n$ with $y_n \in X_\beta$ and $z_n \in l^2 \oplus l^2 \oplus \ldots \oplus l^2$ for every $n \in \mathbb{N}$. Hence $c_0\text{-SM}[(x_n)] = c_0\text{-SM}[(y_n)]$ and $c_0\text{-SM}[X_\xi] = c_0\text{-SM}[X_\beta]$. Thus $c_0\text{-SM}[X_\xi] = c_0\text{-SM}[l^2] = 0$ for every $\xi < \omega_1$.

§3. In this section we define an index for every Baire-1 function on a compact metric space, classifying the Baire-1/4 functions. In fact we prove that every function in V_ξ has index greater than ξ.

Definition 3.1. Let (f_n) be a uniformly bounded sequence of continuous functions on a compact metric space K. For every real number $\delta > 0$ we set

$$T_s((f_n), \delta) = \bigcup_{k=1}^{\infty} \Bigg\{ (f_{n_1}, \ldots, f_{n_k}) : n_1 < \ldots < n_k \text{ and, for all } \lambda_1, \ldots, \lambda_k \in \mathbb{R},$$

$$(1/1+\delta) \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|_\infty \leqslant \left\| \sum_{i=1}^{k} \lambda_i f_{n_i} \right\|_\infty \leqslant (1+\delta) \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|_\infty \Bigg\}.$$

It is easy to see that $T_s((f_n), \delta)$ is a tree on $C(K)$.

We define the *$\delta-s$-ordinal* of (f_n) as follows.

$s[(f_n), \delta] = O[T_s((f_n), \delta)] < \omega_1$, if $T_s((f_n), \delta)$ is well founded, and

$s[(f_n), \delta] = \omega_1$, otherwise.

The *s-index* of (f_n) is $s[(f_n)] = \sup_{\delta > 0} s[(f_n), \delta]$.

It is easy to see that $s[(f_n)] < \omega_1$, if, and only if, (f_n) has no subsequence equivalent to the summing basis of c_0. If (f_n) converges pointwise to a function F, then $F \in \text{DBSC}(K)$ if $s[(f_n)] = \omega_1$ but the converse is not true.

We will define another index of (f_n) which is smaller than the *s-index* and it is analogous to the spreading model index which we defined in Section 2.

Definition 3.2. Let K be a compact metric space and (f_n) a uniformly bounded sequence of continuous functions on K. For $\delta > 0$ let

$$S[(f_n), \delta] = \{1 \leqslant \xi < \omega_1 : \text{there exists } (n_1, \ldots, n_k) \in \mathcal{F}_\xi \text{ such that } (f_{n_1}, \ldots, f_{n_k}) \notin T_s((f_n), \delta)\}.$$

We define the *$\delta - s$-spreading model index* of (f_n) as follows.

$$s\text{-SM}[(f_n), \delta] = \inf S((f_n), \delta) \quad \text{if } S((f_n), \delta) \neq \varnothing \quad \text{and}$$

$$s\text{-SM}[(f_n), \delta] = \omega_1 \quad \text{otherwise.}$$

The *s-spreading model index* of (f_n) is

$$s\text{-SM}[(f_n)] = \sup_{\delta > 0} s\text{-SM}[(f_n), \delta].$$

If s-SM$[(f_n)] > 1$ and (f_n) generates a spreading model then the spreading model is equivalent to the summing basis of c_0. Also if s-SM$[(f_n)] > \xi$ then $s[(f_n)] \geqslant \omega^{\xi}$. Hence if (f_n) converges pointwise to a function F and s-SM$[(f_n)] = \omega_1$ then $F \in \text{DBSC}(K)$.

It is well known that $F \in \text{DBSC}(K)$, if, and only if, every bounded sequence $(f_n) \subseteq C(K)$ converging pointwise to F has a convex block subsequence equivalent to the summing basis of c_0.

Definition 3.3. Let K be a compact metric space and $(f_n) \subseteq C(K)$ a bounded sequence. Let

$$X = \{((n_1, \ldots, n_k), \lambda_1, \ldots, \lambda_k) : n_1 < \ldots < n_k \in \mathbb{N}, \lambda_1, \ldots, \lambda_k \in \mathbb{R}$$
$$\text{with } \lambda_i \geqslant 0 \text{ for every } 1 \leqslant i \leqslant k \text{ and } \lambda_1 + \ldots + \lambda_k = 1\}.$$

For every $x = ((n_1, \ldots, n_k), \lambda_1, \ldots, \lambda_k)$, $y = ((m_1, \ldots, m_p), \mu_1, \ldots, \mu_p)$ in X we define $x \leqslant y$ if $n_k \leqslant m_1$. Also we define a metric d on X by

$$d(x, y) = 1 \quad \text{if} \quad (n_1, \ldots, n_k) \neq (m_1, \ldots, m_p) \quad \text{and}$$

$$d(x, y) = \max_{1 \leqslant i \leqslant k} |\lambda_i - \mu_i| \text{ otherwise.}$$

The set X with the metric d is a Polish (separable, complete) space.

On the space (X, d) we will define a tree $T_{bs}((f_n), \delta)$ for every $\delta > 0$. We denote $\sum_{i=1}^{k} \lambda_i f_{n_i}$ by $b(x)$.

$$T_{bs}((f_n), \delta) = \Big\{ (x_1, \ldots, x_n) \in X^n : x_1 < \ldots < x_n \text{ and if } g_i = b(x_i) \text{ for } 1 \leqslant i \leqslant n \text{ then}$$
$$(1/1+\delta) \Big\| \sum_{i=1}^{n} a_i s_i \Big\|_\infty \leqslant \Big\| \sum_{i=1}^{n} a_i g_i \Big\|_\infty \leqslant (1+\delta) \Big\| \sum_{i=1}^{n} a_i s_i \Big\|_\infty$$
$$\text{for all } a_1, \ldots, a_n \in \mathbb{R} \Big\}.$$

This tree is closed on X and it is well founded, if, and only if, (f_n) has no convex block subsequence δ-equivalent to the summing basis of c_0.

We define the *δ-block-s-index* of (f_n) as follows.

$$bs[(f_n), \delta] = O[T_{bs}((f_n), \delta)] < \omega_1 \quad \text{if} \quad T_{bs}((f_n), \delta)$$

is well founded and

$$bs[(f_n), \delta] = \omega_1 \quad \text{otherwise.}$$

It is easy to see that $bs[(f_n), \delta] \geqslant s[(f_n), \delta]$ and that $bs[(f_n), \delta] = \omega_1$ if, and only if, (f_n) has a convex block subsequence δ-equivalent to the summing basis to c_0.

Now we will define ordinal indices for every Baire-1 function.

Definition 3.4. Let K be a compact metric space and $F \in B_1(K)$. We set

$$A[F] = \{1 \leqslant \xi < \omega_1 : \text{every bounded sequence } (f_n) \subseteq C(K) \text{ converging pointwise to } F \text{ satisfies } s[(f_n)] < \xi\}.$$

The *s-index* of F is

$$s\text{-}[F] = \inf A[F] \quad \text{if } A[F] \neq \varnothing \quad \text{and}$$

$$s\text{-}[F] = \omega_1 \quad \text{otherwise.}$$

The *$\delta - s$-spreading model index* of F for $\delta > 0$ is

$$s\text{-SM}[F, \delta] = \inf \{1 \leqslant \xi \leqslant \omega_1 : \text{there exists a bounded sequence } (f_n) \text{ in } C(K) \text{ converging pointwise to } F \text{ such that every convex block subsequence } (g_n) \text{ has } s\text{-SM}[(g_n), \delta] \leqslant \xi\}.$$

The *s-spreading model index* of F is

$$s\text{-SM}[F] = \sup_{\delta > 0} s\text{-SM}[F, \delta].$$

The *δ-block s-index* of F for $\delta > 0$ is

$$bs[F, \delta] = \inf \{bs[(f_n), \delta] : (f_n) \subseteq C(K) \text{ is a bounded sequence converging pointwise to } F\}.$$

The *block s-index* of F is

$$bs[F] = \sup_{\delta > 0} bs[F, \delta]$$

Of course if $s\text{-SM}[F] > \xi$ for some $1 \leqslant \xi < \omega_1$ then $bs[F] \geqslant \omega^\xi$ and $s[F] \geqslant \omega^\xi$. Also $F \in \text{DBSC}(K)$, if, and only if, $bs[F] = \omega_1$.

PROPOSITION 3.5. *Let K be a compact metric space and $F \in B_1(K) \setminus C(K)$. Then*

(1) *$F \in B_{1/4}(K)$, if, and only if, $s\text{-SM}[F] > 1$;*
(2) *$F \in \text{DBSC}(K)$, if, and only if, $s\text{-SM}[F] = \omega_1$; and*
(3) *if $F \in V_\xi(K)$ then $s\text{-SM}[F] > \xi$.*

Proof. (1) According to Definition 3.4, $s\text{-SM}[F] > 1$, if, and only if, every bounded sequence $(f_n) \subseteq C(K)$ converging pointwise to F has a convex block subsequence (g_n) with $s\text{-SM}[(g_n)] > 1$. Also similarly to Proposition 2.4 we can prove that (f_n) has a convex block subsequence (g_n) with $s\text{-SM}[(g_n)] > 1$, if, and only if, it has a convex block subsequence generating a spreading model equivalent to the summing basis of c_0. Hence from Theorem 1.1 we have the proof of (1).

(2) It is easy to see that if $F \in \text{DBSC}(K)$ then $s\text{-SM}[F] = \omega_1$. On the other hand if $s\text{-SM}[F] = \omega_1$ then $bs[F] = \omega_1$, hence as we notice in Definition 3.3, $F \in \text{DBSC}(K)$.

(3) Theorem 1.3 implies this result.

COROLLARY 3.6. *Let K be a compact metric space. The intersection of all spaces $V_\xi(K)$ for $1 \leqslant \xi < \omega_1$ coincides with the* DBSC(K).

Problem 3.7. Let K be a compact metric space, $F \in B_1(K) \setminus C(K)$ and ξ a countable ordinal. If $s\text{-SM}[F] > \xi$, is F in $V_\xi(K)$?

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