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Characterizations of elements of a double dual Banach space and their canonical reproductions

by

VASSILIKI FARMAKI (Athens)

Abstract. For every element x^{**} in the double dual of a separable Banach space X there exists the sequence $(x^{(2n)})$ of the canonical reproductions of x^{**} in the even-order duals of X. In this paper we prove that every such sequence defines a spreading model for X. Using this result we characterize the elements of $X^{**} \setminus X$ which belong to the class $B_1(X) \setminus B_{1/2}(X)$ (resp. to the class $B_{1/4}(X)$) as the elements with the sequence $(x^{(2n)})$ equivalent to the usual basis of ℓ^1 (resp. as the elements with the sequence $(x^{(4n-2)} - x^{(4n)})$ equivalent to the usual basis of c_0). Also, by analogous conditions but of isometric nature, we characterize the embeddability of ℓ^1 (resp. c_0) in X.

Introduction. In the last few years, the theory of spreading models ([1], [8]) has proved fruitful in the study of Banach spaces. For example, it was used by R. Haydon, E. Odell and H. Rosenthal in [5] to give characterizations of certain elements of the second dual X^{**} of a separable Banach space X.

In this paper we prove (Theorem 3) that for every element $x^{**} \in X^{**} \setminus X$ the sequence $(x^{(2n)})$ of its canonical reproductions defines a spreading model for X.

There are many possible applications of this result. In this paper we are able to determine when an element $x^{**} \in X^{**} \setminus X$ belongs to the class $B_1(X) \setminus B_{1/2}(X)$ (resp. the class $B_{1/4}(X)$) exclusively in terms of the sequence $(x^{(2n)})$ (Theorems 11 and 12). More precisely, $x^{**} \in B_1(X) \setminus B_{1/2}(X)$ if and only if the sequence $(x^{(2n)})$ is equivalent to the usual basis of ℓ^1, and $x^{**} \in B_{1/4}(X)$ if and only if the sequence $(x^{(4n-2)} - x^{(4n)})$ is equivalent to the usual basis of c_0. In the proofs of these results we use the characterizations of elements in $B_1(X) \setminus B_{1/2}(X)$ (resp. $B_{1/4}(X)$) given in [5].

We also characterize the embeddability of ℓ^1 (resp. the embeddability of c_0) in X in terms of the properties of the canonical reproductions $(x^{(2n)})$ of some element $x^{**} \in X^{**}$ (Propositions 6 and 8). Unlike the characterizations of Baire-1 elements of X^{**} given above, these characterizations are of an

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isometric nature. Precisely, ℓ^1 embeds in X (resp. c_0 embeds in X) if and only if there exists $x^{**} \in X^{**} \setminus X$ such that

$$\Big\|x + \sum_{i=1}^{n} a_i x^{(2i)}\Big\| = \Big\|x + \Big(\sum_{i=1}^{n} |a_i|\Big) x^{**}\Big\|$$

$$(\text{resp. } \Big\|x + \sum_{i=1}^{n} a_i (x^{(4i-2)} - x^{(4i)})\Big\| = \|x + \max_{1\le i\le n} |a_i| (x^{**} - x^{(4)})\|)$$

for every $x \in X$, $n \in \mathbb{N}$ and scalars $a_1, \ldots, a_n$. These characterizations are influenced by the deep results of Maurey in [6], where it is proved that the embeddability of ℓ^1 in X is equivalent to the existence of an element x^{**} in $X^{**} \setminus X$ such that $\|x + x^{**}\| = \|x - x^{**}\|$ for every $x \in X$ (for the case of c_0 an analogous characterization is given). In the proof of these results we use some results and techniques of the theory of types and especially from the papers [9] and [4].

Finally, in Propositions 7 and 10 we prove some new characterizations for the embeddability of c_0 in a Banach space.

Throughout this article we denote by X a real separable infinite-dimensional Banach space. $X^{**}, X^{(3)}, X^{(4)}, \ldots$ are the second, third, fourth, … duals of X respectively. For a subset A of X, $\operatorname{conv} A$, $[A]$ and $\overline{A}$ denote the convex hull, linear span and $\|\ \|$-closure of A respectively. For any subset A of $X^{(2n)}$ $(n \ge 1)$ we denote by $\widetilde{A}$ the weak*-closure of A in $X^{(2n)}$.

DEFINITION 1. Let X be a Banach space and $x^{(2)} \in X^{**}$. If $I_k : X \to X^{(2k-2)}$ is the canonical embedding of X in the $(2k-2)$-dual of X then we define $x^{(2k)} = I_k^{**}(x^{(2)})$ for every $k > 1$. The elements $x^{(2k)}$ are the *canonical reproductions* of $x^{(2)}$ in the duals $X^{(2k)}$ of even order for every $k > 1$.

It is easy to see that, if $x^{(2)} = w^*\text{-}\lim_i x_i$ for some net (x_i) in X, then $x^{(2k)} = w^*\text{-}\lim_i x_i$ if (x_i) is considered as a net in $X^{(2k)}$.

In the following theorem we will prove that the canonical reproductions can be considered as the fundamental sequence of a spreading model for X. In the proof we will use the following lemma which is a generalization of an analogous result in [9] and [4].

LEMMA 2. *Let X be a separable Banach space, $k \in \mathbb{N}$, $g \in X^{(2k)}$ and $W_1 \supseteq W_2 \supseteq \ldots$ a sequence of bounded, convex subsets of $X^{(2k-2)}$ so that $g \in \bigcap_{n=1}^{\infty} \widetilde{W}_n$. Then there exists a sequence $L_1 \supseteq L_2 \supseteq \ldots$ of convex subsets of $X^{(2k-2)}$ such that:*

(i) *$W_n \supseteq L_n$ for every $n \in \mathbb{N}$.*

(ii) *If $(x_n) \subseteq X^{(2k-2)}$ is such that $x_n \in L_n$ for every $n \in \mathbb{N}$, then $\|x + g\| = \lim_n \|x + x_n\|$ for every $x \in X$.*

(iii) *$g \in \widetilde{L}_n$ for every $n \in \mathbb{N}$.*

Proof. The lemma is proved in [9] for the case $k = 1$ and in [4] for $k = 2$. For $k > 2$ the proof is analogous.

THEOREM 3. *Let X be a separable Banach space and $x^{(2)} \in X^{**} \setminus X$. If $x^{(2k)}$ are the canonical reproductions of $x^{(2)}$ in the duals $X^{(2k)}$ of even order $(k > 1)$ then there exists a sequence (x_n) in X such that*

$$\|x + a_1 x^{(2)} + \ldots + a_k x^{(2k)}\| = \lim_{n_k} \ldots \lim_{n_1} \|x + a_1 z_{n_1} + \ldots + a_k z_{n_k}\|$$

for every convex block subsequence (z_n) of (x_n), $x \in X$, $k \in \mathbb{N}$ and scalars $a_1, \ldots, a_k$.

Proof. Let $x^{(2)} \in X^{**} \setminus X$. Using Lemma 2 for $W_n = \{x \in X : \|x\| \le \|x^{(2)}\|\}$ $(n \in \mathbb{N})$, we can find a sequence (L_n^1) of convex subsets of X with the properties (i)–(iii) of the lemma. From (iii) we have $x^{(2)} \in \bigcap_{n=1}^{\infty} \widetilde{L}_n^1$, hence $x^{(4)} \in \bigcap_{n=1}^{\infty} \widetilde{L}_n^1$ if L_n^1 for $n \ge 1$ are considered as subsets of X^{**}. Using Lemma 2 again for the space $X \oplus [x^{(2)}]$ and $W_n = L_n^1$, $n \in \mathbb{N}$, we can find a sequence (L_n^2) of convex subsets of X^{**} with the properties (i)–(iii). The next step is to use Lemma 2 for $X \oplus [x^{(2)}, x^{(4)}]$, $W_n = L_n^2$, $n \in \mathbb{N}$, and $x^{(6)} \in X^{(6)}$. We continue in the obvious manner.

We select $x_n \in L_n^n$ for every $n \in \mathbb{N}$. It is easy to see that $x_n \in L_n^k$ for every $n, k \in \mathbb{N}$ with $n \ge k$. Hence for every $x \in X$, $k \in \mathbb{N}$ and scalars $a_1, \ldots, a_k$ we have

$$\begin{aligned}\|x + a_1 x^{(2)} + \ldots + a_k x^{(2k)}\| &= \lim_{n_k} \|x + a_1 x^{(2)} + \ldots + a_{k-1} x^{(2k-1)} + a_k x_{n_k}\| \\ &= \lim_{n_k} \lim_{n_{k-1}} \|x + a_1 x^{(2)} + \ldots + a_{k-1} x_{n_{k-1}} + a_k x_{n_k}\| \\ &= \lim_{n_k} \ldots \lim_{n_1} \|x + a_1 x_{n_1} + \ldots + a_k x_{n_k}\|.\end{aligned}$$

If (z_n) is a convex block subsequence of (x_n) then, since the L_n^n are convex and $L_n^n \supseteq L_{n+1}^{n+1}$ for every $n \in \mathbb{N}$, we conclude that (z_n) is a subsequence of some sequence $(y_n) \subseteq X$ such that $y_n \in L_n^n$ for every $n \in \mathbb{N}$.

We will give a corollary of the previous theorem for the case of Baire-1 elements of a double dual space. Recall that $g \in X^{**} \setminus X$ is said to be a *Baire-1 element* of X^{**} if there exists a sequence (x_n) in X weak*-converging in X^{**} to g ($w^*\text{-}\lim_n x_n = g$).

COROLLARY 4. *Let X be a separable Banach space and $x^{(2)}$ a Baire-1 element of $X^{**} \setminus X$. If $x^{(2k)} \in X^{(2k)}$ $(k > 1)$ are the canonical reproductions of $x^{(2)}$, then there exists a sequence (x_n) in X such that $x^{(2)} = w^*\text{-}\lim_n x_n$ and*

$$\|x + a_1 x^{(2)} + \ldots + a_k x^{(2k)}\| = \lim_{n_k} \ldots \lim_{n_1} \|x + a_1 z_{n_1} + \ldots + a_k z_{n_k}\|$$

for every convex block subsequence (z_n) of (x_n), $k \in \mathbb{N}$, $x \in X$ and scalars $a_1, \ldots, a_k$.

Proof. Let (y_n) be a sequence in X, weak*-converging in X^{**} to $x^{(2)}$. We set $W_n = \mathrm{conv}\{y_i : i \geq n\}$ for $n \in \mathbb{N}$ and in the same way as in the proof of Theorem 3 we can find a convex block subsequence (x_n) of (y_n) such that

$$\|x + a_1 x^{(2)} + \ldots + a_k x^{(2k)}\| = \lim_{n_k} \ldots \lim_{n_1} \|x + a_1 z_{n_1} + \ldots + a_k z_{n_k}\|$$

for every convex block subsequence (z_n) of (x_n), $k \in \mathbb{N}$, $x \in X$ and scalars $a_1, \ldots, a_k$. Moreover, $x^{(2)} = w^*\text{-}\lim_n x_n$.

Let us recall the definition of a spreading model for a Banach space X (see [1], [3]).

DEFINITION 5. The Banach space Y is called a *spreading model* for X if there exist a sequence (e_n) in Y such that $Y = \overline{\mathrm{span}}(X \cup \{e_n : n \in \mathbb{N}\})$ and a sequence (x_n) in X so that (x_n) has no norm-convergent subsequence and

$$\|x + a_1 e_1 + \ldots + a_n e_n\| = \lim_{m_1} \ldots \lim_{m_n} \|x + a_1 x_{m_1} + \ldots + a_n x_{m_n}\|$$

for all $x \in X$, $n \in \mathbb{N}$ and scalars $a_1, \ldots, a_n$. The sequence (e_n) is called the *fundamental sequence* of the spreading model Y which is generated by the (x_n).

The spreading model is 1-*unconditional* over X if

$$\|x + \varepsilon_1 a_1 e_1 + \ldots + \varepsilon_n a_n e_n\| = \|x + a_1 e_1 + \ldots + a_n e_n\|$$

for every $x \in X$, $n \in \mathbb{N}$, scalars $a_1, \ldots, a_n$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$.

As we showed in Theorem 3, the canonical reproductions of an element $x^{(2)}$ in the double dual of a separable Banach space X give an inverse spreading model for X. Precisely, if L is the linear space with basis $(e_i)_{i=1}^{\infty}$ then we define on $X \oplus L$ the norm $\|x + \sum_{i=1}^{k} a_i e_i\| = \|x + \sum_{i=1}^{k} a_{k-i+1} x^{(2i)}\|$ for every $x \in X$, $k \in \mathbb{N}$ and scalars $a_1, \ldots, a_k$. The completion $Y_{x^{(2)}}$ of $X \oplus L$ under this norm is a spreading model for X, according to Theorem 3.

Many known results related to spreading models can be described via the canonical reproductions. It is known (see [1]) that the existence of a spreading model in X with fundamental sequence (e_n) such that

$$\Big\|x + \sum_{i=1}^{k} a_i e_i\Big\| = \Big\|x + \Big(\sum_{i=1}^{k} |a_i|^p\Big)^{1/p} e_1\Big\| \quad \text{for } 1 \leq p < \infty$$

$$\Big(\text{resp. } \Big\|x + \sum_{i=1}^{k} a_i e_i\Big\| = \|x + (\max_{1 \leq i \leq k} |a_i|) e_1\|\Big)$$

for every $x \in X$, $k \in \mathbb{N}$ and scalars $a_1, \ldots, a_k$ implies the embeddability of ℓ^p for $1 \leq p < \infty$ (resp. of c_0).

We characterize the embeddability of ℓ^1 and c_0 via the canonical reproductions in Propositions 6 and 8 respectively. These results are consequences of the deep results of Maurey in [6] on the embeddability of ℓ^1 and c_0 in X, and of Theorem 3. In the proofs we use some results and techniques of the theory of types which can be found in [9] and [4].

PROPOSITION 6. *Let X be a separable Banach space. The following conditions are equivalent*:

(i) *X contains a subspace isomorphic to ℓ^1.*

(ii) *There exists $x^{(2)} \in X^{**} \setminus X$ such that*

$$\Big\|x + \sum_{i=1}^{k} a_i x^{(2i)}\Big\| = \Big\|x + \Big(\sum_{i=1}^{k} |a_i|\Big) x^{(2)}\Big\|$$

for every $x \in X$, $k \in \mathbb{N}$ and scalars $a_1, \ldots, a_k$.

(iii) *There exists $x^{(2)} \in X^{**} \setminus X$ such that the spreading model $Y_{x^{(2)}}$ is 1-unconditional over X.*

Proof. (i)⇒(ii). From [6], if ℓ^1 is embeddable in X there exists $x^{(2)} \in X^{**} \setminus X$ such that $\|x + x^{(2)}\| = \|x - x^{(2)}\|$ for every $x \in X$. According to Theorem 3 there exists a sequence (x_n) in X such that

$$\Big\|x + \sum_{i=1}^{k} a_i x^{(2i)}\Big\| = \lim_{n_k} \ldots \lim_{n_1} \|x + a_1 z_{n_1} + \ldots + a_k z_{n_k}\|$$

for every convex block subsequence (z_n) of (x_n), $x \in X$, $k \in \mathbb{N}$ and scalars $a_1, \ldots, a_k$.

Thus $\lim_n \lim_m \|x + a x_n + b x_m\| = \lim_n \|x + (a+b) x_n\|$ for every $x \in X$ and $a, b \geq 0$. Indeed, let $a, b > 0$ with $a + b = 1$ and $x \in X$ such that $\lim_n \lim_m \|x + a x_n + b x_m\|$ is not equal to $\lim_n \|x + x_n\| = \|x + x^{(2)}\|$. By Ramsey's principle [7] we can choose a sequence $n(1) < n(2) < \ldots$ of natural numbers so that

$$\lim_n \lim_m \|x + a x_n + b x_m\| = \lim_{j > i \to \infty} \|x + a x_{n(i)} + b x_{n(j)}\|.$$

Let $z_i = a x_{n(2i)} + b x_{n(2i+1)}$ for every $i \in \mathbb{N}$. Since (z_i) is a convex block subsequence of (x_n) we have $\lim_i \|x + z_i\| = \|x + x^{(2)}\|$. On the other hand, $\lim_i \|x + z_i\| = \lim_n \lim_m \|x + a x_n + b x_m\|$, a contradiction.

Thus

$$\Big\|x + \sum_{i=1}^{k} a_i x^{(2i)}\Big\| = \Big\|x + \Big(\sum_{i=1}^{k} a_i\Big) x^{(2)}\Big\|$$

for every $x \in X$, $k \in \mathbb{N}$ and scalars $a_1, \ldots, a_k \geq 0$.

Since $\|x + x^{(2)}\| = \|x - x^{(2)}\|$ we have

$$\lim_n \lim_m \|x + a x_n + b x_m\| = \lim_n \|x + (|a| + |b|) x_n\|$$

for every $x \in X$ and scalars a, b. Hence

$$\Big\|x+\sum_{i=1}^{k} a_i x^{(2i)}\Big\| = \Big\|x+\Big(\sum_{i=1}^{k}|a_i|\Big)x^{(2)}\Big\|$$

for every $x \in X$, $k \in \mathbb{N}$ and scalars $a_1,\ldots,a_k$.

(ii)⇒(iii). For every $x\in X$, $k\in\mathbb{N}$, scalars $a_1,\ldots,a_k$, and signs $\varepsilon_1,\ldots,\varepsilon_k$ we have the equalities

$$\Big\|x+\sum_{i=1}^{k}\varepsilon_i a_i e_i\Big\| = \Big\|x+\sum_{i=1}^{k}\varepsilon_{k-i+1}a_{k-i+1}x^{(2i)}\Big\| = \Big\|x+\Big(\sum_{i=1}^{k}|a_i|\Big)x^{(2)}\Big\|$$
$$= \Big\|x+\sum_{i=1}^{k}a_{k-i+1}x^{(2i)}\Big\| = \Big\|x+\sum_{i=1}^{k}a_i e_i\Big\|.$$

(iii)⇒(i). Obvious from [6].

In the following we give criteria for a Banach space to contain c_0. If $A,B\in\mathbb{R}$ and $\varepsilon>0$ we write $A\overset{\varepsilon}{\sim}B$ whenever $|A-B|<\varepsilon$.

Proposition 7. *Let X be a Banach space. Then c_0 embeds in X if and only if there exists a net (x_i) in X such that $\lim_i\|x_i\|>0$ and, for every $x\in X$, we have $\lim_i\lim_j\|x+x_i+x_j\|=\lim_i\|x+x_i\|$.*

Proof. Let (x_i) be a bounded net as in the statement of the proposition. For every $\varepsilon>0$ we can find a sequence (y_n) in X such that $y_n=x_{i_n}$ for $n\in\mathbb{N}$, $i_1<i_2<\ldots$ and

$$(*)\qquad \|y_{n_1}+\ldots+y_{n_k}\|\overset{\varepsilon}{\sim}\lim_i\|x_i\|$$

for every increasing sequence (n_m) of natural numbers and $k\in\mathbb{N}$.

Indeed, let (ε_n) be a sequence of positive numbers so that $\sum_{n=1}^{\infty}\varepsilon_n<\varepsilon$ and let $A=\sup_i\|x_i\|$. Using Ascoli's Theorem, we select inductively a sequence (y_n) with $y_n=x_{i_n}$ for every $n\in\mathbb{N}$ so that $i_{n-1}<i_n$ and

$$\lim_j\|x+y_n+x_j\|\overset{\varepsilon_n}{\sim}\lim_i\|x+x_i\|$$

for every $x\in[y_1,y_2,\ldots,y_{n-1}]$ with $\|x\|\le nA$.

Hence for every increasing sequence (n_m) of natural numbers and $k\in\mathbb{N}$ we have

$$\Big\|\sum_{m=1}^{k}y_{n_m}\Big\|\overset{\varepsilon_{n_k}}{\sim}\lim_i\Big\|\sum_{m=1}^{k-1}y_{n_m}+x_i\Big\|\overset{\varepsilon_{n_k}+\varepsilon_{n_{k-1}}}{\sim}\lim_i\Big\|\sum_{m=1}^{k-2}y_{n_m}+x_i\Big\|$$

and finally

$$\Big\|\sum_{m=1}^{k}y_{n_m}\Big\|\overset{\varepsilon}{\sim}\lim_i\|x_i\|.$$

From [2], (y_n) has a basic subsequence equivalent to the usual basis of c_0, as required.

The converse follows from [6] (Theorem 2, (i)⇒(ii)).

Proposition 8. *Let X be a separable Banach space. The following conditions are equivalent:*

(i) *X contains a subspace isomorphic to c_0.*

(ii) *There exists a Baire-1 element $x^{(2)}$ of X^{**} such that*

$$\Big\|x+\sum_{i=1}^{k}a_i(x^{(4i-2)}-x^{(4i)})\Big\| = \|x+\max\{|a_i|:1\le i\le k\}(x^{(2)}-x^{(4)})\|$$

for every $x\in X$, $k\in\mathbb{N}$ and scalars $a_1,\ldots,a_k$.

(iii) *There exists $x^{(2)}\in X^{**}\setminus X$ such that*

$$\|x+x^{(2)}-x^{(4)}+x^{(6)}-x^{(8)}\| = \|x+x^{(2)}-x^{(4)}\|$$

for every $x\in X$.

(iv) *There exists $x^{(2)}\in X^{**}\setminus X$ and $k\in\mathbb{N}$ such that*

$$\Big\|x+\sum_{i=1}^{k+1}(x^{(4i-2)}-x^{(4i)})\Big\| = \Big\|x+\sum_{i=1}^{k}(x^{(4i-2)}-x^{(4i)})\Big\|$$

for every $x\in X$.

Proof. (i)⇒(ii). If c_0 embeds in X, by [9] there exists a sequence (x_n) in X which c_0-strongly generates a nontrivial type τ ($\tau(x)=\lim_n\|x+x_n\|$ for every $x\in X$). This means that for every $\varepsilon>0$ and $x\in X$ there exists $m\in\mathbb{N}$ such that

$$(*)\qquad \Big|\Big\|x+\sum_{i=m}^{k}a_ix_i\Big\|-\tau(x)\Big|<\varepsilon$$

for every $k\in\mathbb{N}$ with $m\le k$ and scalars $a_m,\ldots,a_k$ with $\max\{|a_m|,\ldots,|a_k|\}=1$. Thus, for $\varepsilon=\tau(0)/2$ and $x=0$, there exists $m_0\in\mathbb{N}$ such that

$$\frac{\tau(0)}{2}\le\Big\|\sum_{i=m_0}^{k}a_ix_i\Big\|\le\frac{3\tau(0)}{2}$$

for every $k\in\mathbb{N}$ with $m_0\le k$ and $a_{m_0},\ldots,a_k\in\{0,1\}$.

Take $y_{nm}=x_{m_0+n}+\ldots+x_m$ for every $n,m\in\mathbb{N}$ with $m_0+n\le m$. The sequences $(y_{nm})_{m=1}^{\infty}$ are bounded and weakly Cauchy for every $n\in\mathbb{N}$. Indeed, if for some $k\in\mathbb{N}$ the sequence (y_{km}) is not weakly Cauchy, then there exist $f\in X^*$, $\varepsilon>0$ and a sequence (z_n) with $z_n=x_{k(n)}+\ldots+x_{m(n)}$, $n\le k(n)\le m(n)$ for all $n\in\mathbb{N}$, such that $|f(z_n)|>\varepsilon$ for every $n\in\mathbb{N}$. But the sequence (z_n) c_0-strongly generates τ, hence must be weakly null according to Proposition 1.7 of [4], a contradiction.

We set $y_n^{**} = w^*\text{-}\lim_m y_{nm}$ for every $n \in \mathbb{N}$. According to [4], the sequence (y_n^{**}) strongly dually generates τ and hence $\tau(x) = \|x+g\|$ for every $x \in X$, where $g = w^*\text{-}\lim_n y_n^{**} = x^{(2)} - x^{(4)}$ with $x^{(2)} = y_1^{**}$, because $y_n^{**} = y_1^{**} - y_{1n}$ for every $n \in \mathbb{N}$.

From Corollary 4 there exists a convex block subsequence (z_n) of (y_{1n}) such that $x^{(2)} = y_1^{**} = w^*\text{-}\lim_n z_n$ and

$$\Big\|x + \sum_{i=1}^{k} a_i x^{(2i)}\Big\| = \lim_{n_k} \ldots \lim_{n_1} \|x + a_1 z_{n_1} + \ldots + a_k z_{n_k}\|$$

for every $x \in X$, $k \in \mathbb{N}$ and scalars $a_1, \ldots, a_k$.

Let $x \in X$, $k \in \mathbb{N}$ and let $a_1, \ldots, a_k$ be scalars with $\max\{|a_1|, \ldots, |a_k|\} = 1$. By the Ramsey principle ([7], [8]) there exists a subsequence (w_n) of (x_n) such that

$$\Big\|x + \sum_{i=1}^{k} a_i (x^{(4i-2)} - x^{(4i)})\Big\| = \lim_{n_1 > \ldots > n_{2k} \to \infty} \|x + a_1 w_{n_1} - a_1 w_{n_2} + \ldots + a_k w_{n_{2k-1}} - a_k w_{n_{2k}}\|.$$

Since

$$a_1 w_{n_1} - a_1 w_{n_2} + \ldots + a_k w_{n_{2k-1}} - a_k w_{n_{2k}} \in c_0\text{-}\operatorname{conv}\{x_n : n \in \mathbb{N}\}$$

for every $n_1 > n_2 > \ldots > n_{2k} \in \mathbb{N}$ (where $x \in c_0\text{-}\operatorname{conv}\{x_n : n \in \mathbb{N}\}$ if and only if $x = \sum_{i=1}^{k} c_i x_{n_i}$ with $\max\{|c_1|, \ldots, |c_k|\} = 1$), from $(*)$ we have

$$\Big\|x + \sum_{i=1}^{k} a_i (x^{(4i-2)} - x^{(4i)})\Big\| = \tau(x) = \|x + x^{(2)} - x^{(4)}\|.$$

Hence for every $x \in X$, $k \in \mathbb{N}$ and scalars $a_1, \ldots, a_k$ we have

$$\Big\|x + \sum_{i=1}^{k} a_i (x^{(4i-2)} - x^{(4i)})\Big\| = \|x + \max\{|a_1|, \ldots, |a_k|\}(x^{(2)} - x^{(4)})\|.$$

(ii)⇒(iii). This is obvious.

(iii)⇒(iv). Take $k = 1$.

(iv)⇒(i). We set $u = \sum_{i=1}^{2k-2} (-1)^{i+1} x^{(2i)}$ if $k > 1$ and $u = 0$ if $k = 1$. Then we have

$$\|x + u + x^{(4k-2)} - x^{(4k)}\| = \|x + u + (x^{(4k-2)} - x^{(4k)}) + (x^{(4k+2)} - x^{(4k+4)})\|$$

for every $x \in X$. From Theorem 3 we can find a sequence $(x_n) \subseteq X$ such that

$$\lim_i \lim_j \|x + u + x_j - x_i\| = \lim_i \lim_j \lim_n \lim_m \|x + u + x_m - x_n + x_j - x_i\|$$

for every $x \in X$.

By the Ramsey principle ([7]) there exists a subsequence (w_n) of (x_n) such that

$$\lim_n \|x + u + z_n\| = \lim_n \lim_m \|x + u + z_m + z_n\|$$

for every $x \in X$, where $z_n = w_{2n} - w_{2n-1}$ for all $n \geq 1$.

Using methods analogous to Proposition 7 we can find a subsequence (y_n) of (z_n) such that

$$\|u + y_{n_1} + \ldots + y_{n_m}\| \overset{1}{\sim} \|u + x^{(4k-2)} - x^{(4k)}\|$$

for every $m \in \mathbb{N}$, $n_1 < \ldots < n_m \in \mathbb{N}$. Since $\lim_n \|y_n\| = \lim_n \|z_n\| = \|x^{(2)} - x^{(4)}\| > 0$ we conclude from [2] that (y_n) has a subsequence equivalent to the usual basis of c_0, as required.

Remark 9. (i) Let $x^{(2)} \in X^{**}$. If for some $k \in \mathbb{N}$, we have

$$(*) \qquad \Big\|x + \sum_{i=1}^{2k+2} (-1)^{i+1} x^{(2i)}\Big\| = \Big\|x + \sum_{i=1}^{2k} (-1)^{i+1} x^{(2i)}\Big\|,$$

then $(*)$ holds for every $n \in \mathbb{N}$ with $k \leq n$. This is easy to see, because from Theorem 3, there exists a sequence $(x_n) \subseteq X$ such that

$$\Big\|x + \sum_{i=1}^{2k+4} (-1)^{i+1} x^{(2i)}\Big\| = \lim_i \lim_j \Big\|x + \sum_{i=1}^{2k+2} (-1)^{i+1} x^{(2i)} + x_i - x_j\Big\|$$
$$= \lim_i \lim_j \Big\|x + \sum_{i=1}^{2k} (-1)^{i+1} x^{(2i)} + x_j - x_i\Big\| = \Big\|x + \sum_{i=1}^{2k+2} (-1)^{i+1} x^{(2i)}\Big\|$$

for every $x \in X$.

(ii) If $x^{(2)} \in X^{**} \setminus X$ where X is a separable Banach space then from Theorem 3 and the Ramsey principle we can find a sequence (z_n) such that

$$\|x + a_1(x^{(2)} - x^{(4)}) + \ldots + a_k(x^{(2k)} - x^{(2k+2)})\| = \lim_{n_k} \ldots \lim_{n_1} \|x + a_1 z_{n_1} + \ldots + a_k z_{n_k}\|$$

for every $x \in X$, $k \in \mathbb{N}$ and scalars $a_1, \ldots, a_k$.

Hence, the sequence $(x^{(4n-2)} - x^{(4n)})$ is the fundamental sequence of an inverse spreading model for X.

The inversion of this spreading model is always 1-unconditional over X ([1]).

In the next proposition we give a characteristic property of spreading models for X which is equivalent to the embeddability of c_0 in the separable Banach space X.

PROPOSITION 10. *Let X be a separable Banach space. Then c_0 embeds in X if and only if there exists a spreading model $Y = \overline{X \oplus [e_i : i \in \mathbb{N}]}$ for*

X such that for some $k \in \mathbb{N}$ the equality

$$\Big\|x + \sum_{i=1}^{k+1} e_i\Big\| = \Big\|x + \sum_{i=1}^{k} e_i\Big\|$$

holds for every $x \in X$.

Proof. If c_0 embeds in X, then from Proposition 8 there exist $x^{(2)} \in X^{**} \setminus X$ and $k \in \mathbb{N}$ such that

$$\Big\|x + \sum_{i=1}^{k+1}(x^{(4i-2)} - x^{(4i)})\Big\| = \Big\|x + \sum_{i=1}^{k}(x^{(4i-2)} - x^{(4i)})\Big\|$$

for every $x \in X$. According to Remark 9(ii) there exists a spreading model $Y = \overline{X \oplus [e_i : i \in \mathbb{N}]}$ for X such that

$$\Big\|x + \sum_{i=1}^{k} e_i\Big\| = \Big\|x + \sum_{i=1}^{k}(x^{(4i-2)} - x^{(4i)})\Big\|$$

for every $x \in X$ and $k \in \mathbb{N}$.

Conversely, if (x_n) generates a spreading model $Y = \overline{X \oplus [e_i : i \in \mathbb{N}]}$ for X such that, for some $k \in \mathbb{N}$, $\|x + \sum_{i=1}^{k+1} e_i\| = \|x + \sum_{i=1}^{k} e_i\|$ holds for every $x \in X$, then

$$\lim_m \lim_n \Big\|x + \sum_{i=1}^{k-1} e_i + x_m + x_n\Big\| = \lim_n \Big\|x + \sum_{i=1}^{k-1} e_i + x_n\Big\|$$

for every $x \in X$. Using methods similar to the proof of Proposition 7 we can find a subsequence (y_n) of (x_n) such that

$$\Big\|\sum_{i=1}^{k-1} e_i + y_{n_1} + \ldots + y_{n_k}\Big\| \overset{1}{\sim} \Big\|\sum_{i=1}^{k} e_i\Big\|$$

for every $k \in \mathbb{N}$ and $n_1 < \ldots < n_k \in \mathbb{N}$. Hence (y_n) has a subsequence equivalent to the usual basis of c_0 ([2]).

The Baire-1 functions $B_1(K)$ on a compact metric space K were classified by Haydon, Odell and Rosenthal ([5]) by defining the subclasses $B_{1/2}(K)$ and $B_{1/4}(K)$. Let us recall the definitions.

The class $B_1(K)$ of Baire-1 functions contains the pointwise limits of uniformly bounded sequences of continuous functions on K. By $\mathrm{DBSC}(K)$ we denote the class of differences of bounded semicontinuous functions on K and it is easy to see that

$$\mathrm{DBSC}(K) = \Big\{F \in B_1(K) : \text{there exists } (f_n) \subseteq C(K) \text{ converging pointwise to } F \text{ with } f_0 = 0 \text{ and } \sum_{n=0}^{\infty} |f_{n+1}(k) - f_n(k)| < \infty\Big\}.$$

The vector space $\mathrm{DBSC}(K)$ is a Banach space with the norm

$$|F|_{\mathrm{D}} = \inf\Big\{C > 0 : \text{there exists } (f_n) \subseteq C(K) \text{ converging pointwise to } F \text{ with } f_0 = 0 \text{ and } \sum_{n=0}^{\infty} |f_{n+1}(k) - f_n(k)| \le C \text{ for all } k \in K\Big\}.$$

It is easy to see that $\|F\|_\infty \le |F|_{\mathrm{D}}$ for every $F \in \mathrm{DBSC}(K)$, but the two norms are not equivalent in general. Hence we have the definitions:

$$B_{1/2}(K) = \{F \in B_1(K) : \text{there exists a sequence } (F_n) \subseteq \mathrm{DBSC}(K) \text{ converging uniformly to } F\},$$

$$B_{1/4}(K) = \{F \in B_1(K) : \text{there exists a sequence } (F_n) \subseteq \mathrm{DBSC}(K) \text{ converging uniformly to } F \text{ and } \sup_n |F_n|_{\mathrm{D}} < \infty\}.$$

Let X be a separable Banach space and K the unit ball of the dual space X^* with the w^*-topology. We define $B_{1/2}(X) = B_1(X) \cap B_{1/2}(K)$ and $B_{1/4}(X) = B_1(X) \cap B_{1/4}(K)$. In [5], some examples are presented from which it follows that in general

$$X \subsetneq B_{1/4}(X) \subsetneq B_{1/2}(X) \subsetneq B_1(X).$$

In the next theorem we characterize the elements in $B_1(X) \setminus B_{1/2}(X)$ via their canonical reproductions. The proof of this theorem is a consequence of the characterization of the functions in $B_1(K) \setminus B_{1/2}(K)$ given by R. Haydon, E. Odell and H. Rosenthal in [5], and of Theorem 3. According to [5], F belongs to $B_1(K) \setminus B_{1/2}(K)$ if and only if there exists a uniformly bounded sequence (f_n) of continuous functions converging pointwise to F such that every convex block subsequence has a subsequence generating a spreading model with the fundamental sequence equivalent to the usual basis of ℓ^1.

THEOREM 11. *Let X be a separable Banach space and $x^{(2)}$ a Baire-1 element of $X^{**} \setminus X$. Then $x^{(2)} \in B_1(X) \setminus B_{1/2}(X)$ if and only if the sequence $(x^{(2n)})_{n=1}^{\infty}$ of the canonical reproductions of $x^{(2)}$ is equivalent to the usual basis of ℓ^1.*

Proof. Let $x^{(2)} \in B_1(X) \setminus X$. From Corollary 4 there exists a bounded sequence (x_n) in X converging to $x^{(2)}$ in the w^*-topology and such that its convex block subsequences (y_n) satisfy

$$\|x + a_1 x^{(2)} + \ldots + a_k x^{(2k)}\| = \lim_{n_k} \ldots \lim_{n_1} \|x + a_1 y_{n_1} + \ldots + a_k y_{n_k}\|$$

for every $x \in X$, $k \in \mathbb{N}$ and scalars $a_1, \ldots, a_k$.

If $x^{(2)} \notin B_{1/2}(X)$ then from [5] there exists a subsequence (y_n) of (x_n) which generates a spreading model with the fundamental sequence (e_n) equivalent to the usual basis of ℓ^1. Since

$$\|a_1 x^{(2)} + \ldots + a_k x^{(2k)}\| = \|a_k e_1 + \ldots + a_1 e_k\|$$

for every $k \in \mathbb{N}$ and scalars $a_1, \ldots, a_k$, we see that the sequence $(x^{(2n)})$ is equivalent to the usual basis of ℓ^1.

On the other hand, let $(x^{(2n)})$ be equivalent to the usual basis of ℓ^1. If (e_n) is the fundamental sequence of the spreading model which is generated by (x_n), then (e_n) is equivalent to the usual basis of ℓ^1. Of course every convex block subsequence of (x_n) generates the same spreading model for X. Hence from [5] we conclude that $x^{(2)} \notin B_{1/2}(X)$.

In the following theorem we characterize the elements in $B_{1/4}(X)$ via their canonical reproductions. In the proof we will use a characterization of the functions in $B_{1/4}(K)$ given in [5] and also the fact that every sequence of continuous functions converging pointwise to such a function has a convex block subsequence generating a spreading model with the fundamental sequence equivalent to the summing basis of c_0.

THEOREM 12. *Let X be a separable Banach space and $x^{(2)}$ a Baire-1 element of $X^{**} \setminus X$. Then $x^{(2)} \in B_{1/4}(X)$ if and only if the sequence $(x^{(4n-2)} - x^{(4n)})_{n=1}^{\infty}$ is equivalent to the usual basis of c_0.*

Proof. Let $x^{(2)}$ be a Baire-1 element of $X^{**} \setminus X$. From Corollary 4 there exists a bounded sequence (x_n) in X converging to $x^{(2)}$ in the w^*-topology and such that its convex block subsequences (y_n) satisfy

$$\|x + a_1 x^{(2)} + \ldots + a_k x^{(2k)}\| = \lim_{n_k} \ldots \lim_{n_1} \|x + a_1 y_{n_1} + \ldots + a_k y_{n_k}\|$$

for every $x \in X$, $k \in \mathbb{N}$ and scalars $a_1, \ldots, a_k$.

If $x^{(2)} \in B_{1/4}(K)$, then from [5] there exists a convex block subsequence (y_n) of (x_n) generating a spreading model with the fundamental sequence (e_n) equivalent to the summing basis of c_0. Since

$$(*) \qquad \|a_k e_1 + \ldots + a_1 e_k\| = \|a_1 x^{(2)} + \ldots + a_k x^{(2k)}\|$$

we have

$$\|a_1(x^{(2)} - x^{(4)}) + \ldots + a_k(x^{(4k-2)} - x^{(4k)})\| = \|a_k(e_2 - e_1) + \ldots + a_1(e_{2k} - e_{2k-1})\|$$

for every $k \in \mathbb{N}$ and scalars $a_1, \ldots, a_k$.

Hence the sequence $(x^{(4n-2)} - x^{(4n)})$ is a 1-unconditional basic sequence equivalent to the usual basis of c_0.

On the other hand, let the sequence $(x^{(4n-2)} - x^{(4n)})$ be equivalent to the usual basis of c_0. If (e_n) is the fundamental sequence of the spreading model which is given by the sequence (x_n) then by Ramsey's principle ([7]) there exists a subsequence (y_n) of (x_n) such that

$$(**) \qquad \Big\|\sum_{i=1}^{k} a_i e_i\Big\| = \lim_{n_k > \ldots > n_1 \to \infty} \Big\|\sum_{i=1}^{k} a_i y_{n_i}\Big\|.$$

According to [5], $x^{(2)}$ is in $B_{1/4}(X)$ if there exist $0 < M < \infty$ and $(y_n) \subseteq X$ with $y_0 = 0$, converging to $x^{(2)}$ in the w^*-topology, with the property that for all $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that if (y_{n_i}) is a subsequence of $(y_n)_{n=m}^{\infty}$ then

$$\sum_{i \in B((n_i), x^*)} |x^*(y_{n_{i+1}} - y_{n_i})| \le M$$

for every $x^* \in X^*$ with $\|x^*\| \le 1$, where

$$B((n_i), x^*) = \{i \in \mathbb{N} : |x^*(y_{n_{i+1}} - y_{n_i})| \ge \varepsilon\}.$$

The basic sequence $(e_{2n} - e_{2n-1})$ is equivalent to the usual basis of c_0 by $(*)$. Hence according to $(**)$ there exists $0 < C < \infty$ such that for every $k \in \mathbb{N}$ there exists $m(k) \in \mathbb{N}$ with $k \le m(k)$ such that if $m(k) \le n_1 < \ldots < n_{2k}$ then

$$(***) \qquad \Big\|\sum_{i=1}^{k} a_i (y_{n_{2i}} - y_{n_{2i-1}})\Big\| \le C \max\{|a_1|, \ldots, |a_k|\}.$$

This immediately yields

$$(****) \qquad \sum_{i=1}^{k} |x^*(y_{n_{2i}} - y_{n_{2i-1}})| \le C$$

for every $x^* \in X^*$ with $\|x^*\| \le 1$ and $m(k) \le n_1 < \ldots < n_{2k} \in \mathbb{N}$.

Let $\varepsilon > 0$ and $k > 2C/\varepsilon$. If (y_{n_i}) is a subsequence of $(y_n)_{n=m(k)}^{\infty}$ and $x^* \in X^*$ with $\|x^*\| \le 1$, then the set $B((n_i), x^*)$ contains fewer than k elements. Indeed, if $i_1 < \ldots < i_k$ are in $B((n_i), x^*)$ then

$$\sum_{j=1}^{k} |x^*(y_{n_{i_j+1}} - y_{n_{i_j}})| \ge k\varepsilon.$$

On the other hand, from $(****)$ we have

$$\sum_{j=1}^{k} |x^*(y_{n_{i_j+1}} - y_{n_{i_j}})| = \sum_{\substack{1 \le j \le k \\ j \text{ odd}}} |x^*(y_{n_{i_j+1}} - y_{n_{i_j}})| + \sum_{\substack{1 \le j \le k \\ j \text{ even}}} |x^*(y_{n_{i_j+1}} - y_{n_{i_j}})| \le 2C.$$

Since $k > 2C/\varepsilon$ we have a contradiction.

Thus $B((n_i), x^*)$ contains fewer than k elements and then

$$\sum_{i \in B((n_i), x^*)} |x^*(y_{n_{i+1}} - y_{n_i})| \le 2C.$$

Hence, since (y_n) converges to $x^{(2)}$ in the w^*-topology, we conclude from [5] that $x^{(2)} \in B_{1/4}(X)$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ATHENS
PANEPISTEMIOPOLIS-ILISIA
GR-15784 ATHENS, GREECE
E-mail: SMA20@GRATHUN1.BITNET

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Isometries of Musielak–Orlicz spaces II

by

J. E. JAMISON, A. KAMIŃSKA and
PEI-KEE LIN (Memphis, Tenn.)

Abstract. A characterization of isometries of complex Musielak–Orlicz spaces L_Φ is given. If L_Φ is not a Hilbert space and $U : L_\Phi \to L_\Phi$ is a surjective isometry, then there exist a regular set isomorphism τ from (T, Σ, μ) onto itself and a measurable function w such that $U(f) = w \cdot (f \circ \tau)$ for all $f \in L_\Phi$. Isometries of real Nakano spaces, a particular case of Musielak–Orlicz spaces, are also studied.

1. Introduction. For any σ-finite atomless measure space (T, Σ, μ), a nonnegative function $\Phi : \mathbb{R}_+ \times T \to \mathbb{R}_+ \cup \{\infty\}$ is said to be a *Young function* if

(1) $\Phi(0, t) = 0$ for all $t \in T$;

(2) for any $t \in T$, $\Phi(\cdot, t)$ is a left continuous nondecreasing convex function;

(3) for any $u \in \mathbb{R}_+$, $\Phi(u, \cdot)$ is a Σ-measurable function;

(4) $\mu(\{t : \Phi(u, t) = 0 \text{ for all } u > 0\}) = 0 = \mu(\{t : \Phi(u, t) = \infty \text{ for all } u > 0\})$.

For any Young function Φ, the *Musielak–Orlicz space* L_Φ associated with Φ is the set of all (complex- or real-valued) measurable functions such that

$$I_\Phi(\lambda f) = \int_T \Phi(|\lambda f(t)|, t)\, d\mu(t) < \infty$$

for some $\lambda > 0$. The space L_Φ is equipped with the Luxemburg norm, that is, the norm of $f \in L_\Phi$ is given by $\|f\|_\Phi = \inf\{\varepsilon > 0 : I_\Phi(\frac{f}{\varepsilon}) \le 1\}$ [10, 13].

If Φ does not depend on t, i.e. $\Phi(u, t) = \varphi(u)$, then we shall call L_Φ the *Orlicz space* L_φ [11]. In [5], Fleming and the first two authors studied the isometries of complex Musielak–Orlicz spaces. They proved that if Φ satisfies the following condition:

$$\text{for almost all } t \in T, \text{ the function } u \to \frac{\Phi'(u, t)}{u} \text{ is monotone},$$

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