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The structure of Eberlein, uniformly Eberlein and Talagrand compact spaces in $\Sigma(R^\Gamma)$

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Abstract. Well known compact spaces such as Eberlein, uniformly Eberlein and Talagrand compact spaces can be considered as subsets of $\Sigma(R^\Gamma)$. In this paper we give characterizations of these classes of sets distinguishing them by their structure inside $\Sigma(R^\Gamma)$. A consequence of these results is an easy proof of Talagrand's example of a Talagrand compact space which is not Eberlein compact.

1. Introduction and preliminaries. Since the early sixties, various classes of compact spaces generalizing compact metric spaces have been studied by many authors. Thus, classes of Eberlein, uniformly Eberlein and Talagrand compact spaces were defined and distinguished by means of examples. However, there did not exist a characterization distinguishing them by their *structure*. Such characterizations are contained in the present paper.

A compact Hausdorff space is called *Eberlein compact* (E. C.) iff it is homeomorphic to a weakly compact subset of a Banach space. Amir and Lindenstrauss ([1]) proved that every E. C. space is homeomorphic to a weakly compact subset of $c_0(\Gamma)$ for some set Γ, where

$$c_0(\Gamma) = \{f \in R^\Gamma\colon \text{for each } \varepsilon > 0 \text{ the set } \{\gamma \in \Gamma\colon |f(\gamma)| > \varepsilon\} \text{ is finite}\}$$

(The weak topology on a weakly compact subset of $c_0(\Gamma)$ is exactly the topology of point-wise convergence.)

An E. C. space will be called *uniformly Eberlein compact* (U. E. C.) iff it is homeomorphic to a weakly compact subset of a Hilbert space. U. E. C. spaces have been introduced by Benyamini–Starbird in [4], where they give an example of an E. C. space which is not U. E. C.

A compact space K is called *Talagrand compact* (T. C.) iff the Banach space $C(K)$ in its weak topology is $\mathscr{K}$-analytic.

S. Mercourakis proved in [7] that a compact space K is T. C. iff it is homeomorphic to a compact subset of a space

$$c(\Gamma_\sigma\colon \sigma \in \Sigma) = \{f \in R^\Gamma: f \text{ bounded and } f|_{\Gamma_\sigma} \in c_0(\Gamma_\sigma) \text{ for all } \sigma \in \Sigma\}$$

where $\Sigma = N^N$ and $\{\Gamma_\sigma: \sigma \in \Sigma\}$ is a decomposition of a set Γ with the property $\Gamma_{\sigma_1} \subseteq \Gamma_{\sigma_2}$ if $\sigma_1 < \sigma_2$. The space $c(\Gamma_\sigma: \sigma \in \Sigma)$ has the topology of pointwise convergence.

The classes of E. C., U. E. C., T. C. spaces have interesting stability properties such as being invariant under continuous images ([3], [11]). M. Talagrand proved ([10]) that E. C. spaces are T. C. spaces and Vašak ([12]) (implicitly) and Gulko ([5]) that every T. C. space is homeomorphic to a subset of $\Sigma(R^\Gamma)$ for some set Γ where

$$\Sigma(R^\Gamma) = \{f: \Gamma \to R: \text{the set } \{\gamma \in \Gamma: f(\gamma) \neq 0\} \text{ is at most countable}\}$$

Thus all these compact spaces may be considered as subsets of the space $\Sigma(R^\Gamma)$.

The present paper continues the study of E. C., U. E. C. and T. C. spaces separating these classes by means of their structure inside $\Sigma(R^\Gamma)$. In a previous paper by S. Argyros and the present author ([2]) we gave a characterization of U. E. C. sets in the Banach space $c_0(\Gamma)$. Extending the techniques of [2] to the present situation, we give a similar characterization of the structure of U. E. C. sets in the locally convex space $\Sigma(R^\Gamma)$ (Theorem 2.10). We also characterize the structure of E. C. and T. C. subsets of $\Sigma(R^\Gamma)$ (Theorems 2.9 and 2.11).

These characterizations separate the various classes of compact subsets of $\Sigma(R^\Gamma)$. Therefore imply a very simple proof of the properties of an example of Talagrand [11] of a T. C. space which is not E. C.

If A is a set, card$A$ or $|A|$ will denote its cardinality. ω denotes the first infinite cardinal. I_A denotes the characteristic function of A, $e_\gamma = I_{\{\gamma\}}$, and $x|_A = x \cdot I_A$, for $x \in R^\Gamma$ and $A \subseteq \Gamma$.

Let $\mathscr{K}$ be a family of subsets of a set K.

(1) The family $\mathscr{K}$ *separates the points* of K iff for any $x, y \in K$, $x \neq y$, there exists $A \in \mathscr{K}$ such that $I_A(x) \neq I_A(y)$.

(2) The family $\mathscr{K}$ is *point finite* (*point countable*) iff every $x \in K$ belongs to finitely many (countable many) elements of $\mathscr{K}$.

(3) The family $\mathscr{K}$ is k-finite ($k \in N$) iff any $x \in K$ belongs to at most k elements of $\mathscr{K}$.

PROPOSITION 1.1 ([9], [3]). *A compact space K is* E. C. (U. E. C.) *iff there exists a family $\mathscr{K}$ of open F_σ subsets of K with the properties*:

(i) *$\mathscr{K}$ separates the points of K.*

(ii) *There exists a decomposition $\{\mathscr{K}_m: m \in N\}$ of $\mathscr{K}$ such that, for every $m \in N$, $\mathscr{K}_m$ is point finite ($\mathscr{K}_m$ is $k(m)$-finite for some $k(m) \in N$).*

PROPOSITION 1.2 [7]. *A compact space K is* T. C. *iff there exists a point-countable family $\mathscr{K}$ of open F_σ subsets of K with the properties*:

(i) *$\mathscr{K}$ separates the points of K.*

(ii) *There exists a decomposition $\{\mathscr{K}_\sigma: \sigma \in \Sigma\}$ ($\Sigma = N^N$) of $\mathscr{K}$ such that $\mathscr{K}_{\sigma_1} \subseteq \mathscr{K}_{\sigma_2}$ if $\sigma_1 < \sigma_2$ and the families $\mathscr{K}_\sigma$ are point-finite for every $\sigma \in \Sigma$.*

2. The characterizations of E. C., U. E. C. and T. C. sets.

2.1. DEFINITION. For each subset K of $\Sigma(R^\Gamma)$, we denote by $\mathscr{E}(K)$ the set

$$\mathscr{E}(K) = \{f \in \Sigma(R^\Gamma): \exists g \in K, |g| = |f|\}$$

2.2. PROPOSITION. *Let K be a compact subset of the space $\Sigma(R^\Gamma)$.*

(a) *If K is* E. C. *then $\mathscr{E}(K)$ is* E. C.

(b) *If K is* U. E. C. *then $\mathscr{E}(K)$ is* U. E. C.

(c) *If K is* T. C. *then $\mathscr{E}(K)$ is* T. C.

The proof will be based on the following Lemmas 2.4, 2.5.

2.3. Notation. Consider a compact subset K of $\Sigma(R^\Gamma)$ and an enumeration (a_n, b_n): $n \in N\}$ of the open intervals with rational ends such that $0 \notin [a_n, b_n]$. For any $(\gamma, n) \in \Gamma \times N$ put

$$V_{(\gamma,n)} = \{f \in K: f(\gamma) \in (a_n, b_n)\}.$$

It is easy to see that the family

$$\mathscr{K} = \{V_{(\gamma,n)}: (\gamma, n) \in \Gamma \times N\}$$

atisfies the following properties:

(i) $\mathscr{K}$ consists of open, F_σ subsets of $\Sigma(R^\Gamma)$.

(ii) $\mathscr{K}$ separates the points of K.

(iii) $\mathscr{K}$ is a point-countable family.

2.4. LEMMA. *For every weakly compact subset K of the space $c_0(\Lambda)$ and every ›oint countable family $\{V_t: t \in T\}$ of open F_σ subsets of K, such that $0 \notin V_t$ for all $t \in T$, here exists a family $\{U_\delta: \delta \in \Delta\}$ of open F_σ subsets of K with the following properties*:

(1) *For each $\delta \in \Delta$ there exists $t \in T$ so that $U_\delta \subseteq V_t$.*

(2) *For each $t \in T$ there exists a countable $\Delta_t \subseteq \Delta$ such that $V_t = \bigcup \{U_\delta: \delta \in \Delta_t\}$.*

(3) *There exists a decomposition $\{\Delta_k: k \in N\}$ of Δ so that the family $\{U_\delta: \delta \in \Delta_k\}$'s point finite for every $k \in N$.*

Proof. For every $t \in T$ and $x \in V_t$ we can choose an open basic neighborhood W_x of x contained in V_t. Let

$$W_x = \{y \in K: |y(\lambda_i^x) - x(\lambda_i^x)| < \varepsilon_x \text{ for } 1 \leqslant i \leqslant n_x\}$$

where $n_x \in N$, $\{\lambda_i^x: 1 \leqslant i \leqslant n_x\} \subseteq \Lambda$ and ε_x is a rational number. Furthermore we can choose ε_x so that $|x(\lambda)| > \frac{\varepsilon_x}{2}$ for every $\lambda \in M_x$ where $M_x = \{\lambda_i^x: x(\lambda_i^x) > 0\}$.

Each V_t is weakly Lindelöf, hence there exists a countable subset $\{x_k^t: k \in N\}$ of V_t so that $V_t = \bigcup_{k \in N} W_{x_k^t}$. Define $\Delta = T \times N$ and $U_\delta = W_{x_k^t}$ where $\delta = (k, t)$.

It is easy to check that the family $\{U_\delta: \delta \in \Delta\}$ satisfies properties (1) and (2).

For every finite subset M of Λ and $k \in N$ the set $L = \{t \in T: M_{x_k^t} = M\}$ is at most countable (for the proof see Lemma 1.4 of [2]).

Let $\{W(M,r): r\in N\}$ be an enumeration of L.

For every $(k,m,r)\in N^3$ we define

$$\Delta_{(k,m,r)} = \left\{\delta = (t,k)\in\Delta:\ \varepsilon_{x_k^t} > \frac{1}{m},\ M_{x_k^t} = W(M,r)\right\}.$$

It remains to show that the family $\{U_\delta: \delta\in\Delta(k,m,r)\}$ is point finite. Indeed, if there exists $y\in\bigcap_{i\in I} U_{\delta_i}$ where $\{\delta_i: i\in I\}$ is an infinite subset of $\Delta(k,m,r)$, then $|y(\lambda)| > \frac{1}{2m}$ for every $\lambda\in M$ where $M = \bigcup_{i\in I} M_{\delta_i}$ $(M_\delta = M_{x_k^t}$ for $\delta = (t,k))$. But the set M is infinite, because the sets $\{M_{\delta_i}: i\in I\}$ are pairwise different. This is a contradiction and therefore condition (3) is also true.

2.5. Lemma. *For every compact subset K of the space $c(\Lambda_\sigma: \sigma\in\Sigma)$ and each point countable family $\{V_t: t\in T\}$ of open F_σ subsets of K such that $0\notin V_t$ for all $t\in T$, there exists a point countable family $\{U_\delta: \delta\in\Delta\}$ of open F_σ subsets of K with the properties* (1), (2) *of Lemma* 2.4 *and*

(4) *There exists a decomposition $\{\Delta_\sigma: \sigma\in\Sigma\}$ $(\Sigma = N^N)$ of Δ, so that $\Delta_{\sigma_1}\subseteq\Delta_{\sigma_2}$ if $\sigma_1<\sigma_2$ and the family $\{U_\delta: \delta\in\Delta_\sigma\}$ is point finite for every $\sigma\in\Sigma$.*

Proof. As in the proof of the previous lemma, let $\Delta = T\times N$ and $U_\delta = W_{x_k^t}$ for every $\delta = (t,k)\in\Delta$. Obviously the family $\{U_\delta: \delta\in\Delta\}$ is point countable and has the properties (1), (2).

For every $\sigma\in\Sigma$ and $(k,m,r)\in N^3$ define the sets

$$\Delta^\sigma_{(k,m,r)} = \left\{\delta = (t,k)\in\Delta:\ \varepsilon_{x_k^t} > \frac{1}{m},\ M_{x_k^t} = W(M,r),\ \text{and}\ M_{x_k^t}\subseteq\Lambda_\sigma\right\}.$$

The family $\{U_\delta: \delta\in\Delta^\sigma_{(k,m,r)}\}$ is point finite. This is true because if there exists $y\in\bigcap_{i\in I} U_{\delta_i}$ where $\{\delta_i: i\in I\}$ is an infinite subset of $\Delta^\sigma_{(k,m,r)}$, then $|y(\lambda)| > \frac{1}{m}$ for every $\lambda\in\bigcup_{i\in I} M_{\delta_i}$. This is a contradiction because $\bigcup_{i\in I} M_{\delta_i}$ is an infinite subset of Λ_σ and $y|_{\Lambda_\sigma}\in c_0(\Lambda_\sigma)$.

Furthermore, it is clear that if $\sigma_1<\sigma_2$ then $\Delta^{\sigma_1}_{(k,m,r)}\subseteq\Delta^{\sigma_2}_{(k,m,r)}$.

For every $\sigma = (\sigma(1),\sigma(2),\ldots)\in\Sigma$ denote by Δ_σ the set $\bigcup_{\substack{l\leqslant k\leqslant\sigma(1)\\ l\leqslant m\leqslant\sigma(2)\\ l\leqslant r\leqslant\sigma(3)}} \Delta^{(\sigma(4),\sigma(5),\ldots)}_{(k,m,r)}$.

Obviously, this decomposition $\{\Delta_\sigma: \sigma\in\Sigma\}$ of Δ satisfies property (4), and the proof of the lemma is complete.

Proof of Proposition 2.2. We can assume that the set K satisfies the property $x(\gamma)\geqslant 0$ for any $x\in K$, $\gamma\in\Gamma$. (Otherwise we consider the continuous mapping $G: K\to K_1$ defined by $G(x) = |x|$. By [3], [11], the set $K_1 = G(K)$ is E. C., U. E. C., T. C., if K is E. C., U. E. C., T. C. respectively.) In the sequel

$$\mathscr{K} = \{V_{(\gamma,n)}: (\gamma,n)\in\Gamma\times N\}$$

is the family described in 2.3.

(a) If K is E. C. there exists a homeomorphism $f: K\to c_0(\Lambda)$ for some Λ, with the property $f(0) = 0$. The family $\{f(V_{(\gamma,n)}): (\gamma,n)\in\Gamma\times N\}$ consists of open F_σ subsets of the set $f(K)$, separates its points and is point countable. From Lemma 2.4 there exists another family $\{A_\delta: \delta\in\Delta\}$ of open F_σ subsets of $f(K)$, which satisfies properties (1), (2), (3).

We set $U_\delta = f^{-1}(A_\delta)$ for every $\delta\in\Delta$. It is easy to check that the families $\{V_{(\gamma,n)}: (\gamma,n)\in\Gamma\times N\}$ and $\{U_\delta: \delta\in\Delta\}$ also satisfy the properties (1), (2), (3) of Lemma 2.4.

Let $\delta\in\Delta$ and $x\in U_\delta$. We can choose an open neighborhood V_x of x which is contained in U_δ and has the form

$$V_x = \{y\in K: y(\gamma_i^x)\in(c_i^x, d_i^x),\ 1\leqslant i\leqslant n_x\}$$

where, for every $1\leqslant i\leqslant n_x$, c_i^x, d_i^x are rational numbers, positive if $x(\gamma_i^x)>0$ and $c_i^x = -d_i^x$ otherwise.

The family $\{V_x: x\in U_\delta\}$ is an open cover of U_δ so there exists a countable subcover $\{V_{x(\delta,\mu)}: \mu\in N\}$.

For every $s = (\varepsilon_1^s,\ldots,\varepsilon_{n_x}^s)\in\{-1,1\}^{n_x}$ let

$$V_x^s = \{y\in\mathscr{E}(K): y(\gamma_i^x)\in(\varepsilon_i^s c_i^x, \varepsilon_i^s d_i^x),\ 1\leqslant i\leqslant n_x\}$$

and

$$E = \{V^s_{x(\delta,\mu)}: (\delta,\mu)\in\Delta\times N,\ s\in\{-1,1\}^{n_{x(\delta,\mu)}}\}.$$

E consists of open F_σ subsets of $\mathscr{E}(K)$ and separates its points.

Define a decomposition of E as follows:

$$E_{(k,\mu)} = \{V^s_{x(\delta,\mu)}: \delta\in\Delta_k,\ s\in\{-1,1\}^{n_{x(\delta,\mu)}}\}\ \text{for}\ (k,\mu)\in N^2$$

The family $E_{(k,\mu)}$ is point finite, because if $y\in\bigcap_{i\in N} V^{s_i}_{x(\delta_i,\mu)}$, where

$$\{V^{s_i}_{x(\delta_i,\mu)}: i\in N\}\subseteq E(k,\mu),$$

then the set

$$B = \{\delta\in\Delta: \delta = \delta_i \text{ for some } i\in N\}$$

is an infinite subset of Δ_k.

This implies that $|y|\in\bigcap_{\delta\in B} V_{x(\delta,\mu)}$, so $\bigcap_{\delta\in B} U_\delta\neq\emptyset$, which is a contradiction.

The claim (a) now follows from Rosenthal's characterization of E. C. sets (1.1).

(b) This is analogous to the proof of Proposition 1.3 of [2], which dealt with the special case of a U. E. C. subset of $c_0(\Gamma)$.

(c) If K is a T. C. set there exists a homeomorphism $f: K\to c(\Lambda_\sigma: \sigma\in\Sigma)$ for some set $\Lambda = \bigcup_{\sigma\in\Sigma}\Lambda_\sigma$ ([7]), with the property $f(0) = 0$.

The family $\{f(V_{(\gamma,n)}: (\gamma,n)\in\Gamma\times N)\}$ satisfies the assumption of Lemma 2.5, so there exists another family $\{A_\delta: \delta\in\Delta\}$ of open F_σ subsets of $f(K)$ which satisfies its conclusions.

If we set $U_\delta = f^{-1}(A_\delta)$ for every $\delta \in \Delta$, the families $\{V_{(\gamma,n)}: (\gamma, n) \in \Gamma \times N\}$ and $\{U_\delta: \delta \in \Delta\}$ satisfy properties (1), (2) and (4) of Lemma 2.5.

As in (a), we define, for every $\delta \in \Delta$ and $x \in U_\delta$, the open neighborhoods V_x of x and the corresponding open F_σ subsets V_x^s of $\mathscr{E}(K)$ for every $s \in \{-1, 1\}^{n_x}$.

The set U_δ is Lindelöf, therefore there exists a countable subset $\{x_{(\delta,\mu)}: \mu \in N\}$ of U_δ such that $U_\delta = \bigcup_{\mu \in N} V_{x(\delta,\mu)}$.

The family $E = \{V^s_{x(\delta,\mu)}: (\delta, \mu) \in \Delta \times N, s \in \{-1, 1\}^{n_{x(\delta,\mu)}}\}$ separates the points of $\mathscr{E}(K)$ and is point countable. Decompose E into sets $E_{(\mu,\sigma)}$, $(\mu, \sigma) \in N \times \Sigma$, where

$$E_{(\mu,\sigma)} = \{V^s_{x(\delta,\mu)}: \delta \in \Delta_\sigma\}$$

Recall that the family $\{U_\delta: \delta \in \Delta_\sigma\}$ is point finite and if $\sigma_1 < \sigma_2$ then $\Delta_{\sigma_1} \subseteq \Delta_{\sigma_2}$. It is therefore easy to check that the family $E_{(\mu,\sigma)}$ is point finite and if $\sigma_1 < \sigma_2$ then $E_{(\mu,\sigma_1)} \subseteq E_{(\mu,\sigma_2)}$.

For every $\sigma = (\sigma(1), \sigma(2), \ldots) \in \Sigma$ set $E_\sigma = \bigcup_{\mu \leqslant \sigma(1)} E_{(\mu,\sigma')}$ where $\sigma' = (\sigma(2), \sigma(3), \ldots) \in \Sigma$.

The families E_σ are point finite for any $\sigma \in \Sigma$ and $E_{\sigma_1} \subseteq E_{\sigma_2}$ if $\sigma_1 < \sigma_2$.

The fact that $\mathscr{E}(K)$ is T. C. now follows by Proposition 1.2.

PROPOSITION 2.6. *Let K be a compact subset of $\Sigma(\mathbf{R}^\Gamma)$ and $\overline{\text{co}}(K)$ its closed convex hull.*

(a) *If K is* E. C. *then $\overline{\text{co}}(K)$ is* E. C.

(b) *If K is* U. E. C. *then $\overline{\text{co}}(K)$ is* U. E. C.

(c) *If K is* T. C. *then $\overline{\text{co}}(K)$ is* T. C.

The proof is an immediate consequence of results of S. Mercourakis [7].

LEMMA 2.7. *Let K be a subset of $\Sigma(\mathbf{R}^\Gamma)$.*

(a) *If $x \in \overline{\text{co}}(\mathscr{E}(K))$ and A is a subset of Γ then $x|_A \in \overline{\text{co}}(\mathscr{E}(K))$.*

(b) *If $L_1, L_2, \ldots, L_n$ are pairwise disjoint, finite subsets of Γ, $p_1 I_{L_1} + \ldots + p_n I_{L_n} \in \overline{\text{co}}(\mathscr{E}(K))$ and $0 \leqslant a \leqslant \min\{p_1, \ldots, p_n\}$ then $a(I_{L_1} + \ldots + I_{L_n}) \in \overline{\text{co}}(\mathscr{E}(K))$.*

(c) *If $x \in \overline{\text{co}}(\mathscr{E}(K))$ and $x(\gamma) > a$ for every $\gamma \in \operatorname{supp} x$ then $a \cdot I_{\operatorname{supp} x} \in \overline{\text{co}}(\mathscr{E}(K))$.* ($\operatorname{supp} x = \{\gamma \in \Gamma: x(\gamma) \neq 0\}$).

Proof. (a) and (b) are easy (see [2], Lemma 1.6).

(c) The support of x is countable, say $\operatorname{supp} x = \{\gamma_k: k \in N\}$. If $y = aI_{\operatorname{supp} x}$ then y is the pointwise limit of the sequence $(y_n)_{n\in N}$ where $y_n = a \sum_{k=1}^{n} e_{\gamma_k}$. According to (a) and (b) $y_n \in \overline{\text{co}}(\mathscr{E}(K))$, so $y \in \overline{\text{co}}(\mathscr{E}(K))$.

The next combinatorial lemma is a special case of Hajnal's theorem [6].

LEMMA 2.8 ([8]). *Let $m \in N$ and $\{A_i: i \in I\}$ be a countable family* ($\operatorname{card} I = \omega$) *of subsets of I with $\operatorname{card} A_i \leqslant m$ $(m \in N)$ for $i \in I$. Then there exists an infinite subset I_1 of I, such that if $i_1 \neq i_2$ are elements of I_1, then $i_1 \notin A_{i_2}$.*

We are now able to give the characterization of the structure of Eberlein compact subsets of $\Sigma(\mathbf{R}^\Gamma)$.

THEOREM 2.9. *Let K be a compact subset of $\Sigma(\mathbf{R}^\Gamma)$. The following are equivalent:*

1. *The set K is* E. C.

2. *For every $\varepsilon > 0$ there exists a decomposition $\{\Gamma_m^{(\varepsilon)}: m \in N\}$ of Γ, such that for every $x \in K$ and $m \in N$*

$$\operatorname{card}\{\gamma \in \Gamma_m^{(\varepsilon)}: |x(\gamma)| > \varepsilon\} < \omega .$$

Proof. $(2 \Rightarrow 1)$ By compactness of K, for every $\gamma \in \Gamma$ there exists an $M_\gamma \in N$ such that $|x(\gamma)| \leqslant M_\gamma$ for every $x \in K$.

Let

$$V^n_{\gamma,j} = \left\{x \in K: \frac{j-2}{n} < x(\gamma) < \frac{j}{n}\right\} \quad \text{where}$$

$$j \in A_n^\gamma = \{j \in Z: -nM_\gamma + 1 \leqslant j \leqslant -1 \text{ or } 3 \leqslant j \leqslant nM_\gamma + 1\}.$$

The sets $V^n_{\gamma,j}$ are open F_σ subsets of K and $\bigcup_{j \in A_n^\gamma} V^n_{\gamma,j} = \left\{x \in K: |x(\gamma)| > \frac{1}{n}\right\}$.

The family $\mathscr{K} = \{V^n_{\gamma,j}: \gamma \in \Gamma, n \in N, j \in A_n^\gamma\}$ separates the points of K. Let $\{\Gamma_m^{(\frac{1}{n})}: m \in N\}$ be the decomposition of Γ corresponding to $\varepsilon = \frac{1}{n}$. Decompose $\mathscr{K}$ into the families $\{\mathscr{K}_{n,m}\}$ where, for every $(n, m) \in N^2$,

$$\mathscr{K}_{(n,m)} = \{V^n_{\gamma,j}: \gamma \in \Gamma_m^{(\frac{1}{n})} \text{ and } j \in A_n^\gamma\}.$$

We shall prove that every $\mathscr{K}_{(n,m)}$ is point finite. Suppose, to the contrary, that there exists $x \in \bigcap_{i \in I} V^n_{\gamma_i, j_i}$ where $\{V^n_{\gamma_i,j_i}: i \in I\}$ is an infinite subset of $\mathscr{K}_{(n,m)}$. Then $|x(\gamma_i)| > \frac{1}{n}$ for every $i \in I$. This is a contradiction, because $\{\gamma_i: i \in I\}$ is an infinite subset of $\Gamma_m^{(\frac{1}{n})}$.

$(1 \Rightarrow 2)$ Denote by K_1 the set $\overline{\text{co}}(\mathscr{E}(K \cup \{e_\gamma: \gamma \in \Gamma\}))$. Using Propositions 2.2 and 2.6, the set K_1 is E. C., so there exists a homeomorphism $f: K_1 \to c_0(\Lambda)$ satisfying $f(0) = 0$ and $\|f(x)\| \leqslant 1$ for every $x \in K_1$.

For a fixed number ε, $0 < \varepsilon < 1$ and any $\gamma \in \Gamma$, define the open sets

$$V_\gamma = \left\{x \in K_1: |x(\gamma)| > \frac{\varepsilon}{4}\right\}.$$

Note that $0 \notin V_\gamma$ and $d_\gamma = \frac{\varepsilon}{2} e_\gamma$ belongs to V_γ by Lemma 2.7. Since V_γ is open there exists a basic neighborhood W_γ of $f(d_\gamma)$ with $W_\gamma \subseteq f(V_\gamma)$. Let

$$W_\gamma = \{y \in f(K_1): |f(d_\gamma)(\lambda_i^\gamma) - y(\lambda_i^\gamma)| < \varepsilon_\gamma \text{ for every } 1 \leqslant i \leqslant n_\gamma\}$$

where $\{\lambda_1^\gamma, \ldots, \lambda_{n_\gamma}^\gamma\} \subseteq \Lambda$ and $\varepsilon_\gamma > 0$ for every $\gamma \in \Gamma$. Since $0 \notin W_\gamma$ there is $\lambda_i^\gamma = \lambda_\gamma$ such that $|f(d_\gamma)(\lambda_\gamma)| > 0$ and we can assume that $0 < \varepsilon_\gamma < \frac{|f(d_\gamma)(\lambda_\gamma)|}{2}$.

Hence it is easy to see that

$$(*) \qquad |y(\lambda_\gamma)| > \frac{|f(d_\gamma)(\lambda_\gamma)|}{2} > \varepsilon_\gamma \quad \text{for all } y \in W_\gamma .$$

For each $\gamma \in \Gamma$ we can choose a finite subset S_γ of Γ and $0 < \varepsilon_\gamma^1 < \frac{\varepsilon}{4}$ such that the set

$$U_\gamma = \left\{ x \in K_1 : |x(\delta)| < \varepsilon_\gamma^1 \text{ for any } \delta \in S_\gamma - \{\gamma\} \text{ and } \left| x(\gamma) - \frac{\varepsilon}{2} \right| < \varepsilon_\gamma^1 \right\}$$

is an open neighborhood of d_γ, contained in the set $f^{-1}(W_\gamma)$.

Notice that the set

$$A_\gamma = \{\delta \in \Gamma : |f(d_\delta)(\lambda_\gamma)| > 2\varepsilon_\gamma\} \text{ is finite}.$$

For every $(n, m) \in N^2$ put

$$\Gamma_{(n,m)}^{(\varepsilon)} = \left\{ \gamma \in \Gamma : |f(d_\gamma)(\lambda_\gamma)| > \frac{1}{n} \text{ and } \operatorname{card}(S_\gamma \cup A_\gamma) < m \right\}.$$

The family $\{\Gamma_{(n,m)}^{(\varepsilon)} : (n, m) \in N^2\}$ is a decomposition of Γ.

The proof of the theorem will be complete if we can prove the claim:

For any $x \in K_1$ and $(n, m) \in N^2$

$$\operatorname{card}\{\gamma \in \Gamma_{(n,m)}^{(\varepsilon)} : |x(\gamma)| > \varepsilon\} < \omega .$$

Suppose, to the contrary, that there exists an $x \in K_1$ and $(n, m) \in N^2$ such that the set $A = \{\gamma \in \Gamma_{(n,m)}^{(\varepsilon)} : |x(\gamma)| > \varepsilon\}$ is infinite.

According to Lemma 2.8, there exists an infinite subset B of A such that for γ_1, γ_2 elements of B, with $\gamma_1 \neq \gamma_2$, $\gamma_1 \notin A_{\gamma_2} \cup S_{\gamma_2}$.

Let $x_1 = |x|$. Then the functions x_1, $x_1|_B$ and $y = \frac{\varepsilon}{2} \sum_{\gamma \in B} e_\gamma$ belong to the set K_1 according to Lemma 2.7.

Therefore $y \in \bigcap_{\gamma \in B} U_\gamma$ because for every $\gamma_1 \neq \gamma_2$ in B holds $\gamma_1 \notin S_{\gamma_2}$. Consequently $f(y) \in \bigcap_{\gamma \in B} W_\gamma$ and by $(*)$

$$|f(y)(\lambda_\gamma)| > \frac{|f(d_\gamma)(\lambda_\gamma)|}{2} > \frac{1}{2n}$$

for every $\gamma \in B$.

This is a contradiction because for every $\gamma_1 \neq \gamma_2$ in B, $\lambda_{\gamma_1} \neq \lambda_{\gamma_2}$ ($\gamma_1 \notin A_{\gamma_2}$) and $f(y) \in c_0(\Lambda)$.

If K is U. E. C., the cardinality of the sets $\{\gamma \in \Gamma_m^{(\varepsilon)} : |x(\gamma)| > \varepsilon\}$ can be controlled. In fact, we have the following characterization:

THEOREM 2.10. *Let K be a compact subset of $\Sigma(\boldsymbol{R}^\Gamma)$. The following are equivalent:*

1. *The set K is* U. E. C.
2. *For every $\varepsilon > 0$ there exists a decomposition $\{\Gamma_m^{(\varepsilon)} : m \in N\}$ of Γ and a sequence $\{k(m) : m \in N\}$, such that for every $x \in K$ and $m \in N$*

$$\operatorname{card}\{\gamma \in \Gamma_m^{(\varepsilon)} : |x(\gamma)| > \varepsilon\} < k(m).$$

Proof. $(2 \Rightarrow 1)$ As in Theorem 2.9 we define the family $\mathscr{K}$ of open F_σ subsets of K, which separate the points of K. For $(n, m, l) \in N^3$, let

$$\mathscr{K}_{(n,m,l)} = \{V_{\gamma,j}^n : \lambda \in \Gamma_m^{(\frac{1}{n})} \text{ such that } \operatorname{card} A_n^\gamma \leqslant l \text{ and } j \in A_n^\gamma\}.$$

Clearly $\{\mathscr{K}_{(n,m,l)}\}$ is a decomposition of $\mathscr{K}$. Further, $\mathscr{K}_{(m,n,l)}$ is $k\left(m, \frac{1}{n}\right) \cdot l$-finite. (The proof is analogous to the corresponding one in 2.9). Hence Proposition 1.1 implies that K is U. E. C.

$(1 \Rightarrow 2)$ Let K be the set $\overline{\operatorname{co}}(\mathscr{E}(K \cup \{e_\gamma : \gamma \in \Gamma\}))$. Using the Propositions 2.2 and 2.6 the set K_1 is U. E. C. So there exists a homeomorphism $f: K_1 \to l_2(\Lambda)$ satisfying $f(0) = 0$ and $\|f(x)\| \leqslant 1$ for every $x \in K_1$.

For a fixed $0 < \varepsilon < 1$ we define the decomposition $\{\Gamma_{(n,m)}^{(\varepsilon)} : (n, m) \in N^2\}$ of Γ as in the proof of Theorem 2.9. It now follows, using methods analogous to the ones in [2], (Theorem 1.7), that, for any $x \in K_1$

$$\operatorname{card}\{\gamma \in \Gamma_{(n,m)}^{(\varepsilon)} : |x(\gamma)| > \varepsilon\} < (2n)^{2m}$$

and the proof of the theorem is complete.

In the following theorem, we characterize T. C. subsets of $\Sigma(\boldsymbol{R}^\Gamma)$ as *subsets* of a space $c(\Gamma_\sigma : \sigma \in \Sigma)$, thus improving a result of Mercourakis [7], to the effect that T. C. spaces are *homeomorphic* to compact subsets of $c(\Gamma_\sigma : \sigma \in \Sigma)$.

THEOREM 2.11. *Let K be a compact subset of $\Sigma(\boldsymbol{R}^\Gamma)$. The following are equivalent:*

(1) *The set K is* T. C.

(2) *There exists a decomposition $\{\Gamma_\sigma : \sigma \in \Sigma\}$ of Γ such that $\Gamma_{\sigma_1} \subseteq \Gamma_{\sigma_2}$ if $\sigma_1 < \sigma_2$ and $\{x|_{\Gamma_\sigma} : x \in K\} \subseteq C_0(\Gamma_\sigma)$ for every $\sigma \in \Sigma$.*

Proof. $(2) \Rightarrow (1)$ K is a subset of $c(\Gamma_\sigma : \sigma \in \Sigma)$ so K is T. C.

$(1) \Rightarrow (2)$ If $K_1 = \overline{\operatorname{co}}(\mathscr{E}(K \cup \{e_\gamma : \gamma \in \Gamma\})$ then K_1 is T. C. according to Propositions 2.2 and 2.6. So there exists an homeomorphism $f: K_1 \to c(\Lambda_\sigma : \sigma \in \Sigma)$ satisfying $f(0) = 0$ and $\|f(x)\| \leqslant 1$ for every $x \in K_1$.

As in the proof of Theorem 2.9, we define for fixed $0 < \varepsilon < 1$ and $\lambda \in \Gamma$, the open sets $V_\gamma \subseteq K_1$, $W_\gamma \subseteq f(V_\gamma)$, $U_\gamma \subseteq f^{-1}(W_\gamma)$ and the finite subsets A_γ of Γ.

For every $(n, m) \in N^2$ and $\sigma \in \Sigma$, set

$$\Gamma^{(\varepsilon)}_{(n,m,\sigma)} = \left\{\gamma \in \Gamma:\ \lambda_\gamma \in \Lambda_\sigma,\ |f(d_\gamma)(\lambda_\gamma)| > \frac{1}{n} \text{ and } \operatorname{card}(s_\gamma \cup A_\gamma) < m\right\}.$$

We claim that:

(1) $\Gamma = \bigcup \{\Gamma_{(n,m,\sigma)}:\ (n, m) \in N^2, \sigma \in \Sigma\}$;

(2) If $\sigma_1 < \sigma_2$ then $\Gamma_{(n,m,\sigma_1)} \subseteq \Gamma_{(n,m,\sigma_2)}$;

(3) $\operatorname{card}\{\gamma \in \Gamma^{(\varepsilon)}_{(n,m,\sigma)}:\ |x(\gamma)| > \varepsilon\} < \omega$;

for every $x \in K$, $(n, m) \in N^2$ and $\sigma \in \Sigma$.

It is easy to check that conditions 1 and 2 are true. Suppose that condition 3 is false and that the set $A = \{\gamma \in \Gamma^{(\varepsilon)}_{(n,m,\sigma)}:\ x(\gamma)| > \varepsilon\}$ is infinite for some $x \in K$ and $(n, m, \sigma) \in \Sigma$. Then, by Lemma 2.8, we can choose an infinite subset B of A such that for γ_1, γ_2 elements of B, $\gamma_1 \notin A_{\gamma_2} \cup S_{\gamma_2}$. Therefore, the element $y = \frac{\varepsilon}{2} \sum_{\gamma \in B} e_\gamma$ of K_1 satisfies the relation $y \in \bigcap_{\gamma \in B} U_\gamma$ and for every $\gamma_1 \neq \gamma_2$ in B, $\lambda_{\gamma_1} \neq \lambda_{\gamma_2}$.

Consequently

$$f(y) \in \bigcap_{\gamma \in B} W_\gamma \quad \text{and therefore} \quad |f(y)(\lambda_\gamma)| > \frac{1}{2n}$$

for every $\gamma \in B$. This is a contradiction because $\{\lambda_\gamma:\ \gamma \in B\}$ is an infinite subset of Λ_σ and $f(y)|_{\Lambda_\sigma} \in c_0(\Lambda_\sigma)$. Therefore condition 3 holds.

For every $\sigma \in \Sigma$, set $\sigma^* = (\sigma(3), \sigma(4), \ldots) \in \Sigma$

$$\Gamma^{(\varepsilon)}_\sigma = \bigcup_{\substack{1 \leqslant n \leqslant \sigma(1) \\ 1 \leqslant m \leqslant \sigma(2)}} \Gamma^{(\varepsilon)}_{(n,m,\sigma^*)}$$

and

$$\Gamma_{(\sigma_1, \sigma_2, \ldots)} = \bigcap_{n \in N} \Gamma^{(\frac{1}{n})}_{\sigma_n} \quad \text{for} \quad (\sigma_1, \sigma_2, \ldots) \in \Sigma^N.$$

We then have

(4) $\Gamma = \bigcup \{\Gamma_{(\sigma_1, \sigma_2, \ldots)}:\ (\sigma_1, \sigma_2, \ldots) \in \Sigma^N\}$;

(5) If $(\sigma_1, \sigma_2, \ldots) < (\sigma_1^1, \sigma_2^1, \ldots)$ then

$$\Gamma_{(\sigma_1, \sigma_2, \ldots)} \subseteq \Gamma_{(\sigma_1^1, \sigma_2^1, \ldots)};$$

(6) $\operatorname{card}\{\gamma \in \Gamma_{(\sigma_1, \sigma_2, \ldots)}:\ |x(\gamma)| > \varepsilon\} < \omega$

for every $x \in K$, $\varepsilon > 0$ and $(\sigma_1, \sigma_2, \ldots) \in \Sigma^N$.

Conditions 4 and 5 are consequences of 1 and 2. Condition 6 is true because for every $\varepsilon > 0$ there exists an $n \in N$ such that $\varepsilon > \frac{1}{n}$. But, for all $x \in K$.

$$\operatorname{card}\{\gamma \in \Gamma_{(\sigma_1, \ldots \sigma_n \ldots)}:\ |x(\gamma)| > \varepsilon\} \leqslant \operatorname{card}\{\gamma \in \Gamma^{(\frac{1}{n})}_{\sigma_n}:\ |x(\gamma)| > \varepsilon\} \leqslant \operatorname{card}\left\{\gamma \in \Gamma^{(\frac{1}{n})}_{\sigma_n}:\ |x(\gamma)| > \frac{1}{n}\right\}$$

which is finite by condition 3 and the definition of $\Gamma^{(\frac{1}{n})}_{\sigma_n}$. Hence

$$x|_{\Gamma_{(\sigma_1, \sigma_2, \ldots)}} \in c_0(\Gamma_{(\sigma_1, \sigma_2, \ldots)})$$

and the proof of the theorem is complete.

After the characterization of E. C. sets described in Theorem 2.9, the question arises whether the decomposition of Γ can be made independent of ε, as for T. C. sets. (i.e. the $x|_{\Gamma_m} \in c_0(\Gamma_m)$ for all $x \in K$).

This is impossible in general, as shown by Example 2.15 below. However, if K consists of characteristic functions of subsets of Γ, then the decomposition can easily be made independent of ε, by choosing $\varepsilon = \frac{1}{2}$ in Theorem 2.9. Thus, we have:

Corollary 2.12. *Let Γ be a nonempty set and $\mathscr{A}$ a family of subsets of Γ such that the set $K = \{I_S:\ S \in \mathscr{A}\}$ is a compact subset of $\Sigma(R^\Gamma)$. The set K is* E. C. *iff there exists a decomposition $\{\Gamma_m:\ m \in N\}$ of Γ satisfying the condition*

$$\operatorname{card}(S \cap \Gamma_m) < \omega$$

for every $S \in \mathscr{A}$ and $m \in N$.

For the case of U. E. C. subsets of $\Sigma(R^\Gamma)$ which consist of characteristic functions, we have from Theorem 2.10 the following corollary. This corollary was proved in [2] for the special case where K consists of characteristic functions of finite subsets of Γ.

Corollary 2.13. *Let $\mathscr{A}$ be a family of subsets of a set Γ such that the set $K = \{I_S:\ S \in \mathscr{A}\}$ is a compact subset of $\Sigma(R^\Gamma)$. The set K is* U. E. C. *iff there exists a decomposition $\{\Gamma_m:\ m \in N\}$ of Γ and a sequence $\{k(m)\}$ of natural numbers satisfying the condition*

$$\operatorname{card}(S \cap \Gamma_m) < k(m)$$

for every $S \in \mathscr{A}$ and $m \in N$.

The characterization of E. C. and T. C. subsets of $\Sigma(R^\Gamma)$ given by Theorems 2.9 and 2.11 gives a very simple proof that Talagrand's example [11] is a T. C. space which is not E. C.

Example 2.14. Denote the set N^N by Σ and define the families $\mathscr{B} = \{\{\sigma\}:\ \sigma \in \Sigma\}$ and, for every $n \in N$,

$$\mathscr{A}_n = \{S \subset \Sigma:\ |S| \geqslant 2, \forall \sigma, \varrho \in S\ \sigma|_n = \varrho|_n \text{ and } \sigma(n+1) \neq \varrho(n+1)\}$$

where $\sigma|_n = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in N^n$

Let $\mathscr{A} = \mathscr{B} \cup \left(\bigcup_{n \in N} \mathscr{A}_n\right)$ and $K = \{I_S:\ S \in \mathscr{A}\}$. It is easy to check that the set K is a compact subset of $\Sigma(R^\Sigma)$.

For every $\sigma \in \Sigma$ define

$$\Sigma_\sigma = \{\varrho \in \Sigma:\ \varrho \leqslant \sigma\}.$$

Then $\{\Sigma_\sigma\colon \sigma \in \Sigma\}$ is a decomposition of Σ and $\Sigma_{\sigma_1} \subseteq \Sigma_{\sigma_2}$ if $\sigma_1 < \sigma_2$. Also,

$$\{x|_{\Sigma_\sigma}\colon x \in K\} \subseteq c_0(\Sigma_\sigma) \quad \text{because}$$

$$\operatorname{card}(S \cap \Sigma_\sigma) \leqslant \sigma(n+1) \quad \text{for every } S \in \mathscr{A}.$$

Hence K is T. C. by Theorem 2.11.

Assume K is E. C. Then from Corollary 2.12 there exists a decomposition $\{\Sigma_m\colon m \in N\}$ of Σ such that

$$\operatorname{card}(S \cap \Sigma_n) < \omega \quad \text{for every } S \in \mathscr{A} \text{ and } m \in N.$$

From Baire's theorem there exists an $m_0 \in N$ such that the closure of Σ_{m_0} has a non-empty interior. If $x \in (\overline{\Sigma_{m_0}})^0$ we can find an open set $V_x = \{y \in \Sigma\colon y|_{n_0} = x|_{n_0}\}$ so that $V_x \subseteq \overline{\Sigma_{m_0}}$. Hence for every $k \in N$ there exists $y_k \in V_x^k \cap \Sigma_{m_0}$ where

$$V_x^k = \{y \in \Sigma\colon y|_{n_0} = x|_{n_0} \text{ and } y(n_0+1) = k\}.$$

The set $S = \{y_k\colon k \in N\}$ belong to $\mathscr{A}$ and is a subset of Σ_{m_0}. Therefore $\operatorname{card}(S \cap \Sigma_{m_0}) = \omega$. This is a contradiction, hence K is not an E. C. space.

In the following example we describe a compact subset of a space $\Sigma(\boldsymbol{R}^\Gamma)$ which is E. C., in fact U. E. C., but there does not exist a decomposition $\{\Gamma_m\colon m \in N\}$ of Γ such that

$$\{x|_{\Gamma_m}\colon x \in K\} \subseteq c_0(\Gamma_m) \quad \text{for every } m \in N.$$

EXAMPLE 2.15 (in cooperation with S. Argyros). As in Example 2.14 we define the family $\mathscr{A} = \mathscr{B} \cup (\bigcup_{n \in N} \mathscr{A}_n)$ of subsets of Σ. For every S in $\bigcup_{n \in N} \mathscr{A}_n$ there exists only one $n \in N$ such that $S \in \mathscr{A}_n$. If $S \in \mathscr{A}_n$ we define the function $\psi_S = \frac{1}{n} I_S$. For every $\sigma \in \Sigma$ and $k \in N$ we define $\psi_\sigma^k = \frac{1}{k} e_\sigma$.

Denote by

$$K = \{\psi_s\colon S \in \bigcup_{n \in N} \mathscr{A}_n\} \cup \{\psi_\sigma^k\colon k \in N, \sigma \in \Sigma\} \cup \{0\}$$

The set K is a compact subset of $\Sigma(\boldsymbol{R}^\Sigma)$. To prove this it is enough to show that if $\{x_i\}_{i \in I}$ is a net in K which converges to $x \in [-1, 1]^\Sigma$ then $x \in K$. It is clear that $\{x(\sigma)\colon \sigma \in \Sigma\} \subseteq \left\{\frac{1}{n}\colon n \in N\right\} \cup \{0\}$.

If $x(\sigma_1) = \frac{1}{k}$ and $x(\sigma_2) = \frac{1}{\lambda}$ for $k \neq \lambda$ in N then there exists $i \in I$ such that $x_i(\sigma_1) = \frac{1}{k}$ and $x_i(\sigma_2) = \frac{1}{\lambda}$. This is a contradiction, therefore $x = 0$ or $x = \frac{1}{k} I_S$ for some $k \in N$ and $S \subseteq \Sigma$.

Let $\operatorname{card} S \geqslant 2$ and $\sigma_1 \neq \sigma_2$ in S. Then there exists $i \in I$ such that $x_i(\sigma_1) = x_i(\sigma_2) = \frac{1}{k}$. Therefore $x_i = \frac{1}{k}\psi_{S_i}$ where $S_i \in \mathscr{A}_k$ and then $\{\sigma_1, \sigma_2\} \in \mathscr{A}_k$. Hence $x \in K$ and consequently K is compact.

We will prove that K is U. E. C. using Theorem 2.10. For every $\varepsilon = \frac{1}{n}$ define the sets

$$\Sigma_{k_1,\ldots,k_{n+1}}^{(\frac{1}{n})} = \{\sigma \in \Sigma\colon \sigma|_{n+1} = (k_1, \ldots, k_{n+1})\}$$

for every $(k_1, \ldots, k_{n+1}) \in N^{n+1}$.

These sets form a decomposition of Σ and it is easy to check that

$$\operatorname{card}\left\{\sigma \in \Sigma_{(k_1,\ldots,k_{n+1})}^{(\frac{1}{n})}\colon x(\sigma) > \frac{1}{n}\right\} \leqslant 1$$

for every $x \in K$ and $(k_1, \ldots, k_{n+1}) \in N^{n+1}$.

In the sequel we shall show that there doesn't exist a decomposition $\{\Sigma_m\colon m \in N\}$ of Σ such that

$$\{x|_{\Sigma_m}\colon x \in K\} \subseteq c_0(\Sigma_m) \quad \text{for every } m \in N.$$

Let $\{\Sigma_m\colon m \in N\}$ be a decomposition of Σ. According to Baire's theorem, there exists an $m_0 \in N$ so that $(\overline{\Sigma_{m_0}})^0 \neq \emptyset$. In the same way as in Example 2.14 there exists $n_0 \in N$ and $S \in \mathscr{A}_{n_0}$ such that $\operatorname{card}(S \cap \Sigma_{m_0}) = \omega$.

If $\psi_s = \frac{1}{n_0} I_S$ and $\varepsilon < \frac{1}{n_0}$, then

$$\operatorname{card}\{\sigma \in \Sigma_{m_0}\colon |\psi_s(\sigma)| > \varepsilon\} = \operatorname{card}(S \cap \Sigma_{m_0}) = \omega$$

so $\psi_s|_{\Sigma_{m_0}}$ does not belong to $c_0(\Sigma_{m_0})$.

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Automorphisms of the Loeb algebra

by

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Abstract. Let $(\Omega, L(\Omega), L(\mu))$ be a uniform hyperfinite probability space in a sufficiently saturated nonstandard model of analysis. We prove: 1. Every automorphism of the measure algebra over Ω is induced by an invertible point transformation. 2. Some automorphisms are *not* given by *internal* transformations. 3. The restriction of every automorphism to a small subalgebra *is* given by an internal transformation.

We discuss applications to ergodic theory and hyperfinite measure theory.

1. Introduction. Suppose T is an invertible transformation, measurable in both senses, of a probability space $(X, \mathfrak{B}, m)$. T induces a Boolean (σ-) automorphism $\Phi = \Phi_T$ of the measure algebra $[\![\mathfrak{B}]\!]$ associated with $(X, \mathfrak{B}, m)$. Considerations from Ergodic Theory motivate the converse question: When is a given automorphism Φ induced by a transformation T?

The answer in "always" for most common spaces (von Neumann [14], Choksi [4]). For those spaces $(X, \mathfrak{B}, m)$ and automorphisms Φ of $[\![\mathfrak{B}]\!]$ *not* induced by a transformation, some authors have asked weaker questions, for example (Panzone and Segovia [15]), whether Φ is induced by a transformation T of a thick subset of X.

We consider here the question when $(X, \mathfrak{B}, m)$ is the uniform hyperfinite probability space $(\Omega, L(\Omega), L(\mu))$ deeply investigated by Loeb [13], Anderson [1] and others. This space has a variety of "universality" properties (Anderson [1], Hoover [9], Keisler [12]) which allow questions about more general or common spaces to be reduced to questions about Ω. (For a further discussion, see Section 5.)

Our main result, Theorem 4.1, is that in the presence of sufficient saturation, every measure algebra automorphism is indeed given by a permutation of Ω.

Since in application the most useful transformations of Ω are the internal ones, we consider whether the transformation in Theorem 4.1 can always be taken to be internal. Theorem 4.3 gives a negative answer. However, the restriction of Φ to any sufficiently small subset of $[\![L(\Omega)]\!]$ *is* induced by an internal permutation; this is Theorem 4.4. (Another proof of Theorem 4.4, using Hall's "Marriage Lemma", appears in Ross [16].)

We give some applications of these results in Section 5. Proposition 5.1 shows