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Classes of nonseparable Banach spaces with no universal element

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Abstract. In this paper we show that certain classes of nonseparable Banach spaces have no universal element. In particular, we show that for every infinite cardinal α the class $\mathscr{A}(\alpha)$ of all weakly compactly generated Banach spaces which have topological weight at most α and whose dual spaces have the Radon–Nikodým property contains no universal element. The same is true for the class $\mathscr{R}(\alpha)$ of all reflexive Banach spaces with topological weight at most α.

Introduction. A Banach space X is said to be *universal* for a class $\mathscr{A}$ of Banach spaces iff each member of $\mathscr{A}$ is isomorphic to a closed linear subspace of X. The Banach–Mazur theorem ([3]) asserts that the space of all continuous scalar-valued functions on the closed unit interval is universal for the class of separable Banach spaces. W. Szlenk proved in [13] that the class of all Banach spaces with separable dual contains no universal element. The basic tool in his proof is the "index" of a Banach space. He also showed that there is no separable reflexive Banach space universal for the class of all separable reflexive Banach spaces.

The purpose of the present paper is to extend these results to the nonseparable case. First we show (Theorem 9) that for every infinite cardinal α the class $\mathscr{R}(\alpha)$ of all reflexive Banach spaces with topological weight at most α contains no universal element.

A Banach space X is said to have the *Radon–Nikodým property* (R.N.P.) iff given any finite measure space (S, Σ, μ) and any X-valued measure m on Σ with finite total variation and absolutely continuous with respect to μ, m is the indefinite integral of an X-valued Bochner integrable function on S with respect to μ. There are several equivalent formulations of the R.N.P. (see [6]). C. Stegall [11] characterized conjugate Banach spaces X^* having the R.N.P. as those spaces for which any separable subspace of X has a separable dual (see [9] and [12] for other equivalences). It is known that if X^* is separable, or if X is reflexive, then X^* has the R.N.P. ([7], [10]).

A Banach space X is said to be *weakly compactly generated* (W.C.G.) whenever there exists a weakly compact subset K of X whose linear span is dense in X (see [6]). An especially interesting property of a W.C.G. Banach space X is that the closed unit ball of its dual X^* is, in its weak-star topology, Eberlein compact (i.e. homeomorphic to a weakly compact subset of some Banach space) ([1]). In particular, the closed unit ball of X^* is weak-star sequentially compact. It is clear that every separable Banach space and every reflexive Banach space is W.C.G.

Extending Szlenk's index, we prove the results which are obtained in [13] in the more general setting of W.C.G. Banach spaces whose dual spaces have the R.N.P. More precisely, we show (Theorem 8) that for every infinite cardinal α there is no universal element in the class $\mathscr{A}(\alpha)$ of all W.C.G. Banach spaces which have topological weight at most α and whose dual spaces have the R.N.P.

This result gives a partial answer to the problem posed by Benyamini, Rudin and Wage in [4], namely, whether for every infinite cardinal α the class $W(\alpha)$ of all W.C.G. Banach spaces with topological weight at most α has a universal element.

Notation. Throughout this paper, capital letters $X, Y, \ldots$ denote Banach spaces and X^* denotes the (topological) dual of X. By the w-topology of X we mean the X^*-topology on X and by the w*-topology of X^* the X-topology on X^* ([8]). The symbol $\text{w-}\lim_n x_n = x$ (resp. $\text{w*-}\lim_n f_n = f$) denotes that the sequence $\{x_n\}$ in X converges to x in the w-topology (resp. the sequence $\{f_n\}$ in X^* converges to f in the w*-topology).

Ordinal numbers are denoted by $\xi, \zeta, \eta, \varrho$. We shall denote the first nonzero ordinal number which has no predecessor by ω. Cardinal numbers are denoted by $\alpha, \beta, \gamma, \delta$. For every cardinal α, we denote by α^+ the least cardinal β such that $\alpha < \beta$ ([5], § 1).

For every topological space X, we denote by $w(X)$ the topological weight of X (i.e. the cardinality of a minimal base of the space) and by $d(X)$ the density character of X (i.e. the cardinality of a minimal dense subset of X).

We start by defining an index for every Banach space which is similar to Szlenk's index ([13]).

1. Definition. Let A be a bounded subset of a Banach space X and B a bounded w*-compact subset of the dual space X^*. To a fixed $\varepsilon > 0$ and to each ordinal number ξ we assign by transfinite induction a set $P_\xi(\varepsilon, A, B)$ as follows:

(1.1) $P_0(\varepsilon, A, B) = B$;

(1.2) $P_{\xi+1}(\varepsilon, A, B) = \{f \in X^*$: there are sequences $\{x_n\}$ in A and $\{f_n\}$ in $P_\xi(\varepsilon, A, B)$ such that $\text{w-}\lim_n x_n = 0$, $\text{w*-}\lim_n f_n = f$ and $\limsup_n |f_n(x_n)| \geqslant \varepsilon\}$;

(1.3) For every ordinal number ξ which has no predecessor

$$P_\xi(\varepsilon, A, B) = \bigcap_{\zeta<\xi} P_\zeta(\varepsilon, A, B).$$

If $w(X) = \alpha$ then we define

$$\eta(\varepsilon, A, B) = \sup\{\xi < \alpha^+ : P_\xi(\varepsilon, A, B) \neq \emptyset\}.$$

Let S and S^* be the closed unit balls of X and X^* respectively. For every $\varepsilon > 0$ we define $\eta(\varepsilon, X) = \eta(\varepsilon, S, S^*)$, the *$\varepsilon$-index* of X.

The ordinal number $\eta(X) = \sup_{\varepsilon>0} \eta(\varepsilon, X)$ is called the *index* of the Banach space X. The following is immediate:

Lemma 2 ([13]). (a) *If $\varepsilon_1 \geqslant \varepsilon_2 > 0$, $A_1 \subseteq A_2 \subseteq X$ and $B_1 \subseteq B_2 \subseteq X^*$ (A_1, A_2, B_1, B_2 are bounded and B_1, B_2 w*-compact) then for every ordinal ξ*

$$P_\xi(\varepsilon_1, A_1, B_1) \subseteq P_\xi(\varepsilon_2, A_2, B_2), \tag{2.1}$$

$$\eta(\varepsilon_1, A_1, B_1) \leqslant \eta(\varepsilon_2, A_2, B_2). \tag{2.2}$$

(b) *If $T: X \to Y$ is an isomorphism, A a bounded subset of X and B a bounded w*-compact subset of X^*, then for every ordinal ξ*

$$P_\xi(\varepsilon, A, B) = T^*\big(P_\xi(\varepsilon, T(A), (T^*)^{-1}(B))\big), \tag{2.3}$$

$$\eta(\varepsilon, A, B) = \eta(\varepsilon, T(A), (T^*)^{-1}(B)) \tag{2.4}$$

where $T^*: Y^* \to X^*$ is the adjoint of T.

Lemma 3. *Let X be a W.C.G. Banach space with $w(X) = \alpha$ such that its dual X^* has the R.N.P. If A is a bounded subset of X, B a bounded w*-compact subset of X^* and $\varepsilon > 0$, the sets $P_\xi(\varepsilon, A, B)$, where $0 < \xi < \alpha^+$, have the following properties:*

(3.1) *$P_\xi(\varepsilon, A, B)$ is w*-compact.*

(3.2) *$P_{\xi+1}(\varepsilon, A, B) \subseteq P_\xi(\varepsilon, A, B)$ and if $P_\xi(\varepsilon, A, B) \neq \emptyset$ then $P_{\xi+1}(\varepsilon, A, B) \neq P_\xi(\varepsilon, A, B)$.*

Proof. We shall first prove condition (3.1) by transfinite induction. The inductive hypothesis is obviously true for $\xi = 0$, and it is also true for every ordinal ξ which has no predecessor if it holds for every ordinal $0 < \zeta < \xi$.

Now suppose that, for an arbitrary ordinal ξ ($0 \leqslant \xi < \alpha^+$), the set $P_\xi(\varepsilon, A, B)$ is w*-compact. The space X is W.C.G. so the set B with the w*-topology is homeomorphic to a weakly compact subset of some Banach space ([1]). The set $P_{\xi+1}(\varepsilon, A, B)$ is a subset of B, so it is w*-compact if and only if for every sequence $\{f_m\}$ of $P_{\xi+1}(\varepsilon, A, B)$ which converges to some $f \in B$ we have $f \in P_{\xi+1}(\varepsilon, A, B)$.

Let $\{f_m\} \subseteq P_{\xi+1}(\varepsilon, A, B)$ and $\text{w*-}\lim_m f_m = f$. By Definition 1 there exist sequences $\{f_{m,n}\}$ in $P_\xi(\varepsilon, A, B)$ and $\{x_{m,n}\}$ in A such that

$$\text{w*-}\lim_n f_{m,n} = f_m, \quad \text{w-}\lim_n x_{m,n} = 0,$$

$$\limsup_n |f_{m,n}(x_{m,n})| \geqslant \varepsilon \quad \text{for every } m \in N.$$

We set

$$C = \{x_{m,n}: (m, n) \in N^2\}, \quad D = \{f_{m,n}: (m, n) \in N^2\},$$

$$E = \bar{C}^w \text{ (the w-closure of } C), \quad F = \bar{D}^{w^*} \text{ (the w*-closure of } D).$$

The w-topology in E is metrizable, because E is a bounded subset of the Banach space X_1 generated by C, which has a separable dual because X^* has the R.N.P.

The w*-topology in F is also metrizable because F with the w*-topology is homeomorphic to a weakly compact subset of some Banach space; hence $d(F) = w(F) = \omega$.

Therefore one can choose a sequence $\{(m_k, n_k)\}$ of pairs of integers such that

$$\text{w*-}\lim_k f_{m_k,n_k} = f, \quad \text{w-}\lim_k x_{m_k,n_k} = 0, \quad \limsup_k |f_{m_k,n_k}(x_{m_k,n_k})| \geqslant \varepsilon.$$

Thus $f \in P_{\xi+1}(\varepsilon, A, B)$. Hence the set $P_{\xi+1}(\varepsilon, A, B)$ is w*-compact and the proof of condition 3.1 is complete.

For every ordinal number $0 \leqslant \xi < \alpha^+$ each $f \in P_{\xi+1}(\varepsilon, A, B)$ is a w*-limit of elements of $P_\xi(\varepsilon, A, B)$, hence, by 3.1, $f \in P_\xi(\varepsilon, A, B)$. Therefore $P_{\xi+1}(\varepsilon, A, B)$ is a subset of $P_\xi(\varepsilon, A, B)$. Finally, since X^* has the R.N.P., X is an Asplund space ([12]). It is therefore an easy consequence of Proposition 5 of [2] that $P_{\xi+1}(\varepsilon, A, B) \neq P_\xi(\varepsilon, A, B)$ for every $0 \leqslant \xi < \alpha^+$.

Proposition 4. *If X is a W.C.G. Banach space with $w(X) = \alpha$ and if its dual space X^* has the R.N.P., then $\eta(X) < \alpha^+$.*

Proof. If A and B are bounded subsets of X and X^* respectively with B w*-compact, and if $\varepsilon > 0$, then $\eta(\varepsilon, A, B) < \alpha^+$, because from Stegall's theorem [11] we have $w(X) = w(X^*)$ and according to Lemma 3 and the Cantor–Baire theorem there exists an ordinal $\xi < \alpha^+$ such that

$$P_\xi(\varepsilon, A, B) \neq \emptyset \quad \text{and} \quad P_{\xi+1}(\varepsilon, A, B) = \emptyset.$$

Therefore according to Lemma 2 it is true that

$$\eta(X) = \sup_{k\in N} \eta(1/k, X) = \sup_{k\in N} \eta(1/k, S, S^*) < \alpha^+$$

and the proof is complete.

Proposition 5. *If T is an isomorphism from a Banach space X onto a subspace of a W.C.G. Banach space Y, then $\eta(X) \leqslant \eta(Y)$. Moreover, if T is isometric, then $\eta(\varepsilon, X) \leqslant \eta(\varepsilon, Y)$ for every $\varepsilon > 0$.*

Proof. Let $w(Y) = \alpha$, $Y_1 = T(X)$ and $K, K^*, S, S^*, K_1, K_1^*$ be the closed unit balls of $Y, Y^*, X, X^*, Y_1, Y_1^*$ respectively.

We shall first show that

$$(*) \qquad \eta(\varepsilon, Y_1) \leqslant \eta(\varepsilon, Y) \quad \text{for every } \varepsilon > 0.$$

This will follow immediately if we can prove the claim:

If $f \in P_\xi(\varepsilon, K_1, K_1^*)$ $(0 \leqslant \xi < \alpha^+)$ then there exists an extension f^1 of f such that $f^1 \in P_\xi(\varepsilon, K, K^*)$.

Proof of the claim. The claim is obviously true for $\xi = 0$ and if it holds for every ordinal $0 \leqslant \zeta < \xi$ where ξ has no predecessor, then it also holds for ξ. This is an immediate consequence of Definition 1.

Let us assume that the claim is true for some ordinal $0 < \xi < \alpha^+$. If $f \in P_{\xi+1}(\varepsilon, K_1, K_1^*)$ then there are sequences $\{f_n\}$ in $P_\xi(\varepsilon, K_1, K_1^*)$ and $\{x_n\}$ in K_1 such that

$$\text{w*-}\lim_n f_n = f, \quad \text{w-}\lim_n x_n = 0 \quad \text{and} \quad \limsup_n |f_n(x_n)| \geqslant \varepsilon.$$

By the inductive hypothesis there exists, for every $n \in N$, an extension f_n^1 of f_n such that $f_n^1 \in P_\xi(\varepsilon, K, K^*)$. The sequence $\{f_n^1\}$ has a w*-convergent subsequence $\{f_{n_k}^1\}$, because the set K^* with the w*-topology is homeomorphic to a weakly compact subset of a Banach space ([1]). Let $\text{w*-}\lim_k f_{n_k}^1 = f^1$. It is easy to check that f^1 is an extension of f and $f^1 \in P_{\xi+1}(\varepsilon, K, K^*)$. This proves the claim.

Since $T: X \to Y_1$ is an isomorphism, there exist real numbers $M, N > 0$ such that

$$(**) \qquad T(M\cdot S) \subseteq K_1 \subseteq T(N \cdot S)$$

(if T is isometric, then $M = N = 1$). Therefore according to Lemma 2 we get

$$\eta(\varepsilon, X) \leqslant \eta\left(\frac{M}{N}\varepsilon, Y_1\right).$$

Finally, using condition (*), we have $\eta(X) \leqslant \eta(Y)$ and if T is isometric, $\eta(\varepsilon, X) \leqslant \eta(\varepsilon, Y)$ for every $\varepsilon > 0$.

Let $\{X_\gamma: \gamma \in \Gamma\}$ be a nonempty family of Banach spaces. For $1 \leqslant p < \infty$, the symbol $l_p(X_\gamma)_{\gamma\in\Gamma}$ denotes the Banach space of all elements $x = (x_\gamma)_{\gamma\in\Gamma}$ of the Cartesian product $\prod_{\gamma\in\Gamma} X_\gamma$ such that

$$\|x\|_p = \Big(\sum_{\gamma\in\Gamma} \|x_\gamma\|^p\Big)^{1/p} < \infty.$$

Similarly $(X \times Y)_\infty$ denotes the Cartesian product of the Banach spaces X and Y with the norm

$$\|(x, y)\|_\infty = \max(\|x\|, \|y\|).$$

Lemma 6. *If X is a W.C.G. Banach space whose dual X^* has the R.N.P. then $\eta(\varepsilon, (X \times l_2)_1) \geqslant \eta(\varepsilon, X) + 1$ $(l_2 = l_2(N))$.*

The proof of this lemma is similar to the corresponding one in Szlenk's paper [13].

PROPOSITION 7. *For every cardinal number α and for each ordinal number $0 \leqslant \xi < \alpha^+$ there exists a reflexive Banach space X_ξ such that $w(X_\xi) \leqslant \alpha$ and $\eta(X_\xi) \geqslant \xi$.*

Proof. Let α be an infinite cardinal. We set $X_0 = l_2$,

$$X_{\xi+1} = (X_\xi \times l_2)_1 \quad \text{for every } 0 < \xi < \alpha^+,$$
$$X_\xi = l_2(X_\zeta)_{\zeta<\xi} \quad \text{for each ordinal } 0 < \xi < \alpha^+ \text{ having no predecessor.}$$

Obviously the spaces X_ξ are reflexive for all $0 \leqslant \xi < \alpha^+$ and $w(X_\xi) \leqslant \alpha$. We shall show inductively that $\eta(\varepsilon, X_\xi) \geqslant \xi$ for every $0 \leqslant \xi < \alpha^+$ and $0 < \varepsilon < 1$.

If $\eta(\varepsilon, X_\xi) \geqslant \xi$, then according to Lemma 6, $\eta(\varepsilon, X_{\xi+1}) \geqslant \eta(\varepsilon, X_\xi)+1 \geqslant \xi+1$ for every $0 < \varepsilon < 1$.

If ξ has no predecessor and $\eta(\varepsilon, X_\zeta) \geqslant \zeta$ for every ordinal $0 \leqslant \zeta < \xi$, then, using the fact that the spaces X_ζ are contained isometrically in X_ξ for all $0 \leqslant \zeta < \xi$, and applying Proposition 5 we get

$$\eta(\varepsilon, X_\xi) \geqslant \sup_{0 \leqslant \zeta < \xi} \eta(\varepsilon, X_\zeta) \geqslant \sup_{0 \leqslant \zeta < \xi} \zeta = \xi.$$

Hence for every $0 \leqslant \xi < \alpha^+$ we have $\eta(\varepsilon, X_\xi) \geqslant \xi$ for $0 < \varepsilon < 1$ and hence $\eta(X_\xi) \geqslant \xi$.

THEOREM 8. *Let α be an infinite cardinal number and $\mathscr{A}(\alpha)$ the class of all weakly compactly generated (W.C.G.) Banach spaces with topological weight at most α whose duals have the Radon–Nikodým property (R.N.P.). The class $\mathscr{A}(\alpha)$ has no universal element.*

Proof. If $\mathscr{A}(\alpha)$ has a universal element X, then $\eta(X) < \alpha^+$ by Proposition 4. But by Proposition 7, there exists $Y \in \mathscr{A}(\alpha)$ with $\eta(Y) \geqslant \eta(X)+1 > \eta(X)$, contradicting Proposition 5.

THEOREM 9. *Let α be an infinite cardinal and $\mathscr{R}(\alpha)$ the class of all reflexive Banach spaces with topological weight at most α. Then $\mathscr{R}(\alpha)$ has no universal element.*

Proof. Analogous to the proof of Theorem 8.

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