Mathematika, to appear.

The Ramsey principle for every countable ordinal index

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Abstract

The complete thin Schreier system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ is a collection of families of finite subsets of $\mathbb{N}$ (where, $\mathcal{A}_k$ consists of all k-element subsets of $\mathbb{N}$, for $k \in \mathbb{N}$) with the properties that each $\mathcal{A}_\xi$ is thin (i.e. it does not contain proper initial segments of any of its elements) and the Cantor-Bendixson index, defined for $\mathcal{A}_\xi$, is equal to $\xi+1$. It is notable that $\mathcal{A}_\xi$ is defined for all countable ordinals and not only for ordinals of the form α^ω.

Using this system we establish, for every countable ordinal index, the correct generalization of the classical Ramsey theorem (which corresponds to the finite ordinal indices). Indeed, for a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$, we obtain the following:

(1) For every infinite subset M of $\mathbb{N}$ and every countable ordinal ξ, there is an infinite subset L of M such that
either $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$ or $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$;
(where $[L]^{<\omega}$ denotes the family of all finite subsets of L).

(2) If in addition $\mathcal{F}$ is hereditary and pointwise closed, then for every infinite subset M of $\mathbb{N}$ there is a countable ordinal number $\xi_M^{\mathcal{F}}$ such that:
   (a) for ξ with $\xi+1 < \xi_M^{\mathcal{F}}$ there is an infinite subset L of M such that $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$;
   (b) for ξ with $\xi_M^{\mathcal{F}} < \xi+1$ there is an infinite subset L of M such that $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ and equivalently $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^* \setminus \mathcal{A}_\xi$;
   ($\mathcal{A}_\xi^*$ denotes the family of all initial segments of elements of $\mathcal{A}_\xi$)
   (c) for $\xi+1 = \xi_M^{\mathcal{F}}$, both alternatives ((a) and (b)) may materialize.

(3) A ξ-Pták type theorem is proved for every countable ordinal ξ (the classical Pták's theorem essentially coincides with the ω_1 case) which provides a method to estimate the crucial ordinal $\xi_M^{\mathcal{F}}$ mentioned above.

1991 Mathematics Subject Classification: Primary 05D10; Secondary 46B45, 5A17.
Keywords and phrases: Ramsey principles- Complete thin Schreier system.

Introduction

In this paper we develop a Ramsey-type principle for every countable ordinal ξ (the classical Ramsey theorem corresponds to the finite, less than ω, ordinals). The vehicle for developing this far reaching principle is the complete thin Schreier system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ (Definition 1.3) consisting of families of finite subsets of $\mathbb{N}$, defined not only for ordinals of the form ω^α as a thin (Definition 2.1) version of the generalized Schreier families, but in addition for all intermediate ordinals by a procedure that mimicks the ordinal representation (Proposition 1.2), based on ordinal Euclidean algorithm.

The ξ-Ramsey type theorem, for $1 \leq \xi < \omega_1$, (Theorem 1.6), is proved using a combination of the natural concept of a ξ-uniform family (introduced by Pudlák and Rödl [P-R]) (Definition 1.7 and Proposition 1.9), and the fact that $\mathcal{A}_\xi$ is a ξ-uniform family as proved by a rather delicate procedure in Theorem 1.10.

Here is the statement of the theorem:

ξ-Ramsey type theorem. Let $\mathcal{F}$ be an arbitrary family of finite subsets of $\mathbb{N}$, M an infinite subset of $\mathbb{N}$ and ξ a countable ordinal number. Then, there exists an infinite subset L of M such that

$$\text{either } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F} .$$

We next develop and apply the fundamental criterion for deciding if, for a given partition, the "large homogenous" family will fall in the one or the other element of the partition. The criterion (given in Theorem 3.9), involves the strong Cantor-Bendixson index (introduced in [A-M-T]) whose basic for our purposes properties (such as that the index of the family $\mathcal{A}_\xi$ is precisely equal to $\xi + 1$ and it is stable under restriction to any infinite subset of $\mathbb{N}$) are developed in Section 2; where an interesting result on the "canonical representation" of every finite subset of $\mathbb{N}$ with respect to $\mathcal{A}_\xi$ for $\xi < \omega_1$ is also established.

Here is the statement of Theorems (3.9 and 3.12)

Refined ξ-Ramsey type theorem. Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}$ and M an infinite subset of $\mathbb{N}$. We have the following cases:

Case 1 If the family $\mathcal{F}\cap[M]^{<\omega}$ is not pointwise closed, then there exists $L \in [M]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$.

Case 2 If the family $\mathcal{F} \cap [M]^{<\omega}$ is pointwise closed, then there exists $L \in [M]$ such that $[L]^{<\omega} \subseteq ([\mathbb{N}]^{<\omega} \setminus \mathcal{F})_*$. Moreover setting

$$\xi_M^{\mathcal{F}} = sup\{s_L(\mathcal{F}) : L \in [M]\} ;$$

the following obtain:

2(i) For every countable ordinal ξ with $\xi+1<\xi_M^{\mathcal{F}}$ there exists $L\in[M]$ such that

$$(\mathcal{A}_\xi)_\star\cap[L]^{<\omega}\subseteq\mathcal{F}\ .$$

2(ii) For every countable ordinal ξ with $\xi_M^{\mathcal{F}}<\xi+1$ there exists $L\in[M]$ such that

$$\mathcal{F}\cap[L]^{<\omega}\subseteq(\mathcal{A}_\xi)^\star\setminus\mathcal{A}_\xi\ ;$$

and equivalently,

$$\mathcal{A}_\xi\cap[L]^{<\omega}\subseteq[\mathbb{N}]^{<\omega}\setminus\mathcal{F}\ .$$

2(iii) If $\xi_M^{\mathcal{F}}=\xi+1$, then both alternatives may materialize.

In the final section we give in Theorem 4.2 a criterion and method to find the crucial "separating" countable ordinal $\xi_M^{\mathcal{F}}$ used in the above theorem. We call this theorem the ξ-Ptak type theorem, for $1\leq\xi<\omega_1$, since as a limiting case (ω_1-case) essentially implies the classical Pták's theorem. An important role for the establishment of the ξ-Ptak type theorem is the notion of the weight $w_\xi(F;s)$ defined for every $1\leq\xi<\omega_1$, every finite subset F of $\mathbb{N}$ and every $s\in\mathcal{A}_\xi$. This definition as the proof of the theorem are given recursively, using the fundamental ordinal respesentation.

We are not concerned in this paper with applications of these Ramsey principles; some of them (in Banach space theory) will appear in a forthcoming paper. Some of the ideas in this paper appeared in embryonic form, in our publication [F2], related to the Kechris and Louveau [K-L] treatment of Baire-1 functions. We believe, however, that these are combinatorial principles that will have a wide spectrum of applications.

Notation. We denote by $\mathbb{N}$ the set of all natural numbers. For an infinite subset M of $\mathbb{N}$ we denote by $[M]^{<\omega}$ the set of all finite subsets of M, for $k\in\mathbb{N}$ we denote by $[M]^k$ the set of all k-element subsets of M, and by $[M]$ the set of all infinite subsets of M (considering them as strictly increasing sequences).

If s,t are finite subsets of $\mathbb{N}$, then $s\preceq t$ means that s is an initial segment of t, while $s\prec t$ means that s is a proper initial segment of t. We write $s\leq t$ if $max\ s\leq min\ t$, while $s<t$ if $max\ s<min\ t$.

Identifying every subset of $\mathbb{N}$ with its characteristic function, we topologize the set of all subsets of $\mathbb{N}$ by the topology of pointwise convergence.

1. The complete thin Schreier system and the Ramsey principle for every countable order

Our starting point is to recall Ramsey's classical partition theorem.

Theorem 1.1 (*k*-**Ramsey** ([R]). Let $\mathcal{F}$ be an arbitrary family of finite subsets of $\mathbb{N}$, M an infinite subset of $\mathbb{N}$ and k a natural number. Then there exists an infinite subset L of M such that

$$\text{either } [L]^k \subseteq \mathcal{F} \text{ or } [L]^k \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F} .$$

We will take the view, and this view will be shown to be the correct one, that this classical Ramsey partition theorem for $k \in \mathbb{N}$ (the $k-$Ramsey theorem as we may call it) is but the initial segment of a whole family of Ramsey type partition theorems, one for every countable ordinal ξ.

In order to arrive at the statement of the ξ-**Ramsey type theorem** for ξ any countable ordinal, we note first that

$$[L]^k = [\mathbb{N}]^k \cap [L]^{<\omega} .$$

Here, the dependence on k is transferred from the "relative" family $[L]^k$ to the "absolute" family $[\mathbb{N}]^k = \mathcal{A}_k$. We need a ξ-ordinal analogue $\mathcal{A}_\xi$ of $\mathcal{A}_k$ in order to state the ξ-Ramsey type theorem. Our next task will be to define precisely this family $\mathcal{A}_\xi$, for some $\xi < \omega_1$, by a rather laborious transfinite induction, that depends essentially on a (classical) representation of (limit) ordinals, involving the ordinal analogue of Euclidean algorithm as follows:

Proposition 1.2 (**Representation of ordinals**, [C2] [L]). Let α be a non-zero, countable ordinal. For every limit ordinal ξ, so that $\omega^\alpha < \xi < \omega^{\alpha+1}$, there exist a unique natural number $m \geq 0$, a sequence of ordinals $\alpha > \alpha_1 \ldots > \alpha_m > 0$ and natural numbers $p, p_1, \ldots, p_m \geq 1$ (so that either $p > 1$ or $p = 1$ and $m \geq 1$), such that

$$\xi = p\omega^\alpha + \sum_{i=1}^{m} p_i \omega^{\alpha_i} .$$

We are now ready to define the families $\mathcal{A}_\xi$, for $\xi < \omega_1$, which for reasons that will be explained later, will be collectively called the complete thin Schreier system.

Definition 1.3 (**The complete thin Schreier system** $(\mathcal{A}_\xi)_{\xi<\omega_1}$).

For every non zero, limit ordinal λ we choose and fix a strictly increasing sequence (λ_n) of successor ordinals smaller than λ with $sup_n\ \lambda_n = \lambda$.

We will define the system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ recursively as follows:

(1) [**Case** $\xi = 0$]

$$\mathcal{A}_0 = \{\emptyset\} \ ;$$

(2) [**Case** $\xi = \zeta + 1$]

$$\mathcal{A}_\xi = \mathcal{A}_{\zeta+1} = \{s \subseteq \mathbb{N} : s = \{n\} \cup s_1 \ , \ \text{where } n \in \mathbb{N}, \{n\} < s_1 \text{ and } s_1 \in \mathcal{A}_\zeta\} \ ;$$

(3) [**Case** $\xi = \omega^{\beta+1}$, β countable ordinal]

$$\mathcal{A}_\xi = \mathcal{A}_{\omega^{\beta+1}} = \{s \subseteq \mathbb{N} : s = \bigcup_{i=1}^{n} s_i \text{ with } n = min\ s_1 \ , \quad s_1 < \ldots < s_n,$$

$$\text{and } s_1, \ldots, s_n \in \mathcal{A}_{\omega^\beta}\};$$

(4) [**Case** $\xi = \omega^\lambda$, λ non-zero, countable limit ordinal]

$$\mathcal{A}_\xi = \mathcal{A}_{\omega^\lambda} = \{s \subseteq \mathbb{N} : s \in \mathcal{A}_{\omega^{\lambda_n}} \text{ with } \ n = min\ s\} \ ,$$

(where (λ_n) is the sequence of ordinals, converging to λ , fixed above); and

(5) [**Case** ξ **limit** $,\omega^\alpha < \xi < \omega^{\alpha+1}$ for some $0 < \alpha < \omega_1$]

Let $\xi = p\omega^\alpha + \sum_{i=1}^{m} p_i\omega^{\alpha_i}$ be the above representation (Proposition 1.2).

$$\mathcal{A}_\xi = \{s \subseteq \mathbb{N} : s = s_0 \cup (\bigcup_{i=1}^{m} s_i) \ \text{ with } s_m < \ldots < s_1 < s_0 \ ,$$

$$s_0 = s_1^0 \cup \ldots \cup s_p^0 \text{ with } s_1^0 < \ldots < s_p^0, \ s_j^0 \in \mathcal{A}_{\omega^\alpha}, 1 \leq j \leq p \ ,$$

$$s_i = s_1^i \cup \ldots \cup s_{p_i}^i, \text{ with } s_1^i < \ldots < s_{p_i}^i, \ s_j^i \in \mathcal{A}_{\omega^{\alpha_i}}, 1 \leq i \leq m, \ 1 \leq j \leq p_i \ \} \ .$$

Definition 1.4 We set

$$\mathcal{B}_\alpha = \mathcal{A}_{\omega^\alpha} \text{ for each } 1 \leq \alpha < \omega_1 \ .$$

Remark 1.5 (i) $\mathcal{A}_\xi \subseteq [\mathbb{N}]^{<\omega}$ for every $\xi < \omega_1$.

(ii) $\mathcal{A}_k = [\mathbb{N}]^k$ for $k = 1, 2, \ldots$

(iii) $\mathcal{B}_1 = \mathcal{A}_\omega = \{s \in [\mathbb{N}]^{<\omega} : s = (n_1 < \ldots < n_k) \quad \text{with} \quad n_1 = k\}$.

Thus $\mathcal{A}_\omega$ is a modification of the **classical Schreier family** ([S])

$$\mathcal{F}_1 = \{s \subseteq \mathbb{N} : s = (n_1 < \ldots < n_k) \text{ with } n_1 \geq k\} \ .$$

In this sense $\mathcal{A}_\omega$ is a thin Schreier family (this notion, used also, by Pudlak - Rödl, will be defined precisely later on in Definition 2.1).

(iv) $\mathcal{B}_\alpha = \mathcal{A}_{\omega^\alpha}$, for $\alpha < \omega_1$, is defined using only $\mathcal{B}_\beta = \mathcal{A}_{\omega^\beta}$ for $\beta < \alpha$, and not using all previously defined families $\mathcal{A}_\xi, \xi < \omega^\alpha$. $\mathcal{B}_k$, for $k \in \mathbb{N}$, is a modification of generalized Schreier families defined by Alspach-Odell ([A-O]); and more generally $\mathcal{B}_\alpha$, for $\alpha < \omega_1$ is a modification of the families $\mathcal{F}_\alpha$, defined by Alspach-Argyros ([A-A]).

Now that the definition of the complete thin Schreier system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ is given, we are ready to state the first ξ- Ramsey type theorem, for ξ any countable ordinal, a theorem whose scope can be appreciated by the fact that the classical Ramsey theorem corresponds to a finite ordinal $\xi < \omega$.

Theorem 1.6 (ξ-Ramsey type theorem). Let $\mathcal{F}$ be an arbitrary family of finite subsets of $\mathbb{N}$, M an infinite subset of $\mathbb{N}$ and ξ a countable ordinal number. Then, there exists an infinite subset L of M such that

$$\text{either } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F} .$$

In order to prove this theorem, we must find a way to relate the complete thin Schreier system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ with Ramsey type partition. The connecting concept, that of a ξ-uniform family for $\xi < \omega_1$, isolated and defined by Pudlák and Rödl ([P-R]); is essentially an integral part of the standard inductive proof (from less than k to k) of the classical Ramsey theorem. As it turns out, it is ideally suited for our purposes.

Definition 1.7 (Pudlák and Rödl ([P-R]). Let $M \in [\mathbb{N}]$ and $\mathcal{L}$ a family of finite subsets of M. For $m \in M$ we set $\mathcal{L}(m) = \{s \in [M]^{<\omega} : \{m\} < s \text{ and } \{m\} \cup s \in \mathcal{L}\}$.

(i) $\mathcal{L}$ is 0- **uniform** if $\mathcal{L} = \{\emptyset\}$;

(ii) $\mathcal{L}$ is ξ- **uniform** on M for some non-zero, countable ordinal ξ if $\emptyset \notin \mathcal{L}$ and $\mathcal{L}(m)$ is a $\xi(m)-$ uniform family on $M \cap (m, +\infty)$ for all $m \in M$, where $\xi(m) = \xi - 1$ for every $m \in M$ in case ξ is a successor ordinal and $(\xi(m), m \in M)$ is a strictly increasing sequence of ordinals with $sup_{m\in M}\xi(m) = \xi$ in case ξ is a non-zero, limit countable ordinal.

Remark 1.8 (i) For every $M \in [\mathbb{N}]$ and every natural number k there is exactly one $k-$uniform family on M, namely the family $[M]^k$ of all k-element subsets of M.

(ii) Every ξ-uniform family on M is a non-empty set of finite subsets of $\mathbb{N}$.

(iii) If $\mathcal{L}$ is a ξ- uniform family on M and $L \in [M]$, then, as can be proved by induction on $\xi, \mathcal{L} \cap [L]^{<\omega}$ is ξ-uniform on L (the proof by induction is given in

[P-R]).

(iv) Every uniform family $\mathcal{L}$ on M is a maximal thin subset of $[M]^{<\omega}$, (for the proof see [P-R]).

We now, mimicuing the standard proof of the classical Ramsey theorem, establish the Ramsey character of a ξ- uniform family. This has been initially proved, in a different way, using the Nash-Williams partition theorem, by Pudlák and Rödl ([P-R]). We include our simpler proof, for completeness.

Proposition 1.9 Let $\mathcal{F}$ be an arbitrary family of finite subsets of $\mathbb{N}, M$ an infinite subset of $\mathbb{N}, \xi$ a countable ordinal number and $\mathcal{L}$ a ξ-uniform family on $\mathbb{N}$. Then there exists an infinite subset L of M such that

$$\text{either } \mathcal{L} \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{L} \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F} .$$

Proof We will prove an equivalent formulation of the above result, which is more suitable in inductive arguments, the following:

"If $M \in [\mathbb{N}]$, $\mathcal{F} \subseteq [M]^{<\omega}$, ξ a countable ordinal number and $\mathcal{L}$ a ξ-uniform family on M, then there exists $L \in [M]$ such that

$$\text{either } \mathcal{L} \cup [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{L} \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F} \text{ ."}$$

We will prove it by induction on ξ. Let $\xi = 1$. Then $\mathcal{L} = \{\{m\} : m \in M\}$. We set

$$M_1 = \{m \in M : \{m\} \in \mathcal{F}\}; \text{ and}$$

$$M_2 = \{m \in M : \{m\} \in [\mathbb{N}]^{<\omega} \setminus \mathcal{F}\}.$$

In case M_1 is infinite we set $L = M_1$, otherwise we set $L = M_2$.

Let $1 < \xi$. Assume that the claim in valid for all ordinal ζ less than ξ. Let $M \in [\mathbb{N}]$, $\mathcal{F} \subseteq [M]^{<\omega}$ and $\mathcal{L}$ a ξ-uniform family on M.

We set $m_1 = minM$ and $M_1 = M \cap (m_1, +\infty)$. Then $\mathcal{L}(m_1)$ is $\xi(m_1)$-uniform family on M_1, with $\xi(m_1) < \xi$. Setting

$$\mathcal{F}_1 = \{s \subseteq M_1 : \{m_1\} \cup s \in \mathcal{F}\}$$

and using the induction hypothesis, we can find $L_1 \in [M_1]$ such that

$$\text{either } \mathcal{L}(m_1) \cap [L_1]^{<\omega} \subseteq \mathcal{F}_1 \text{ or } \mathcal{L}(m_1) \cap [L_1]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}_1 \text{ .}$$

Let $m_2 = minL_1 > m_1$ and $M_2 = L_1 \cap (m_2, +\infty)$. Then $\mathcal{L}(m_2) \cap [M_2]^{<\omega}$ is a $\xi(m_2)$-uniform family on M_2 (Remark 1.8 (iii)) with $\xi(m_2) < \xi$. Setting

$$\mathcal{F}_2 = \{s \subseteq M_2 : \{m_2\} \cup s \in \mathcal{F}\}$$

and applying the induction hypothesis, we can find $L_2 \in [M_2]$ such that

$$\text{either } \mathcal{L}(m_2) \cap [L_2]^{<\omega} \subseteq \mathcal{F}_2 \text{ or } \mathcal{L}(m_2) \cup [L_2]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}_2 \,.$$

Set $m_3 = minL_2 > m_2 > m_1$ and $M_3 = L_2 \cap (m_3, +\infty)$ and proceed analogously.

In this way we can construct a strictly increasing sequence $I = (m_n)_{n \in \mathbb{N}} \subseteq M$ and two decreasing sequences $(M_n)_{n \in \mathbb{N}}$ and $(L_n)_{n \in \mathbb{N}}$ of infinite subsets of M with the properties:

(i) $m_k \in L_n$ for every $k > n$;

(ii) $L_n \subseteq M_n$ for every $n \in \mathbb{N}$; and

(iii) if $\mathcal{F}_n = \{s \subseteq M_n : \{m_n\} \cup s \in \mathcal{F}\}$ for every $n \in \mathbb{N}$, then

$$\text{either } \mathcal{L}(m_n) \cap [L_n]^{<\omega} \subseteq \mathcal{F}_n \text{ or } \mathcal{L}(m_n) \cap [L_n]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}_n \,.$$

Set

$$I_1 = \{m_n \in I : \mathcal{L}(m_n) \cap [L_n]^{<\omega} \subseteq \mathcal{F}_n\} \text{ and}$$

$$I_2 = \{m_n \in I : \mathcal{L}(m_n) \cup [L_n]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}_n\} \,.$$

Since $I = I_1 \cup I_2$, either I_1 or I_2 is infinite. We will prove that $\mathcal{L} \cap [I_1]^{<\omega} \subseteq \mathcal{F}$; (analogously can be proved that $\mathcal{L} \cap [I_2]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$). Let $F \in \mathcal{L} \cap [I_1]^{<\omega}$. If $k = minF$, then $k = m_n$ for some $m_n \in I_1$. Since $F \in \mathcal{L}$ we have $F = \{m_n\} \cup s$ for some $s \in \mathcal{L}(m_n)$ with $\{m_n\} < s$ and since $F \in [I_1]^{<\omega} \subseteq [I]^{<\omega}$ and $I \cap (m_n, +\infty) \subseteq L_n$ (property (i)) we have that $s \in [L_n]^{<\omega}$. Hence

$$s \in \mathcal{L}(m_n) \cap [L_n]^{<\omega} \subseteq \mathcal{F}_n \quad (\text{since } m_n \in I_1),$$

and consequently

$$F = \{m_n\} \cup s \in \mathcal{F} \quad (\text{since } s \subseteq L_n \subseteq M_n) \,.$$

This finishes the proof of the claim.

Finally, we apply the claim for the family $\mathcal{F} \cap [M]^{<\omega}$ and the ξ-uniform family $\mathcal{L} \cap [M]^{<\omega}$ on M.

We now come to the main result of this section, namely the proof that every family $\mathcal{A}_\xi$, for $\xi < \omega_1$, is ξ-uniform on $\mathbb{N}$ (and hence of Ramsey character).

Theorem 1.10 (i) $\mathcal{A}_\xi$ is a ξ-uniform family on $\mathbb{N}$ for every $\xi < \omega_1$.
(ii) $\mathcal{B}_\alpha$ is an ω^α-uniform family on $\mathbb{N}$ for every $\alpha < \omega_1$.

Proof (i) Firstly, we will prove, by recursion on ξ, that

$$\mathcal{A}_\xi(n) = \mathcal{A}_{\xi(n)} \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega} \quad \text{for every} \quad n \in \mathbb{N} \ ,$$

where $\xi(n) = \xi - 1$ for every $n \in \mathbb{N}$ in case ξ is a susseccor ordinal and $(\xi(n), n \in \mathbb{N})$ is a strictly increasing sequence of ordinals with $sup_n \xi(n) = \xi$, in case ξ is a limit ordinal.

(1) [**Case** $\xi = 1$] For every $n \in \mathbb{N}$ we have

$$\begin{aligned}\mathcal{A}_1(n) &= \{s \subseteq \mathbb{N} : \{n\} < s \ \text{ and } \{n\} \cup s \in \mathcal{A}_1\} \\ &= \{s \subseteq \mathbb{N} : \{n\} < s \text{ and } \ \{n\} \cup s \in [\mathbb{N}]^1\} \\ &= \{\emptyset\} = \mathcal{A}_0 \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega}.\end{aligned}$$

(2) [**Case** $\xi = \zeta + 1$] For every $n \in \mathbb{N}$ we have

$$\begin{aligned}\mathcal{A}_\xi(n) = \mathcal{A}_{\zeta+1}(n) &= \{s \subseteq \mathbb{N} : \{n\} < s \ \text{ and } \{n\} \cup s \in \mathcal{A}_{\zeta+1}\} \\ &= \{s \subseteq \mathbb{N} : \{n\} < s \ \text{ and } s \in \mathcal{A}_\zeta\} = \mathcal{A}_\zeta \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega} \ .\end{aligned}$$

(3) [**Case** $\xi = \omega^{\beta+1}$ for $0 \leq \beta < \omega_1$] For every $n \in \mathbb{N}$ we have

$$\begin{aligned}\mathcal{A}_\xi(n) &= \mathcal{A}_{\omega^{\beta+1}}(n) = \{s \subseteq \mathbb{N} : \{n\} < s \text{ and } \{n\} \cup s \in \mathcal{A}_{\omega^{\beta+1}}\} \\ &= \{s \subseteq \mathbb{N} : \{n\} < s, \{n\} \cup s = \bigcup_{i=1}^{n} s_i \ , s_1 < \ldots < s_n \text{ and } s_1, \ldots, s_n \in \mathcal{A}_{\omega^\beta}\} \\ &= \{s \subseteq \mathbb{N} : \{n\} < s, s = s_0 \cup (\bigcup_{i=2}^{n} s_i) \text{ with } s_0 \in \mathcal{A}_{\omega^\beta}(n), s_2, \ldots, s_n \in \mathcal{A}_{\omega^\beta} \\ &\quad \text{and } s_0 < s_2 < \ldots < s_n\}\end{aligned}$$

So, according to the induction hypothesis,

$$\begin{aligned}\mathcal{A}_\xi(n) = \{s \subseteq \mathbb{N} : s = s_0 \cup (\bigcup_{i=2}^{n} s_i) &\text{ with } s_0 \in \mathcal{A}_{\omega^\beta(n)} \cap [\mathbb{N} \cap (n + \infty)]^{<\omega}, \\ &s_0 < s_2 < \ldots < s_n \text{ and } s_2, \ldots, s_n \in \mathcal{A}_{\omega^\beta}\} \\ &= \mathcal{A}_{(n-1)\omega^\beta + \omega^\beta(n)} \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega} \ .\end{aligned}$$

Hence, $\xi(n) = \omega^{\beta+1}(n) = (n-1)\omega^\beta + \omega^\beta(n)$ for every $n \in \mathbb{N}$ and obviously $sup_n \ \xi(n) = \omega^{\beta+1}$. We note that in case $\xi = \omega$ we have $\xi(n) = n - 1$, since $\omega^0 = 1$.

(4) **[Case** $\xi = \omega^\lambda$ for λ non-zero, countable limit ordinal] For every $n \in \mathbb{N}$ we have

$$\mathcal{A}_\xi(n) = \mathcal{A}_{\omega^\lambda}(n) = \{s \subseteq \mathbb{N} : \{n\} < s \text{ and } \{n\} \cup s \in \mathcal{A}_{\omega^\lambda}\}$$
$$= \{s \subseteq \mathbb{N} : \{n\} < s \text{ and } \{n\} \cup s \in \mathcal{A}_{\omega^{\lambda_n}}\} ,$$

where (λ_n) is the sequence of successor ordinals converging to λ fixed in the definition of the system $(\mathcal{A}_\xi)_{\xi<\omega_1}$. Hence,

$$\mathcal{A}_\xi(n) = \{s \subseteq \mathbb{N} : s \in \mathcal{A}_{\omega^{\lambda_n}}(n)\} = \mathcal{A}_{\omega^{\lambda_n}(n)} \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega} ,$$

according to the induction hypothesis. If $\lambda_n = \mu_n + 1$ for every $n \in \mathbb{N}$, then

$$\xi(n) = \omega^\lambda(n) = \omega^{\lambda_n}(n) = \omega^{\mu_n+1}(n) = (n-1)\omega^{\mu_n} + \omega^{\mu_n}(n) \text{ for every } n \in \mathbb{N} .$$

Of course $sup_n \xi(n) = \omega^\lambda$, since $\omega^{\mu_n} \leq (n-1)\omega^{\mu_n} \leq (n-1)\omega^{\mu_n} + \omega^{\mu_n}(n)$.

(5) **[Case** ξ limit, $\omega^\alpha < \xi < \omega^{\alpha+1}$ for some $0 < \alpha < \omega_1$] In this case, according to Proposition 1.2, ξ has a unique representation of ordinals as follows: $\xi = p\omega^\alpha + \sum_{i=1}^{m} p_i\omega^{\alpha_i}$, where $m \in \mathbb{N}$, $\alpha > \alpha_1 > \ldots > \alpha_m > 0$ are ordinal numbers and $p, p_1, \ldots p_m \geq 1$ are natural numbers, so that either $p > 1$ or $p = 1$ and $m \geq 1$.

For simplicity we examine firstly the case $m = 0$, namely the case $\xi = p\omega^\alpha$ for $p > 1$. For every $n \in \mathbb{N}$ we have

$$\mathcal{A}_\xi(n) = \mathcal{A}_{p\omega^\alpha}(n) = \{s \subseteq \mathbb{N} : \{n\} < s \text{ and } \{n\} \cup s \in \mathcal{A}_{p\omega^\alpha}\} =$$
$$= \{s \subseteq \mathbb{N} : \{n\} < s \text{ and } \{n\} \cup s = \bigcup_{i=1}^{p} s_i \text{ with } s_1 < \ldots < s_p$$
$$\text{and } s_i \in \mathcal{A}_{\omega^\alpha} \text{ for } 1 \leq i \leq p\}$$
$$= \{s \subseteq \mathbb{N} : s = s_0 \cup (\bigcup_{i=2}^{p} s_i) \text{ with } \{n\} < s_0 < s_2 < \ldots < s_p ,$$
$$s_0 \in \mathcal{A}_{\omega^\alpha}(n) \text{ and } s_2, \ldots, s_p \in \mathcal{A}_{\omega^\alpha}\}$$
$$= \{s \subseteq \mathbb{N} : s = s_0 \cup (\bigcup_{i=2}^{p} s_i) \text{ with } \{n\} < s_0 < s_2 < \ldots < s_p,$$
$$s_0 \in \mathcal{A}_{\omega^\alpha(n)} \text{ and } s_2, \ldots, s_p \in \mathcal{A}_{\omega^\alpha}\}$$
$$= \mathcal{A}_{(p-1)\omega^\alpha + \omega^\alpha(n)} \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega} .$$

Thus $\xi(n) = p\omega^\alpha(n) = (p-1)\omega^\alpha + \omega^\alpha(n)$ for every $n \in \mathbb{N}$. Of course, $sup_n \xi(n) = p\omega^\alpha = \xi$, since $sup_n \omega^\alpha(n) = \omega^\alpha$.

Now, let $m \geq 1$. In this case $\xi = \beta + p_m\omega^{\alpha_m}$, where $\beta = p\omega^{\alpha} + \sum_{i=1}^{m-1} p_i\omega^{\alpha_i}$. Of course $\beta < \xi$. Then for every $n \in \mathbb{N}$ we have

$$\begin{aligned}
\mathcal{A}_\xi(n) &= \{s \subseteq \mathbb{N} : \{n\} < s \text{ and } \{n\} \cup s \in \mathcal{A}_\xi\} \\
&= \{s \subseteq \mathbb{N} : \{n\} < s, \{n\} \cup s = s_1 \cup s_2,\ s_1 < s_2,\ s_1 \in \mathcal{A}_{p_m}\omega^{\alpha_m} \text{ and } s_2 \in \mathcal{A}_\beta\} \\
&= \{s \subseteq \mathbb{N} : s = s_0 \cup s_2 \text{ with } \{n\} < s_0 < s_2\ , s_0 \in \mathcal{A}_{p_m}\omega^{\alpha_m}(n) \text{ and } s_2 \in \mathcal{A}_\beta\} \\
&= \{s \subseteq \mathbb{N} : s = s_0 \cup s_2 \text{ with } \{n\} < s_0 < s_2,\ s_0 \in \mathcal{A}_{p_m}\omega^{\alpha_m}(n) \text{ and } s_2 \in \mathcal{A}_\beta\} \\
&= \mathcal{A}_{\beta + p_m\omega^{\alpha_m}(n)} \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega}\ .
\end{aligned}$$

Hence, $\xi(n) = \beta + p_m\omega^{\alpha_m}(n) = p\omega^{\alpha} + \sum_{i=1}^{m-1} p_i\omega^{\alpha_i} + (p_m - 1)\omega^{\alpha} + \omega^{\alpha}(n)$ for every $n \in \mathbb{N}$. Of course, $sup_n\ \xi(n) = \xi$.

This finishes the proof of our claim.

In order to prove that $\mathcal{A}_\xi$ is a ξ-uniform family on $\mathbb{N}$ for every $\xi < \omega_1$ we will use induction on ξ. Indeed, $\mathcal{A}_0 = \{\emptyset\}$ is 0-uniform on $\mathbb{N}$. Let $1 < \xi < \omega_1$. We asume that $\mathcal{A}_\zeta$ is ζ-uniform on $\mathbb{N}$ for every $\zeta < \xi$. According to our claim, $\mathcal{A}_\xi(n) = \mathcal{A}_{\xi(n)} \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega}$ for every $n \in \mathbb{N}$, where $\xi(n) = \xi - 1$ if ξ is a successor ordinal and $(\xi(n), n \in \mathbb{N})$ is a strictly increasing sequence of ordinals with $sup_n\ \xi(n) = \xi$ if ξ is a limit ordinal. So, $\mathcal{A}_\xi(n)$ is $\xi(n)$-uniform family on $\mathbb{N} \cap (n, +\infty)$ for every $n \in \mathbb{N}$ (Remark 1.8 (ii)). Hence, $\mathcal{A}_\xi$ is ξ-uniform family on $\mathbb{N}$.

This finishes our proof.

(ii) It follows from (i), since $\mathcal{B}_\alpha = \mathcal{A}_{\omega^\alpha}$ for every $\alpha < \omega_1$.

Proof of Theorem 1.6 (ξ- Ramsey type theorem). It follows immediately from Theorem 1.10 and Proposition 1.9.

An equivalent formulation of the ξ-Ramsey type theorem is the following:

Corollary 1.11 Let M be an infinite subset of $\mathbb{N}$, $\{P_1, \ldots, P_n\}$ a finite partition of $[M]^{<\omega}$ and ξ a countable ordinal number. Then there exists $L \in [M]$ and $i \in \{1, \ldots, n\}$ such that

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subset P_i\ .$$

Proof It follows by induction on n, using the ξ-Ramsey type theorem (Theorem 1.6).

As a corollary of the ξ-Ramsey type theorem we can state a condition in order a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ to contain a copy of the family $\mathcal{A}_\xi$ for $\xi < \omega_1$.

Corollary 1.12 Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$, $M \in [\mathbb{N}]$ and ξ a countable ordinal. If $\mathcal{A}_\xi \cap \mathcal{F} \cap [I]^{<\omega} \neq \emptyset$ for every $I \in [M]$, then there exists $L \in [M]$ such that

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}.$$

Proof According to the ξ- Ramsey type theorem (Theorem 1.6) there exists $L \in [M]$ such that either $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$ or $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$. But from our hypothesis the second case is impossible, so $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$.

2. Auxilliary properties of the complete thin Schreier system

In order to prove in Section 3 below, the main ξ-Ramsey type theorem, which gives an effective criterion for deciding in which of the two partition classes the "large homegenous" family falls, we need to look more carefully in thin Schreier families $\mathcal{A}_\xi$, $\xi < \omega_1$.

Besides the thin property of $\mathcal{A}_\xi$, we establish two main properties:

(a) The (unique) canonical representation of every finite subset of $\mathbb{N}$, with respect to each family $\mathcal{A}_\xi$, $\xi < \omega_1$ (in terms of a finite set of elements of $\mathcal{A}_\xi$ and a small remainder) (Theorem 2.5); and

(b) the fact that the strong Cantor-Bendixson index of $\mathcal{A}_\xi$ (given in Definition 2.8 below) is precisely $\xi + 1$; and that this index is stable if $\mathcal{A}_\xi$ is restricted to $\mathcal{A}_\xi \cap [M]^{<\omega}$ for any infinite subset M of $\mathbb{N}$ (Theorem 2.16).

Properties (a) and (b) can indeed be shown, with no extra effort, for general ξ- uniform families.

Firstly we give some definitions which we will use below.

Definition 2.1 Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$.

(i) $\mathcal{F}$ is **thin** if there are no elements $s, t \in \mathcal{F}$ with s a proper initial segment (in the order of the natural numbers) of t.

(ii) $\mathcal{F}^\star = \{t \in [\mathbb{N}]^{<\omega} : t \text{ is an initial segment of some } s \in \mathcal{F}\}$.

(iii) $\mathcal{F}_\star = \{t \in [\mathbb{N}]^{<\omega} : t \text{ is a subset of some } s \in \mathcal{F}\}$.

(iv) $\mathcal{F}$ is **hereditary** if $\mathcal{F}_\star = \mathcal{F}$.

Proposition 2.2 Every family $\mathcal{A}_\xi$ for $\xi < \omega_1$, is thin.

Proof It follows from Theorem 1.10 and Remark 1.8 (iv).

In the following we will prove that every finite subset of $\mathbb{N}$ has a canonical representation with respect to a ξ-uniform family, for $\xi < \omega_1$.

Definition 2.3 Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$. A non-empty, finite subset F of $\mathbb{N}$ has **canonical representation** $R_{\mathcal{F}}(F) = \{s_1, \dots, s_n, s_{n+1}\}$ with type $t_{\mathcal{F}}(F) = n$ with respect to $\mathcal{F}$, if there exist unique $n \in \mathbb{N}$, $s_1, \dots, s_n \in \mathcal{F}$ and s_{n+1} a proper initial segment of some element of $\mathcal{F}$ with $s_1 < \dots < s_n < s_{n+1}$ and such that $F = \bigcup_{i=1}^{n+1} s_i$.

Proposition 2.4 Let M be an infinite subset of $\mathbb{N}, \xi$ a countable ordinal number and $\mathcal{L}$ a ξ-uniform family on M. Every non-empty, finite subset of M has canonical representation with respect to $\mathcal{L}$.

Proof We proceed by induction on ξ. Let $\xi = 1$. The only 1-uniform family on M, for some $M \in [\mathbb{N}]$, is the family $\{\{m\} : m \in M\}$. If $F = \{m_1, \dots, m_k\} \in [M]^{<\omega}$ with $m_1 < \dots < m_k$ and $1 \leq k$, then $R_{\mathcal{L}}(F) = \{\{m_1\}, \dots, \{m_k\}\}$ and $t_{\mathcal{L}}(F) = k$.

Assume that $1 < \xi$ and the assertion holds for every $\zeta < \xi$. Let $\mathcal{L}$ be a ξ-uniform family on M, for some $M \in [\mathbb{N}]$. Then there exists a sequence $(\xi(m), m \in M)$ of ordinal numbers smaller than ξ such that $\mathcal{L}(m)$ is $\xi(m)$–uniform family on $M \cap (m, +\infty)$ for every $m \in M$.

Firstly we will prove that for every $F \in [M]^{<\omega}, F \neq \emptyset$ there exist $n \in \mathbb{N}$, $s_1, \dots, s_n \in \mathcal{L}$, $s_{n+1} \in \mathcal{L}^\star \setminus \mathcal{L}$ with $s_1 < \dots < s_n < s_{n+1}$ such that $F = \bigcup_{i=1}^{n+1} s_i$. Indeed, let $F \in [M]^{<\omega}$, $F \neq \emptyset$. If $F \in \mathcal{L}^\star \setminus \mathcal{L}$, then we set $n = 0$ and $s_1 = F$ and if $F \in \mathcal{L}$ then we set $n = 1$ $s_1 = F$ and $s_2 = \emptyset$. Now we assume that $F \notin \mathcal{L}^\star$. If $m_1 = minF$, then $F = \{m_1\} \cup t^1$ with $\{m_1\} < t^1$ and $t^1 \neq 0$. Indeed, if $t^1 = \emptyset$, then $F = \{m_1\}$ and since $F = \{m_1\} \notin \mathcal{L}^\star$ we have that $\mathcal{L}(m_1) = \emptyset$, which is impossible, since $\mathcal{L}(m_1)$ is a $\xi(m_1)$-uniform family on $M \cap (m_1, +\infty)$ (Remark 1.8 (ii)); hence $t^1 \neq \emptyset$. According to the induction hypothesis, t^1 has canonical representation $R_{\mathcal{L}(m_1)}(t^1) = (t_1^1, \dots, t_{n_1+1}^1)$, with type n_1, with respect to the $\xi(m_1)$-uniform family $\mathcal{L}(m_1)$. If $t^1 = t_1^1 \in \mathcal{L}(m_1)^\star$ (case $n_1 = 0$ or $n_1 = 1$ and $t_2^1 = \emptyset$), then $F = \{m_1\} \cup t^1 \in \mathcal{L}^\star$, which contrary to our assuption for F; hence, $t_1^1 \in \mathcal{L}(m_1)$ and $t_2^1 \neq \emptyset$. Since t_1^1 is a proper initial segment of t^1, the set $s_1 = \{m_1\} \cup t_1^1$ is a proper initial segment of F. Hence,

$$F = s_1 \cup F_1 \text{ with } F_1 \neq \emptyset \ , \quad s_1 < F_1 \text{ and } s_1 \in \mathcal{L} \ .$$

We continue analogously treating F_1 in place of F. In detail the argument goes as follows: If $F_1 \in \mathcal{L}^\star \setminus \mathcal{L}$ we set $s_2 = F_1$ and finally $F = s_1 \cup s_2$; if $F_1 \in \mathcal{L}$ then we set $s_2 = F_1$, $s_3 = \emptyset$ and $F = s_1 \cup s_2 \cup s_3$. If $F_1 \notin \mathcal{L}^\star$ then there exists $s_2 \in \mathcal{L}$ and $F_2 \in [M]^{<\omega}$ with $F_2 \neq \emptyset$ such that $F = s_1 \cup s_2 \cup F_2$ with $s_1 < s_2 < F_2$. We continue in the same way.

Secondly, we will prove that the previous representation of every finite subset of M with respect to $\mathcal{L}$ is unique. We will prove it by induction on n.

[Case $n = 0$] Let $F = s_1$ with $s_1 \in \mathcal{L}^\star \setminus \mathcal{L}$. If $F = \bigcup_{i=1}^{m+1} t_i$ with $t_1, \ldots, t_m \in \mathcal{L}$, $t_{m+1} \in \mathcal{L}^\star \setminus \mathcal{L}$ and $t_1 < \ldots < t_m < t_{m+1}$, then $m = 0$ and $t_1 = s_1$. Indeed, if $m \geq 1$, then $t_1 \in \mathcal{L}$ and also t_1 is a proper initial segment of some element of $\mathcal{L}$ since $F \in \mathcal{L}^\star \setminus \mathcal{L}$. But this is impossible, since $\mathcal{L}$ is thin family. Hence $m = 0$ and $t_1 = F = s_1$.

Let $k > 1$. We assume that the assertion holds for $n = k$.

[Case $n = k + 1$] Let $F = \bigcup_{i=1}^{k+2} s_i$ with $s_1, \ldots, s_{k+1} \in \mathcal{L}$, $s_{k+2} \in \mathcal{L}^\star \setminus \mathcal{L}$ and $s_1 < \ldots < s_{k+1} < s_{k+2}$. Also, let $F = \bigcup_{i=1}^{m+1} t_i$ with $t_1, \ldots, t_m \in \mathcal{L}$, $t_{m+1} \in \mathcal{L}^\star \setminus \mathcal{L}$ and $t_1 < \ldots < t_m < t_{m+1}$. Of course $s_1 \in \mathcal{L}$. We will prove that $m \geq 1$. Indeed, if $m = 0$, then $t_1 = F \in \mathcal{L}^\star \setminus \mathcal{L}$, and consequently s_1 is a proper initial segment of some element of $\mathcal{L}$. This is impossible, since $\mathcal{L}$ is thin; thus $m \geq 1$, which implies that $t_1 \in \mathcal{L}$. So, we have that $s_1, t_1 \in \mathcal{L}$ and that t_1, s_1 are initial segments of F. Since $\mathcal{L}$ is thin family the only possibility is $t_1 = s_1$. Set $F_1 = F \setminus s_1$; then according to the induction hypothesis, we have that $k = m - 1$ and that $t_2 = s_2, \ldots, t_m = s_{k+1}$ and $t_{m+1} = s_{k+2}$.

This finishes the proof of the proposition.

Theorem 2.5 (Canonical representation with respect to $\mathcal{A}_\xi$) Let ξ be a countable ordinal number. Every non-empty, finite subset of $\mathbb{N}$ has canonical representation with respect to the family $\mathcal{A}_\xi$.

Proof It follows immediately from the previous proposition,using the fact that each thin Schreier family $\mathcal{A}_\xi$, for $\xi < \omega_1$, is ξ- uniform on $\mathbb{N}$ (Theorem 1.10).

The principal use of the canonical representation of finite subsets of $\mathbb{N}$ with respect to $\mathcal{A}_\xi$, $\xi < \omega_1$, in Ramsey theory is contained in the following:

Proposition 2.6 Let ξ be a countable ordinal number. For every non-empty, finite set F of $\mathbb{N}$ exactly one of the following possibilites occurs:

either (i) F is a proper initial segment of some element of $\mathcal{A}_\xi$;

or (ii) there exists an element of $\mathcal{A}_\xi$ which is initial segment of F.

Proof Let $F \in [\mathbb{N}]^{<\omega}, F \neq \emptyset$. According to Theorem 2.5, the type $t_{\mathcal{A}_\xi}(F)$ of F with respect to the family $\mathcal{A}_\xi$ is either equal to zero on greater to zero. The case $t_{\mathcal{A}_\xi}(F) = 0$ gives equivalently (i) while the complementary case $t_{\mathcal{A}_\xi}(F) \geq 1$ gives equivalently (ii).

Corollary 2.7 Let ξ be a countable ordinal. If F is a proper initial segment of some element of $\mathcal{A}_\xi$, then for every $n \in \mathbb{N}$ with $F < \{n\}$ the set $F \cup \{n\}$ is also initial segment of some element of $\mathcal{A}_\xi$.

Proof Let $F \in \mathcal{A}_\xi^\star \setminus \mathcal{A}_\xi$ and $n \in \mathbb{N}$ with $F < \{n\}$. We set $A = F \cup \{n\}$. According to Proposition 2.6, either $A \in \mathcal{A}_\xi^\star \setminus \mathcal{A}_\xi$ or there exists $s \in \mathcal{A}_\xi$ which is initial segment of A. In the second case $A \in \mathcal{A}_\xi$. Indeed, if s is a proper initial segment of A, then $s \in \mathcal{A}_\xi$ and also is an initial segment of $F \in \mathcal{A}_\xi^\star \setminus \mathcal{A}_\xi$. A contradiction, since $\mathcal{A}_\xi$ is thin family; hence $s = A \in \mathcal{A}_\xi$.

In the following we will estimate the strong Cantor-Bendixson index of a uniform family.This index (see Definition 2.8 below) is analogous to the well-known Cantor-Bendixson index ([B],[C1]) and has been defined in [A-M-T]. Here, we will use a different notation in order to avoid some misinterpretations.

We will prove in Theorem 2.16 below, that the corresponding hereditary family of the thin Schreier family $\mathcal{A}_\xi$, for $\xi < \omega_1$ has strong Cantor-Bendixson index equal to $\xi + 1$, moreover for every $M \in [\mathbb{N}]$ the restricted family $(\mathcal{A}_\xi \cap [M]^{<\omega})_\star$ has also index equal to $\xi + 1$.

This is the reason we have called $(\mathcal{A}_\xi)_{\xi<\omega_1}$ **complete system.**

Definition 2.8 ([A-M-T]) Let $\mathcal{F}$ be a hereditary and pointwise closed family of finite subsets on $\mathbb{N}$. For $M \in [\mathbb{N}]$ we define the **strong Cantor-Bendixson derivatives** $(\mathcal{F})_M^\xi$ of $\mathcal{F}$ on M for every $\xi < \omega_1$ as follows:

$$(\mathcal{F})_M^1 = \{F \in \mathcal{F}[M] : F \text{ is a cluster point of } \mathcal{F}[F \cup L] \text{ for each } L \in [M]\};$$

(where, $\mathcal{F}[M] = \mathcal{F} \cap [M]^{<\omega}$).

If $(\mathcal{F})^\xi_M$ has been defined, then

$$(\mathcal{F})^{\xi+1}_M = \left((\mathcal{F})^\xi_M\right)^1_M .$$

If ξ is a limit ordinal and $(\mathcal{F})^\beta_M$ have been defined for each $\beta < \xi$, then

$$(\mathcal{F})^\xi_M = \bigcap_{\beta<\xi} (\mathcal{F})^\beta_M .$$

The **strong Cantor-Bendixson index of $\mathcal{F}$ on M** is defined to be the smallest countable ordinal ξ such that $(\mathcal{F})^\xi_M = \emptyset$. We denote this index by $s_M(\mathcal{F})$.

Remark 2.9 (i) The strong Cantor-Bendixson index $s_M(\mathcal{F})$ of a hereditary and pointwise closed family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ on some $M \in [\mathbb{N}]$ is a successor countable ordinal and is less or equal to the "usual" Cantor-Bendixson index $O(\mathcal{F})$ of $\mathcal{F}$ (see [K]).
(ii) If $\mathcal{F}_1, \mathcal{F}_2 \subseteq [\mathbb{N}]^{<\omega}$ are hereditary and closed families with $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then

$$s_M(\mathcal{F}_1) \leq s_M(\mathcal{F}_2) \text{ for every } M \in [\mathbb{N}] .$$

(iii) $s_M(\mathcal{F}) = s_M(\mathcal{F} \cap [M]^{<\omega})$ for every $M \in [\mathbb{N}]$.
(iv) For every $M \in [\mathbb{N}]$ and $F \in [M]^{<\omega}$, according to a remark in [J], we have :

$$F \in (\mathcal{F})^1_M \text{ if and only if the set } \{m \in M : F \cup \{m\} \notin \mathcal{F}\} \text{ is finite} .$$

(v) Using the previous remark (iv), can be proved by induction that for every $L \in [M]$ and $\xi < \omega_1$

$$\text{if } A \in (\mathcal{F})^\xi_M, \text{ then } F \cap L \in (\mathcal{F})^\xi_L .$$

Hence, $s_L(\mathcal{F}) \geq s_M(\mathcal{F})$. (see also [A-M-T]).
(vi) If L is almost contained in M, then

$$s_L(\mathcal{F}) \geq s_M(\mathcal{F}) .$$

In the following we will give the precise relation between the strong Cantor-Bendixson derivatives of the corresponding hereditary family $\mathcal{L}_\star$ of a given family $\mathcal{L} \subseteq [\mathbb{N}]^{<\omega}$ and the derivatives of the corresponding families $(\mathcal{L}(n))_\star$ for every $n \in \mathbb{N}$. After that, we will calculate the strong Cantor-Bendixson index of a uniform family.

Lemma 2.10 Let ξ be a countable ordinal, $M \in [\mathbb{N}]$ and $\mathcal{L}$ a family of finite subsets of $\mathbb{N}$ such that $\mathcal{L}_\star$ and $\mathcal{L}(n)_\star$ are closed for every $n \in \mathbb{N}$. If $F \in \left(\mathcal{L}(n)_\star\right)^\xi_M$ for some $n \in \mathbb{N}$, then $\{n\} \cup F \in (\mathcal{L}_\star)^\xi_M$.

Proof We use induction on ξ. Let $n \in \mathbb{N}$, $M \in [\mathbb{N}]$ and $F \in \left(\mathcal{L}(n)_\star\right)^1_M$. Since

$$\{m \in M : F \cup \{m\} \in \mathcal{L}(n)_\star\} \subseteq \{m \in M : F \cup \{m\} \cup \{n\} \in \mathcal{L}_\star\} ,$$

we have, according to Remark 2.9 (iv), that $F \cup \{n\} \in (\mathcal{L}_\star)^1_M$.

Let $1 < \xi$. Suppose that the assertion holds for all ordinals ζ with $\zeta < \xi$. If $F \in (\mathcal{L}(n)_\star)^{\zeta+1}_M$, then, according to the induction hypothesis,

$$\{m \in M : F \cup \{m\} \in (\mathcal{L}(n)_\star)^\zeta_M\} \subseteq \{m \in M : F \cup \{m\} \cup \{n\} \in (\mathcal{L}_\star)^\zeta_M\} .$$

Hence $\{n\} \cup F \in (\mathcal{L}_\star)^{\zeta+1}_M$ (Remark 2.9 (iv)).

The case where ξ is limit ordinal is trivial.

Proposition 2.11 Let $\mathcal{L} \subseteq [\mathbb{N}]^{<\omega}$ such that $\mathcal{L}_\star$ and $\mathcal{L}(n)_\star$ are pointwise closed for every $n \in \mathbb{N}$ and $M \in [\mathbb{N}]$.

(i) If there exists $L \in [M]$ such that $s_M(\mathcal{L}(n)_\star) = \xi$ for every $n \in L$, then

$$s_L(\mathcal{L}_\star) \geq \xi + 1 .$$

(ii) Let $\xi_n = s_M(\mathcal{L}(n)_\star)$ for every $n \in \mathbb{N}$. If there exists $L \in [M]$ such that $\xi = sup_{n\in L}\ \xi_n < \omega_1$ and $\xi_n < \xi$ for every $n \in L$, then

$$s_M(\mathcal{L}_\star) \geq \xi + 1 .$$

Proof (i) Let $L \in [M]$ such that $s_M(\mathcal{L}(n)_\star) = \xi = \zeta + 1$ for every $n \in L$ (Remark 2.9(i)). Then $\emptyset \in (\mathcal{L}(n)_\star)^\zeta_M$ for every $n \in L$. According to Lemma 2.10, $\{n\} \in (\mathcal{L}_\star)^\zeta_M$ for every $n \in L$. So, $\{n\} \in (\mathcal{L}_\star)^\zeta_L$ for every $n \in L$ (by Remark 2.9 (v)). Thus $\emptyset \in (\mathcal{L}_\star)^{\zeta+1}_L$ (Remark 2.9(iv)) and therefore $s_L(\mathcal{L}_\star) \geq \zeta + 2 = \xi + 1$.

(ii) In this case ξ is a limit ordinal. We set $\xi_n = \zeta_n + 1$ for every $n \in \mathbb{N}$ (Remark 2.9 (i)). According to our hypothesis $\emptyset \in (\mathcal{L}(n)_\star)^{\zeta_n}_M$ for every $n \in L$. Hence $\emptyset \in (\mathcal{L}_\star)^{\zeta_n}_M$ for every $n \in L$ (Lemma 2.10). Since $sup_{n\in L}\ \zeta_n = \xi$ and $\zeta_n < \xi$,we have that $\emptyset \in (\mathcal{L}_\star)^\xi_M$ and therefore $s_M(\mathcal{L}_\star) \geq \xi + 1$, as required.

Lemma 2.12 Let ξ be a countable ordinal and $\mathcal{L} \subseteq [\mathbb{N}]^{<\omega}$ such that $\mathcal{L}_\star$ and $\mathcal{L}(n)_\star$ are closed for every $n \in \mathbb{N}$. If $F \neq \emptyset$ and $F \in (\mathcal{L}_\star)^\xi_M$ for some $M \in [\mathbb{N}]$, then there exist $l \in \mathbb{N}$ with $l \leq minF$ and $L \in [M]$ such that $F \setminus \{l\} \in (\mathcal{L}(l)_\star)^\xi_L$.

Proof We use induction on ξ. Let $F = \emptyset$ and $F \in (\mathcal{L}_\star)^1_M$. According to Remark 2.9 (iv) the set $M_F = \{m \in M : F \cup \{m\} \in \mathcal{L}_\star$ with $minF \leq m\}$ is almost equal to M. For each $m \in M_F$ there exists $s_m \in \mathcal{L}$ such that $F \cup \{m\} \subseteq s_m$. Of course $1 \leq mins_m \leq minF$ for every $m \in M_F$. Set

$$l = min\{n \in \mathbb{N} : \text{ the set } \{m \in M_F : min\ s_m = n\} \text{ is infinite}\}; \text{ and}$$

$$L = \{m \in M_F : \ min\ s_m = l\} \cup F\ .$$

Then, $\{l\} \leq F$, $L \in [M]$ and $F \setminus \{l\} \in (\mathcal{L}(l)_\star)^1_L$, as required.

Suppose now that the assertion holds for all ordinals β with $\beta < \xi$. Firstly we examine the case $\xi = \zeta + 1$. Let $F \neq \emptyset$ and $F \in (\mathcal{L}_\star)^{\zeta+1}_M$. According to Remark 2.9 (iv), the set $M_F = \{m \in M : F \cup \{m\} \in (\mathcal{L}_\star)^\zeta_M$ and $minF \leq m\}$ is almost equal to M. Let $m_1 = minM_F$. By the induction hypothesis there exist $l_1 \in \mathbb{N}$ with $l_1 \leq minF$ and $L_1 \in [M_F]$ such that $F \cup \{m_1\} \setminus \{l_1\} \in \left(\mathcal{L}(l_1)_\star\right)^\zeta_{L_1 \cup F}$, since $F \cup \{m_1\} \in (\mathcal{L}_\star)^\zeta_{M_F \cup F}$ (Remark 2.9 (v)). Choose $m_2 \in L_1$ and $m_2 > m_1$. Since $F \cup \{m_2\} \in (\mathcal{L}_\star)^\zeta_{L_1 \cup F}$,there exist $l_2 \in \mathbb{N}$ with $l_2 \leq minF$ and $L_2 \in [L_1]$ such that $F \cup \{m_2\} \setminus \{l_2\} \in \left(\mathcal{L}(l_2)_\star\right)^\zeta_{L_2 \cup F}$. We continue analogously choosing $m_3 \in L_2$ with $m_3 > m_2$ and so on.

Hence, we construct an increasing sequence $(m_i)_{i=1}^\infty$ in M_F, a sequence $(l_i)_{i=1}^\infty$ in $\mathbb{N}$, with $1 \leq l_i \leq minF$ for every $i \in \mathbb{N}$, and a decreasing sequence $(L_i)_{i=1}^\infty$ in $[M_F]$ such that, for every $i \in \mathbb{N}$

$$F \cup \{m_i\} \setminus \{l_i\} \in (\mathcal{L}(l_i)_\star)^\zeta_{L_i \cup F}\ .$$

Let $l \in \mathbb{N}$ with $1 \leq l \leq minF$ such that the set $I = \{i \in \mathbb{N} : l_i = l\}$ is infinite. Set $L = \{m_i : i \in I\} \cup F$. Then, $F \setminus \{l\} \in (\mathcal{L}(l)_\star)^{\zeta+1}_L$, as required.

In the case where ξ is a limit ordinal we fix a strictly increasing sequence $(\zeta_i)_{i=1}^\infty$ of ordinals with $\zeta_i < \xi$ for every $i \in \mathbb{N}$ and $sup_i\ \zeta_i = \xi$. Let $F \in (\mathcal{L}_\star)^\xi_M$, $F \neq \emptyset$. Then $F \in (\mathcal{L}_\star)^{\zeta_i}_M$ for every $i \in \mathbb{N}$. According to the induction hypothesis, there exist $l_1 \in \mathbb{N}$ with $l_1 \leq minF$ and $L_1 \in [M \cap (minF, +\infty)]$ such that $F \setminus \{l_1\} \in (\mathcal{L}(l_1)_\star)^{\zeta_1}_{L_1 \cup F}$. Since $F \in (\mathcal{L}_\star)^{\zeta_2}_{L_2 \cup F}$ there exists $l_2 \in \mathbb{N}$ with $l_2 \leq minF$ and $L_2 \in [L_1]$ such that $L_2 \neq L_1$ and $F \setminus \{l_2\} \in (\mathcal{L}(l_2)_\star)^{\zeta_2}_{L_2 \cup F}$.

In this way, we construct a sequence $(l_i)_{i=1}^\infty$ with $1 \leq l_i \leq minF$ and a strictly decreasing sequence $(L_i)_{i=1}^\infty$ in $[M]$ such that

$$F \setminus \{l_i\} \in (\mathcal{L}(l_i)_\star)^{\zeta_i}_{L_i \cup F}\ , \text{ for every } i \in \mathbb{N}\ .$$

Let $l \in \mathbb{N}$ with $1 \leq l \leq minF$ such that the set $I = \{i \in \mathbb{N} : l_i = l\}$ is infinite. Set $L = \{minL_i : i \in I\} \cup F$. Then $F \setminus \{l\} \in (\mathcal{L}(l)_\star)^{\zeta_i}_L$ for every $i \in I$.Since $sup_{i \in I}\ \zeta_i = \xi$, we have that $F \setminus \{l\} \in \left((\mathcal{L}(l)_\star\right)^\xi_L$.

This completes the proof.

Proposition 2.13 Let $\mathcal{L} \subseteq [\mathbb{N}]^{<\omega}$ such that $\mathcal{L}_\star$ and $\mathcal{L}(n)_\star$ are pointwise closed for every $n \in \mathbb{N}$ and $M \in [\mathbb{N}]$. If $\xi = sup\{s_L(\mathcal{L}(n)_\star) : n \in \mathbb{N}$ and $L \in [M]\}$, then $(\mathcal{L}_\star)^\xi_M \subseteq \{\emptyset\}$ and therefore $s_M(\mathcal{L}_\star) \leq \xi + 1$.

Proof Let $F \in (\mathcal{L}_\star)^\xi_M$ and $F \neq \emptyset$.According to Lemma 2.12 there exist $n \in \mathbb{N}$ and $L \in [M]$ such that $F \setminus \{n\} \in (\mathcal{L}(n)_\star)^\xi_L$. Hence, $s_L(\mathcal{L}(n)_\star) \geq \xi + 1$. A contradiction; which finishes the proof.

Lemma 2.14 Let $M \in [\mathbb{N}]$ and $\mathcal{L}$ a ξ-uniform family on M, for some $\xi < \omega_1$. Then (i) $\mathcal{L}_\star$ is pointwise closed; and

(ii) $(\mathcal{L}(m))_\star$ are pointwise closed for every $m \in M$.

Proof (i) This is easily proved by induction on ξ.

(ii) It follows from (i), since for every $m \in M$ the family $\mathcal{L}(m)$ is $\xi(m)$-uniform on $M \cap (m, +\infty)$ for some $\xi(m) < \xi$.

Theorem 2.15 (Cantor-Bendixson index of a uniform family). Let ξ be a countable ordinal, $M \in [\mathbb{N}]$ and $\mathcal{L}$ a ξ-uniform family on M. For every $L \in [M]$ we have $(\mathcal{L}_\star)^\xi_L = \{\emptyset\}$ and $s_L(\mathcal{L}_\star) = \xi + 1$.

Proof We use induction on ξ. Let $\xi = 1$. Then $\mathcal{L}_\star = \{\{n\} : n \in M\} \cup \{\emptyset\}$. Hence $(\mathcal{L}_\star)^1_L = \{\emptyset\}$ and therefore $s_L(\mathcal{L}_\star) = 2$ for every $L \in [M]$.

Suppose that $\xi > 1$ and the assertion holds for every ordinal number ζ with $\zeta < \xi$. Let $\xi = \zeta + 1$, and $\mathcal{L}$ be a ξ-uniform family on M, for some $M \in [\mathbb{N}]$. Then $\mathcal{L}(n)$, is ζ-uniform on $M_n = M \cap (n, +\infty)$ for every $n \in M$. According to the induction hypothesis $s_L(\mathcal{L}(n)_\star) = \zeta + 1 = \xi$ for every $L \in [M]$ and $n \in M$ (cf. Remark 2.9 (vi)). From Proposition 2.11 (i) we have that $s_L(\mathcal{L}_\star) \geq \xi + 1$ for every $L \in [M]$ and from Proposition 2.13 we have $(\mathcal{L}_\star)^\xi_L \subseteq \{\emptyset\}$ for every $L \in [M]$. Hence $(\mathcal{L}_\star)^\xi_L = \{\emptyset\}$ and $s_L(\mathcal{L}_\star) = \xi + 1$ for every $L \in [M]$.

Let ξ be a non-zero, limit ordinal and $\mathcal{L}$ a ξ-uniform family on M, for some $M \in \mathbb{N}$. Then $\mathcal{L}(n)$ is $\xi(n)$-uniform on $M \cap (n, +\infty)$ for every $n \in M$, where $(\xi(n))$ is a sequence of ordinals smaller than ξ with $sup_{n \in M}\ \xi(n) = \xi$.According to the induction hypothesis and Remark 2.9 (vi) we have that $s_L(\mathcal{L}(n)_\star) = \xi(n) + 1$ for every $L \in [M]$ and $n \in M$. From Proposition 2.11 (ii) we get that $s_L(\mathcal{L}_\star) \geq \xi + 1$ for every $L \in [M]$ and from Proposition 2.13 that $(\mathcal{L}_\star)^\xi_L \subseteq \{\emptyset\}$ for every $L \in [M]$. Hence $(\mathcal{L}_\star)^\xi_L = \{\emptyset\}$ and therefore $s_L(\mathcal{L}_\star) = \xi + 1$ for every $L \in [M]$.

The proof is complete.

Theorem 2.16 (**Cantor-Bendixson index of $\mathcal{A}_\xi$**)
(i) $s_M\big((\mathcal{A}_\xi)_*\big) = \xi + 1$ for every $\xi < \omega_1$, and $M \in [\mathbb{N}]$.
(ii) $s_M\big((\mathcal{B}_\alpha)_*\big) = \omega^\alpha + 1$ for every $\alpha < \omega_1$ and $M \in [\mathbb{N}]$.

Proof It follows from Theorem 1.10 and Theorem 2.15.

Corollary 2.17 Let M an infinite subset of $\mathbb{N}$. Then

$$s_L\big((\mathcal{A}_\xi \cap [M]^{<\omega})_*\big) = \xi + 1 \text{ for every } \xi < \omega_1 \text{ and } L \in [M] .$$

Proof The family $\mathcal{A}_\xi \cap [M]^{<\omega}$ is ξ-uniform on M, according to Theorem 1.10 and Remark 1.8 (iii). Theorem 2.15 finishes the proof.

3. The stronger form of the ξ-Ramsey type theorem

The aim of this section is to establish an improved version of the ξ-Ramsey type theorem, $\xi < \omega_1$, given in Theorem 1.6 above. The improvement is in the sense that, with the aid of the concept of the strong Cantor-Bendixson index, we have not only a ξ-Ramsey partition for each $\xi < \omega_1$, but in addition (under some additional mild conditions) a criterion for deciding whether the "homegenous" part will fall in the given family or in its complement. This criterion is useful in various applications.

We start with a stronger ξ-Ramsey type dichotomy that applies to a hereditary family (Theorem 3.2). The most satisfying and main result of this section, is for a family which is both hereditary and pointwise closed. (Theorem 3.9).

The results are recapitulated at the end of this section, for convenience.

Lemma 3.1 Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}, L$ an infinite subset of $\mathbb{N}$ and ξ a countable ordinal number. The following are equivallent:
(i) $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$; and
(ii) $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^* \setminus \mathcal{A}_\xi$.

Proof (i) $\Rightarrow$ (ii) Let $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ and $F \in \mathcal{F} \cap [L]^{<\omega}$. According to Proposition 2.6, either there exist $s \in \mathcal{A}_\xi$ such that F is a proper initial segment of s which gives that $F \in (\mathcal{A}_\xi)^* \setminus \mathcal{A}_\xi$, as required, or there exists $t \in \mathcal{A}_\xi$ such that t is an initial segment of F. The second case is impossible. Indeed, since $\mathcal{F}$ is a hereditary family and $F \in \mathcal{F} \cap [L]^{<\omega}$, we have $t \in \mathcal{F} \cap [L]^{<\omega}$. But $t \in \mathcal{A}_\xi$, so $t \in \mathcal{A}_\xi \cap [L]^{<\omega} \cap \mathcal{F}$. This contrary to our assumption that $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$.

(ii) $\Rightarrow$ (i) It is obvious.

Theorem 3.2 (**ξ-Ramsey type theorem for hereditary families**). Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}$, M an infinite subset of $\mathbb{N}$ and ξ a countable ordinal number.

Then there exists $L \in [M]$ such that

$$\text{either } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^* \setminus \mathcal{A}_\xi \ .$$

Proof According to the ξ-Ramsey type theorem (Theorem 1.6) there exists $L \in [M]$ such that

$$\text{either } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F} \text{ ; and}$$

according to Lemma 3.1, $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ if and only if $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^* \setminus \mathcal{A}_\xi$.

Corollary 3.3 Let ξ_1, ξ_2 be countable ordinal numbers with $\xi_1 < \xi_2$. For every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that

$$\mathcal{A}_{\xi_1} \cap [L]^{<\omega} \subseteq (\mathcal{A}_{\xi_2})^* \setminus \mathcal{A}_{\xi_2} \ .$$

Proof Let $\mathcal{F}$ be the corresponding hereditary family $(\mathcal{A}_{\xi_1})_*$ of $\mathcal{A}_{\xi_1}$. According to Theorem 3.2, for every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that

$$\text{either } \mathcal{A}_{\xi_2} \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_{\xi_2})^* \setminus \mathcal{A}_{\xi_2} \ .$$

The first alternative is impossible. Indeed, let $(\mathcal{A}_{\xi_2}) \cap [L]^{<\omega} \subseteq (\mathcal{A}_{\xi_1})_*$. Since $s_L((\mathcal{A}_{\xi_2} \cap [L]^{<\omega})_*) = \xi_2 + 1$ (Corollary 2.17) and $s_L((\mathcal{A}_{\xi_1})_*) = \xi_1 + 1$ (Theorem 2.16) we have

$$\xi_2 + 1 = s_L((\mathcal{A}_{\xi_2} \cap [L]^{<\omega})_*) \le s_L((\mathcal{A}_{\xi_1})_*) = \xi_1 + 1 \ .$$

A contradiction; hence $\mathcal{A}_{\xi_1} \cap [L]^{<\omega} \subseteq (\mathcal{A}_{\xi_2})^* \setminus \mathcal{A}_{\xi_2}$.

In the following, using Theorem 3.2, we indicate the close connection that exists between the generalized Schreier families $(\mathcal{F}_\alpha)_{\alpha<\omega_1}$ and the ω^α- thin Schreier families $\mathcal{A}_{\omega^\alpha} = \mathcal{B}_\alpha$ for $\alpha < \omega_1$. Firstly we will give the appropriate definitions.

Definition 3.4 (i) (Generalized Schreier families [S], [A-O],[A-A])
$\mathcal{F}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$;

$$\mathcal{F}_{\alpha+1} = \{F \subseteq \mathbb{N} : F = \bigcup_{i=1}^{k} F_i \ , \{k\} \le F_1 < \ldots < F_k, \text{ and } F_i \in \mathcal{F}_\alpha\} \cup \{\emptyset\} \ ;$$

If α is a limit ordinal choose and fix $(\alpha_n)_{n\in\mathbb{N}}$ strictly increasing to α and set
$\mathcal{F}_\alpha = \{F \subseteq \mathbb{N} : F \in \mathcal{F}_{\alpha_k} \text{ with } k \leq minF\} \cup \{\emptyset\}$.

(ii) For a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ and $L = (l_n)_{n=1}^\infty \in [\mathbb{N}]$ we set

$$\mathcal{F}(L) = \{(l_{n_1}, \ldots, l_{n_k}) \in [L]^{<\omega} : (n_1, \ldots, n_k) \in \mathcal{F}\} .$$

Corollary 3.5 Let α be a countable ordinal. For every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that

$$\mathcal{F}_\alpha(L) \subseteq (\mathcal{B}_\alpha)^\star \subseteq \mathcal{F}_\alpha .$$

Proof The family $\mathcal{F}_\alpha$ is hereditary, hence, according to Theorem 3.2, for every $M \in [\mathbb{N}]$ there exists $I \in [M]$ such that

$$\text{either } \mathcal{A}_{\omega^\alpha+1} \cap [I]^{<\omega} \subseteq \mathcal{F}_\alpha \text{ or } \mathcal{F}_\alpha \cap [I]^{<\omega} \subseteq (\mathcal{A}_{\omega^\alpha+1})^\star \setminus \mathcal{A}_{\omega^\alpha+1} .$$

In case, $\mathcal{A}_{\omega^\alpha+1} \cap [I]^{<\omega} \subseteq \mathcal{F}_\alpha$, using Corollary 2.17 and the fact that $s_I(\mathcal{F}_\alpha) = \omega^\alpha + 1$ (see [A-M-T]), we have

$$\omega^\alpha + 2 = s_I((\mathcal{A}_{\omega^\alpha+1} \cap [I]^{<\omega})_\star) \leq s_I(\mathcal{F}_\alpha) = \omega^\alpha + 1 .$$

A contradiction; hence $\mathcal{F}_\alpha \cap [I]^{<\omega} \subseteq (\mathcal{A}_{\omega^\alpha+1})^\star \setminus \mathcal{A}_{\omega^\alpha+1}$.

If $I = (i_n)_{n=1}^\infty$ we set $L = (i_n)_{n=3}^\infty = (l_n)_{n=1}^\infty$. We will prove that $\mathcal{F}_\alpha(L) \subseteq (\mathcal{B}_\alpha)^\star$. Indeed, if $(n_1, \ldots, n_k) \in \mathcal{F}_\alpha$, then $(n_1+1, n_1+2, \ldots, n_k+2) \in \mathcal{F}_\alpha$ and consequently $(i_{n_1+1}, i_{n_1+2}, \ldots, i_{n_k+2}) \in \mathcal{F}_\alpha \cap [I]^{<\omega}$ (for the properties of $\mathcal{F}_\alpha$ see [A-M-T]. This gives that $(i_{n_1+1}, l_{n_1}, \ldots l_{n_k}) \in (\mathcal{A}_{\omega^\alpha+1})^\star$ and consequently that $(l_{n_1}, \ldots, l_{n_k}) \in (\mathcal{A}_{\omega^\alpha})^\star$, as required.

R. Judd in [J] had provided, using Schreier games, that for every hereditary family $\mathcal{F}$ of finite subsets of $\mathbb{N}$, $\alpha < \omega_1$ and $M \in [\mathbb{N}]$, either there exists $L \in [M]$ such that $\mathcal{F}_\alpha(L) \subseteq \mathcal{F}$ or there exists $L \in [M]$ and $N \in [\mathbb{N}]$ such that $\mathcal{F} \cap [N]^{<\omega}(L) \subseteq \mathcal{F}_\alpha$.

As a corollary of Theorem 3.2 we will prove a stronger version of this result.

Corolalry 3.6 For every hereditary family $\mathcal{F}$ of finite subsets of $\mathbb{N}$, every countable ordinal α and $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that

$$\text{either } \mathcal{F}_\alpha(L) \subseteq \mathcal{F} \text{ or } \mathcal{F} \cap [L]^{<\omega} \subseteq \mathcal{F}_\alpha .$$

Proof According to Theorem 3.2 there exist $N \in [M]$ such that

$$\text{either } \mathcal{B}_\alpha \cap [N]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{F} \cap [N]^{<\omega} \subseteq (\mathcal{B}_\alpha)^\star \text{ ; and}$$

according to Corollary 3.5 and Proposition 2.4 there exists $L \in [N]$ such that

$$\mathcal{F}_\alpha(L) \subseteq (\mathcal{B}_\alpha)^* \cap [L]^{<\omega} \subseteq \left((\mathcal{B}_\alpha)_* \cap [N]^{<\omega}\right)^* \subseteq \mathcal{F} .$$

Hence, either $\mathcal{F}_\alpha(L) \subseteq \mathcal{F}$, or $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{B}_\alpha)^* \subseteq \mathcal{F}_\alpha$.

Since we will restrict to the hereditary families which in addition are closed in the pointwise topology we will give an elementary characterization of them.

Proposition 3.7 Let $\mathcal{F}$ be a non empty, hereditary family of finite subsets of $\mathbb{N}$. The following are equivalent:

(i) $\mathcal{F}$ is pointwise closed.

(ii) There does not exist an infinite sequence $(s_i)_{i=1}^\infty$ of elements of $\mathcal{F}$ with $s_1 \prec s_2 \prec \dots$.

(iii) There does not exist $M \in [\mathbb{N}]$ such that $[M]^{<\omega} \subseteq \mathcal{F}$.

Proof. (i) $\Rightarrow$ (ii) Let $(s_i)_{i=1}^\infty$ be a sequence in $\mathcal{F}$ with $s_1 \prec s_2 \prec \dots$. Obviously, $(s_i)_{i=1}^\infty$ converges pointwise to an infinite subset s of $\mathbb{N}$. Since $s \notin \mathcal{F}$, $\mathcal{F}$ is not closed.

(ii) $\Rightarrow$ (i) Let $(t_n)_{n=1}^\infty \subseteq \mathcal{F}$ converging pointwise to some subset t of $\mathbb{N}$.

(a) If t is finite, then $t \subseteq t_{n_0}$ for some $n_0 \in \mathbb{N}$. Since $\mathcal{F}$ is hereditary, $t \in \mathcal{F}$; as required.

(b) If $t = (n_1, n_2, \dots)$ with $n_1 < n_2 < \dots$,then we set $s_i = (n_1, n_2, \dots, n_i)$ for every $i \in \mathbb{N}$. Of course $s_1 \prec s_2 \prec \dots$. We observe that for every $i \in I$ the sequence $(t_n \cap [0, n_i])_{n=1}^\infty$ in $\mathcal{F}$ converges pointwise to s_i According to the case (a), $s_i \in \mathcal{F}$, for every $i \in \mathbb{N}$. A contradiction to condition (ii); so, t is finite and $t \in \mathcal{F}$.

(iii) $\Leftrightarrow$ (ii) It is easily proved.

Now, using the strong Cantor-Bendixson index defined for every hereditary and pointwise closed family we will state and prove the main dichotomy result (Theorem 3.9).

Lemma 3.8 Let $\mathcal{F}$ be a pointwise closed and hereditary family of finite subsets of $\mathbb{N}$ and $M \in [\mathbb{N}]$. For every countable ordinal ξ with $\xi + 1 < s_M(\mathcal{F})$ and $I \in [M]$ there exists $L \in [I]$ such that

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} .$$

Proof Let $I \in [M]$. According to Theorem 3.2, there exists $L \in [I]$ such that

$$\text{either } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^* \setminus \mathcal{A}_\xi \subseteq (\mathcal{A}_\xi)_* .$$

The second alternative is impossible. Indeed, let $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)_\star$. Using Theorem 2.16 and Remark 2.9 we have

$$s_M(\mathcal{F}) \leq s_L(\mathcal{F}) = s_L(\mathcal{F} \cap [L]^{<\omega}) \leq s_L((\mathcal{A}_\xi)_\star) = \xi + 1 ;$$

a contradiction to our assumption; hence $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$.

Theorem 3.9 (Refined ξ-Ramsey type theorem) Let $\mathcal{F}$ be a pointwise closed and hereditary family of finite subsets of $\mathbb{N}$, M an infinite subset of $\mathbb{N}$ and ξ a countable ordinal number. We set

$$\xi_M^{\mathcal{F}} = sup\ \{s_I(\mathcal{F}) : I \in [M]\} .$$

Then

(i) $\xi_M^{\mathcal{F}}$ is a countable ordinal.

(ii) If $\xi + 1 < \xi_M^{\mathcal{F}}$, then there exists $L \in [M]$ such that

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} .$$

(iii) If $\xi_M^{\mathcal{F}} < \xi + 1$, then there exists $L \in [M]$ such that

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F} ; \text{ and}$$

equivalently,

$$\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi .$$

(iv) If $\xi + 1 = \xi_M^{\mathcal{F}}$, then

$$\text{either } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi .$$

Both alternatives may materialize.

Proof (i) Let $O(F)$ be the Cantor-Bendixson index of $\mathcal{F}$ (see [K]), which is a countable ordinal (as the family of derived sets of $\mathcal{F}$ is countable). Since $s_I(\mathcal{F}) \leq O(\mathcal{F})$ for every $I \in [\mathbb{N}]$, we have

$$\xi_M^{\mathcal{F}} \leq O(\mathcal{F}) < \omega_1 .$$

(ii) Let $\xi + 1 < \xi_M^{\mathcal{F}}$. Then there exists $I \in [M]$ such that $\xi + 1 < s_I(\mathcal{F})$. According to Lemma 3.8 there exists $L \in [I]$ such that

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} .$$

(iii) Let $\xi_M^{\mathcal{F}} < \xi + 1$. According to the ξ-Ramsey type theorem (Theorem 1.6), there exists $L \in [M]$ such that

$$\text{either } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F} .$$

If $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$, then, using Remark 2.9 and Corollary 2.17, we obtain

$$\xi + 1 = s_L((\mathcal{A}_\xi \cap [L]^{<\omega})_\star) \leq s_L(\mathcal{F}) .$$

A contradiction; hence, $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ and equivalently, $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi)$ (according to Lemma 3.1).

(iv) In the limiting case $\xi_m^{\mathcal{F}} = \xi + 1$ both alternatives of the ξ- Ramsey type theorem for hereditary families (Theorem 3.2) may materialize. Indeed, we have the following two simple examples:

Example 1: Let

$$\mathcal{F} = \{s \in [\mathbb{N}]^{<\omega} : s \neq \emptyset \text{ and } |s| = 2min\ s + 1\},$$

where $|s|$ denotes the cardinality of s.

The family $\mathcal{F}$ is ω-uniform on $\mathbb{N}$, since for every $n \in \mathbb{N}$ we have

$$\mathcal{F}(n) = [\mathbb{N} \cap (n, +\infty)]^{2n} ,$$

which is $2n$-uniform family on $\mathbb{N} \cap (n, +\infty)$ (see Remark 1.8 (i)).

The family $\mathcal{F}_\star$ is pointwise closed and according to Theorem 2.15, $s_I(\mathcal{F}_\star) = \omega + 1$ for every $I \in [\mathbb{N}]$. Hence

$$\xi_{\mathbb{N}}^{\mathcal{F}_\star} = \omega + 1 .$$

It is now easy to verify that

$$\mathcal{A}_\omega \subseteq \mathcal{F}_\star \text{ ; and}$$

$$\mathcal{F}_\star \cap [L]^{<\omega} \nsubseteq (\mathcal{A}_\omega)^\star \setminus \mathcal{A}_\omega \text{ for every } L \in [\mathbb{N}] .$$

Example 2 : Let M be the set of all non zero, even natural numbers. We set

$$\mathcal{F} = \{s \in [M]^{<\omega} : s \neq \emptyset \text{ and } |s| = \frac{min\ s}{2}\} .$$

Since $\mathcal{F}(m) = [M \cap (m, +\infty)]^{\frac{m}{2}}$ for every $m \in M$, the family $\mathcal{F}$ is ω-uniform on M.

From Theorem 2.15 we get that $s_I(\mathcal{F}_\star) = \omega + 1$ for every $I \in [M]$. Thus

$$\xi_M^{\mathcal{F}_\star} = \omega + 1 .$$

It is now easy to verify that

$$\mathcal{F}_\star \subseteq (\mathcal{A}_\omega)^\star \setminus \mathcal{A}_\omega \text{ ; and}$$

$$\mathcal{A}_\omega \cap [L]^{<\omega} \nsubseteq \mathcal{F}_\star \text{ for every } L \in [M] .$$

Remark 3.10 (i) In Theorem 3.7 the thin Schreier family $\mathcal{A}_\xi$ can be replaced by any ξ-uniform family.

As a corollary of Theorem 3.9 we have the following result of Argyros, Merkourakis and Tsarpalias ([A-M-T]).

Corollary 3.11 Let $\mathcal{F}$ be a hereditary and pointwise closed family of finite subsets of $\mathbb{N}$, If there exists $M \in [\mathbb{N}]$ such that $s_M(\mathcal{F}) \geq \omega^\alpha$, then there exists $L \in [M]$ such that $\mathcal{F}_\alpha(L) \subseteq \mathcal{F}$.

Proof If $s_M(\mathcal{F}) > \omega^\alpha + 1$, then, according to Theorem 3.9, there exists $N \in [M]$ such that $\mathcal{B}_\alpha \cap [N]^{<\omega} \subseteq \mathcal{F}$ and according to Corollary 3.5 and Proposition 2.4 there exists $L \in [N]$ such that $\mathcal{F}_\alpha(L) \subseteq (\mathcal{B}_\alpha)^\star \cap [L]^{<\omega} \subseteq (\mathcal{B}_\alpha \cap [\mathbb{N}]^{<\omega})^\star \subseteq \mathcal{F}$. $\mathcal{F}_\xi(L) \subseteq \mathcal{F}$.

Now, if $s_M[\mathcal{F}] = \omega^\alpha + 1$, then we set

$$\mathcal{F}^1 = \{\{m\} \cup s : s \in \mathcal{F}, m \in M \text{ and } \{m\} < s\} .$$

It is easy to see that $s_M(\mathcal{F}^1) > \omega^\alpha + 1$. So applying the previous case to $\mathcal{F}^1$ we can find $N = (n_i)_{i=1}^\infty \in [M]$ such that $\mathcal{F}_\alpha(N) \subseteq \mathcal{F}^1$. Setting $L = (n_i)_{i=3}^\infty$ we have that $\mathcal{F}_\alpha(L) \subseteq \mathcal{F}$, as required.

We can now prove a limiting result, the ω_1-Ramsey type theorem, so to speak above all, the ξ-Ramsey type theorems for countable ordinals ξ. This result is a consequence of the methods employed in this paper. We note that the ω_1-Ramsey type theorem has applications, at least in Banach space theory, similar to applications of the infinitary Ramsey theorem (of Nash-Williams [N-W], Galvin-Prikry [G-P], and Silver [S]).

Theorem 3.12 (**ω_1-Ramsey type theorem**) Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}$. For every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that

$$\text{either } [L]^{<\omega} \subseteq \mathcal{F} \text{ or } [L]^{<\omega} \subseteq ([\mathbb{N}]^{<\omega} \setminus \mathcal{F})_\star .$$

Proof Let $M \in [\mathbb{N}]$. In case $\mathcal{F} \cap [M]^{<\omega}$ is not pointwise closed, according to Proposition 3.7, there exists $L \in [M]$ such that

$$[L]^{<\omega} \subseteq \mathcal{F} \cap [M]^{<\omega} \subseteq \mathcal{F} .$$

Now, in case $\mathcal{F} \cap [M]^{<\omega}$ is pointwise closed, from Theorem 3.9 we get that there exists a countable ordinal $\xi_M^{\mathcal{F}}$ such that for every countable ordinal ξ with $\xi_M^{\mathcal{F}} < \xi + 1$ there exists $L_\xi \in [M]$ so that

$$\mathcal{A}_\xi \cap [L_\xi]^{<\omega} \subseteq [M]^{<\omega} \setminus \mathcal{F} .$$

Let $\mathcal{F}_1 = ([M]^{<\omega} \setminus \mathcal{F})_*$. We will prove that $\mathcal{F}_1$ is not pointwise closed. Indeed, if $\mathcal{F}_1$ is pointwise closed, then the "usual" Cantor-Bendixson index $O(\mathcal{F}_1)$ of $\mathcal{F}_1$ is equal to a countable ordinal ξ_0. Let $\xi < \omega_1$ with $\xi > \xi_0$ and $\xi + 1 > \xi_M^{\mathcal{F}}$. Using Remark 2.9 and Corollary 2.17, we have

$$\xi + 1 = s_{L_\xi}((\mathcal{A}_\xi \cap [L_\xi]^{<\omega})_*) \leq s_{L_\xi}(\mathcal{F}_1) \leq O(\mathcal{F}_1) = \xi_0 .$$

A contradiction; hence the family $\mathcal{F}_1 = ([M]^{<\omega} \setminus \mathcal{F})_*$ is not pointwise closed. According to Proposition 3.7, there exists $L \in [M]$ such that

$$[L]^{<\omega} \subseteq ([M]^{<\omega} \setminus \mathcal{F})_* \subseteq ([\mathbb{N}]^{<\omega} \setminus \mathcal{F})_* .$$

This finishes the proof.

Recapitulation of the main results

Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}$ and M an infinite subset of $\mathbb{N}$. We have the following cases:

Case 1 The family $\mathcal{F} \cap [M]^{<\omega}$ is not pointwise closed. Then there exists $L \in [M]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$ (Proposition 3.7).

Case 2 The family $\mathcal{F} \cap [M]^{<\omega}$ is pointwise closed. Then there exists $L \in [M]$ such that $[L]^{<\omega} \subseteq ([\mathbb{N}]^{<\omega} \setminus \mathcal{F})_*$, (Theorem 3.12). Moreover setting

$$\xi_M^{\mathcal{F}} = sup\{s_L(\mathcal{F}) : L \in [M]\} ;$$

the following obtain:

2(i) For every countable ordinal ξ with $\xi + 1 < \xi_M^{\mathcal{F}}$ there exists $L \in [M]$ such that

$$(\mathcal{A}_\xi)_* \cap [L]^{<\omega} \subseteq \mathcal{F} ;$$

(Theorem 3.9).

2(ii) For every countable ordinal ξ with $\xi_M^{\mathcal{F}} < \xi + 1$ there exists $L \in [M]$ such that

$$\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^* \setminus \mathcal{A}_\xi ;$$

and equivalently,

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F} ;$$

(Theorem 3.9).

2(iii) If $\xi_M^{\mathcal{F}} = \xi + 1$, there exists $L \in [M]$ such that

$$\text{either } (\mathcal{A}_\xi)_* \cap [L]^{<\omega} \subseteq \mathcal{F}, \text{ or } \mathcal{F} \cup [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^* \setminus \mathcal{A}_\xi \,.$$

Both alternatives may materialize. (Theorem 1.6, Theorem 3.9).

4. Pták-type theorem for every countable order

We now, in this final section, turn our attention to an application that shows the power and scope of the (strong form of the) ξ-Ramsey principle, (Theorems 1.6, 3.9) developed in the first three sections. The application we have in mind is toward the correct ordinal generalization of the classical Pták's theorem. We thus prove a ξ-Pták type theorem for every countable ordinal ξ, which as a limiting case (the "ω_1-case") essentially implies the classical Pták's theorem.

It is evident that such ordinal extensions have numerous interesting applications; however, we are not concerned here with applications, some of which will appear in another publication.

The ξ-Pták type theorem (Theorem 4.2) can be considered as a criterion and method to find the crucial, "separating" countable ordinal $\xi_M^{\mathcal{F}}$, for an infinite subset M of N, and for a hereditary and pointwise closed family $\mathcal{F}$ of finite subsets of $\mathbb{N}$, introduced and studied in Theorem 3.9 above.

An important role for the establishment of the ξ-Pták type theorem is the notion of the weight $w_\xi(F;s)$ defined for every countable ordinal ξ, every finite subset F of N and every s in the thin Shreier family $\mathcal{A}_\xi$.

It will be seen that this definition (Definition 4.1), and the subsequent proof of the main theorem (Theorem 4.2), are given recursively, using the same fundamental ordinal representation (stated in Definition 1.2).

A Pták type theorem, only for the countable ordinals ξ of the form $\xi = \alpha^\omega$, $1 \leq \alpha \leq \omega_1$, using weighted averages, and proved by different methods, has been given in [A-M-T].

Definition 4.1 For every finite subset F of $\mathbb{N}$, every countable ordinal ξ, and every $s \in \mathcal{A}_\xi$ we define recursively the **ξ-weight $w_\xi(F;s)$ of F with respect to s**, to be a real (in fact, a rational) number in the real interval [0,1], as follows:

(1) [Case $\xi = 1$] Since $\mathcal{A}_1 = \{\{n\} : n \in \mathbb{N}\}$, we set for every $n \in \mathbb{N}$

$$w_1(F;\{n\}) = 1 \text{ if } n \in F \text{ and } w_1(F;\{n\}) = 0 \text{ otherwise .}$$

(2) [**Case** $\xi = \zeta + 1$] Let $s \in \mathcal{A}_{\zeta+1}$. Then, $s = \{n\} \cup s_1$, where $n \in \mathbb{N}$, $\{n\} < s_1$ and $s_1 \in \mathcal{A}_\zeta$. We set

$$w_{\zeta+1}(F; s) = w_\zeta(F; s_1) \cdot w_1(F; \{n\}) .$$

(3) [**Case** $\xi = \omega^{\beta+1}$ for $0 \leq \beta < \omega_1$] Let $s \in \mathcal{A}_{\omega^{\beta+1}}$. Then $s = s_1 \cup \ldots \cup s_n$, with $n = min\ s_1$, $s_1 < \ldots < s_n$ and $s_1, \ldots, s_n \in \mathcal{A}_{\omega^\beta}$. We set

$$w_{\omega^{\beta+1}}(F; s) = \frac{1}{n} \sum_{i=1}^{n} w_{\omega^\beta}(F; s_i) .$$

(4) [**Case** $\xi = \omega^\lambda$ for λ non-zero, countable limit ordinal] Let $s \in \mathcal{A}_{\omega^\lambda}$. Then $s \in \mathcal{A}_{\omega^{\lambda_n}}$ with $n = min\ s$, where (λ_n) is the fixed sequence of ordinals "converging" to λ, (Definition 1.3). So,

$$w_{\omega^\lambda}(F; s) = w_{\omega^{\lambda_n}}(F; s) \ , \quad n = min\ s \ .$$

(5)[**Case** ξ limit, $\omega^{\alpha_0} < \xi < \omega^{\alpha_0+1}$ for some $0 < \alpha_0 < \omega_1$] In this case, according to Proposition 1.2, ξ has a unique representation of ordinals as follows: $\xi = p_0\omega^{\alpha_0} + \sum_{i=1}^{m} p_i\omega^{\alpha_i}$, where $m \in \mathbb{N}$, $\alpha_0 > \alpha_1 > \ldots > \alpha_m > 0$ are ordinal numbers and $p_0, p_1, \ldots p_m \geq 1$ are natural numbers, so that either $p_0 > 1$ or $p_0 = 1$ and $m \geq 1$.

Let $s \in \mathcal{A}_\xi$. Then $s = s_0 \cup s_1 \cup \ldots \cup s_m$ with $s_m < \ldots < s_1 < s_0$, where $s_i = s_1^i \cup \ldots \cup s_{p_i}^i$ with $s_1^i < \ldots < s_{p_i}^i$ and $s_j^i \in \mathcal{A}_{\omega^{\alpha_i}}$ for every $0 \leq i \leq m$ and $1 \leq j \leq p_i$. We set

$$w_\xi(F; s) = \prod_{i=0}^{m} \prod_{j=1}^{p_i} w_{\omega^{\alpha_i}}(F; s_j^i) .$$

Theorem 4.2 (**ξ-Pták type theorem**). Let $\mathcal{F}$ be a hereditary and pointwise closed family of finite subsets of $\mathbb{N}, M \in [\mathbb{N}]$, ξ a non-zero, countable ordinal and $0 < \varepsilon < 1$. If for every $s \in \mathcal{A}_\xi \cap [M]^{<\omega}$ there exists $F \in \mathcal{F}$ such that $w_\xi(F; s) > \varepsilon$, then:

(i) there exists $L \in [M]$ such that $s_L(\mathcal{F}) \geq \xi + 1$;

(ii) $\xi_M^{\mathcal{F}} \leq \xi + 1$, and

(iii) for every ordinal ζ with $\zeta < \xi$ there exists $L \in [M]$ such that

$$\mathcal{A}_\zeta \cap [L]^{<\omega} \subseteq \mathcal{F} .$$

Proof [**Case** $\xi = 1$] We have $\mathcal{A}_1 \cap [M]^{<\omega} = \{\{m\} : m \in M\}$. Let $m \in M$. Then there exists $F \in \mathcal{F}$ such that $w_1(F; \{m\}) = 1 > \varepsilon$ and equivalently $\{m\} \subseteq F$. Since $\mathcal{F}$ is hereditary $\{m\} \in \mathcal{F}$. Hence $\mathcal{A}_1 \cap [M]^{<\omega} \subseteq \mathcal{F}$ and consequently

$$\xi_M^{\mathcal{F}} \geq s_M(\mathcal{F}) \geq s_M((\mathcal{A}_1 \cap [M]^{<\omega})_*) = 2 \ ;$$

(Remark 2.9 and Corollary 2.17).

Let $1 < \xi < \omega_1$. We assume that the theorem is valid for every ordinal ζ with $1 \leq \zeta < \xi$.

[**Case** $\xi = \zeta + 1$] Let $M = (m_n)_{n \in \mathbb{N}}$ with $m_n < m_{n+1}$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ set

$$\mathcal{F}(m_n) = \{F \in \mathcal{F} : \{m_n\} < F \ \text{ and } \ \{m_n\} \cup F \in \mathcal{F}\} \ ; \text{ and}$$

$$M_n = \{m_i \ : \ i \in \mathbb{N} \ \text{ and } \ i > n\} \ .$$

Let $s_1 \in \mathcal{A}_\zeta \cap [M_1]^{<\omega}$. Then $s = \{m_1\} \cup s_1 \in \mathcal{A}_\xi \cap [M]^{<\omega}$. According to our hypothesis, there exists $F \in \mathcal{F}$ such that $w_{\zeta+1}(F; s) > \varepsilon$. This gives that $w(F, \{m_1\}) = 1 > \varepsilon$, so $\{m_1\} \in \mathcal{F}$ and $w_\zeta(F_1; s_1) = w_\zeta(F; s_1) > \varepsilon$, where $F_1 = \{n \in F \ : \ n > m_1\} \ \in \ \mathcal{F}(m_1)$.

From ζ-Pták type theorem there exists $L_1 \in [M_1]$ such that

$$s_{L_1}\big(\mathcal{F}(m_1)\big) \geq \zeta + 1 \ .$$

Analogously, for every $n \in \mathbb{N}$ we can find $L_n \in [M_n \cap L_{n-1}]$ such that

$$s_{L_n}\big(\mathcal{F}(m_n)\big) \geq \zeta + 1 \ .$$

Set $L = (l_n^n)_{n \in \mathbb{N}}$ if $L_n = (l_i^n)_{i \in \mathbb{N}}$ for every $n \in \mathbb{N}$. Then (Remark 2.9(vi))

$$s_L\big(\mathcal{F}(m_n)\big) \geq \zeta + 1 \ \text{ for every } \ n \in \mathbb{N} \ .$$

Hence $\emptyset \in \big(\mathcal{F}(m_n)\big)_L^\zeta$ for every $n \in \mathbb{N}$. This gives that $\{l\} \subseteq \mathcal{F}_L^\zeta$ for every $l \in L$ (Lemma 2.10) and consequently that $\emptyset \in \mathcal{F}_L^{\zeta+1}$. Thus we have proved that

$$\xi_M^{\mathcal{F}} \geq s_L(\mathcal{F}) \geq \zeta + 2 \ .$$

According to Theorem 3.9 there exists $I \in [M]$ such that $\mathcal{A}_\zeta \cap [I]^{<\omega} \subseteq \mathcal{F}$.

[**Case** ξ **limit**] Let $\xi = \sum_{i=1}^{m} p_i \omega^{\alpha_i}$, where $m, p_1, \ldots, p_m \geq 1$ are natural numbers and $\alpha_1 > \ldots > \alpha_m > 0$ are countable ordinal numbers. Then $\xi = \beta + \omega^{\alpha_m}$ where

$\beta = \sum_{i=1}^{m-1} p_i\omega^{\alpha_i} + (p_m - 1)\omega^{\alpha_m}$ (we set $\beta = 0$ if $m = 1$ and $p_m = 1$ and $\beta = (p_m - 1)\omega^{\alpha_m}$ if $m = 1, p_m \neq 1$).

(i) Let $\alpha_m = \alpha + 1$ for some $\alpha < \omega_1$. For every $n \in \mathbb{N}$ set

$$\mathcal{L}^n = \{s \in \mathcal{A}_{\beta+n\omega^\alpha} \ : \ \text{there exists } F \in \mathcal{F} \ \text{such that } \ w_{\beta+n\omega^\alpha}(F;s) > \left(\frac{\varepsilon}{2}\right)^{n+1}\} \ .$$

Let $s \in \mathcal{A}_\xi$. Then there exist $s_1 \in \mathcal{A}_{\omega^{\alpha+1}}$ and $s_2 \in \mathcal{A}_\beta$ (we take $s_2 \in \mathcal{A}_\beta$ only in case $\beta \neq 0$) such that $s_1 < s_2$ and $s = s_1 \cup s_2$. According to our hypothesis for every $s \in \mathcal{A}_\xi \cap [M]^{<\omega}$ there exists $F \in \mathcal{F}$ such that $w_\xi(F;s) > \varepsilon$. Hence, $w_{\omega^{\alpha+1}}(F;s_1) > \varepsilon$ and $w_\beta(F;s_2) > \varepsilon$. Let $s_1 = \bigcup_{i=1}^{k} s_i^1$, where $k = min\ s_1^1$, $s_1^1 < \ldots < s_k^1$ and $s_1^1, \ldots, s_k^1 \in \mathcal{A}_{\omega^\alpha}$. Since $w_{\omega^{\alpha+1}}(F;s_1) > \varepsilon$, the cardinality $c_k(F;s_1)$ of the set $\{i : 1 \leq i \leq k \text{ and } w_{\omega^\alpha}(F;s_i^1) > \frac{\varepsilon}{2}\}$ is greater than $\frac{\varepsilon k}{2}$. Since $c_k(F;s_1) \to +\infty$ as $k \to +\infty$ we have that

$$\mathcal{A}_{\beta+n\omega^\alpha} \cap \mathcal{L}^n \cap [I]^{<\omega} \neq \emptyset \ ,$$

for every $n \in \mathbb{N}$ and $I \in [M]$.

Using the $\beta + n\omega^\alpha$-Ramsey type theorem (Theorem 1.6) we can find inductively a decreasing sequence $(M_n)_{n\in\mathbb{N}}$ in $[M]$ such that

$$\mathcal{A}_{\beta+n\omega^\alpha} \cap [M_n]^{<\omega} \subseteq \mathcal{L}^n \ \text{ for every } \ n \in \mathbb{N}.$$

According to our assumption that the theorem is valid for every ordinal ζ with $\zeta < \xi$, there exists a decreasing sequence (L_n) in $[M]$ such that

$$s_{L_n}(\mathcal{F}) > \beta + n\omega^\alpha \ \text{ for every } \ n \in \mathbb{N} \ .$$

Set $L = (l_n^n)_{n\in\mathbb{N}}$ if $L_n = (l_i^n)_{i\in\mathbb{N}}$ for $n \in \mathbb{N}$.
Then (Remark 2.9(vi))

$$s_L(\mathcal{F}) \geq s_{L_n}(\mathcal{F}) > \beta + n\omega^\alpha \text{ for every } \ n \in \mathbb{N} \ ;$$

and consequently

$$s_L(\mathcal{F}) \geq \beta + \omega^{\alpha+1} = \xi \ .$$

Since $s_L(\mathcal{F})$ is a susseccor ordinal, we have $\xi_M^{\mathcal{F}} \geq s_L(\mathcal{F}) \geq \xi + 1$.

Theorem 3.9 finishes the proof of this case.

(ii) Let $\alpha_m = \lambda$ for some non-zero, limit ordinal λ, and let $(\lambda_n)_{n\in\mathbb{N}}$ be the fixed, increasing to λ sequence of susseccor ordinal used in the definition of $\mathcal{A}_{\omega^\lambda}$ (Definition 1.3). We set $\lambda_n = \mu_n + 1$ for every $n \in \mathbb{N}$.

For every $n \in M$ let k_n be the smallest integer which satisfies $k_n > \frac{\varepsilon n}{2}$ and $k_n > 1$. Set

$$\mathcal{L}_n = \{s \in \mathcal{A}_{\beta+(k_n-1)\omega^{\mu_n}} \ : \text{ there exists } F \in \mathcal{F} \text{ such that } w_{\beta+(k_n-1)\omega^{\mu_n}}(F;s) > (\frac{\varepsilon}{2})^{k_n}\} .$$

Let $s \in \mathcal{A}_\xi$. Then there exist $s_1 \in \mathcal{A}_{\omega^\lambda}$, $s_2 \in \mathcal{A}_\beta$ (only in case $\beta \neq 0$) with $s_1 < s_2$ and $s = s_1 \cup s_2$.
According to our hypothesis there exists $F \in \mathcal{F}$ such that $w_\xi(F;s) > \varepsilon$. Hence $w_{\omega^\lambda}(F;s_1) > \varepsilon$ and $w_\beta(F;s_2) > \varepsilon$. We note that $s_1 \in \mathcal{A}_{\omega^{\mu_n+1}}$, where $n = min\ s_1$. Hence $s_1 = \bigcup_{i=1}^{n} s_i^1$, where $n = min\ s_1$, $s_1^1 < \ldots < s_n^1$ and $s_1^1, \ldots, s_n^1 \in \mathcal{A}_{\omega^{\mu_n}}$. Since $w_{\omega^{\mu_n+1}}(F;s_1) > \varepsilon$, the cardinality of the set $\{i : 1 \leq i \leq n \text{ and } w_{\omega^{\mu_n}}(F;s_i^1) > \frac{\varepsilon}{2}\}$ is greater than $\frac{\varepsilon n}{2}$.

Hence for every $I \in [M]$ and $n \in M$ we have that

$$\mathcal{L}_n \cap \mathcal{A}_{\beta+(k_n-1)\omega^{\mu_n}} \cap [I]^{<\omega} \neq \emptyset .$$

Using the $\beta + (k_n - 1)\omega^{\mu_n}$-Ramsey type theorem (Theorem 1.6) we can find inductively a decreasing sequence $(M_n)_{n\in M}$ in $[M]$ such that

$$\mathcal{A}_{\beta+(k_n-1)\omega^{\mu_n}} \cap [M_n]^{<\omega} \subseteq \mathcal{L}_n \quad \text{for every} \quad n \in M .$$

Since the theorem is valid for every ordinal ζ with with $\zeta < \xi$, there exists a decreasing sequence (L_n) in $[M]$ such that

$$s_{L_n}(\mathcal{F}) > \beta + (k_n - 1)\omega^{\mu_n} \quad \text{for every} \quad n \in M .$$

Setting $L = (l_n^n)_{n\in M}$ if $L_n = (l_i^n)_{i\in \mathbb{N}}$ for every $n \in M$ we have that

$$s_L(\mathcal{F}) > \beta + (k_n - 1)\omega^{\mu_n} \quad \text{for every} \quad n \in M .$$

Since $k_n \to +\infty$, $\mu_n \to \lambda$ we have that

$$s_L(\mathcal{F}) \geq \beta + \omega^\lambda = \xi ;$$

and since $s_L(\mathcal{F})$ is a susseccor ordinal we have that $\xi_M^{\mathcal{F}} \geq s_L(\mathcal{F}) \geq \xi + 1$, as required.

According to Theorem 3.9 for every ordinal ζ with $\zeta < \xi$ there exists $L_\zeta \in [L]$ such that $\mathcal{A}_\zeta \cap [L_\zeta]^{<\omega} \subseteq \mathcal{F}$.

This finishes the proof of the theorem.

Corollary 4.3 (ω_1**-Pták type theorem**). Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}$, $M \in [\mathbb{N}]$ and $0 < \varepsilon < 1$. If for every countable ordinal α and every $s \in \mathcal{A}_{\omega^\alpha} \cap [M]^{<\omega}$ there exists $F \in \mathcal{F}$ such that $w_{\omega^\alpha}(F; s) > \varepsilon$, then there exists $L \in [\mathbb{N}]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$.

Proof Suppose that there is no $L \in [\mathbb{N}]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$. Then $\mathcal{F}$ is pointwise closed (Proposition 3.7). According to Theorem 4.2, for every $\alpha < \omega_1$ there eixist $L_\alpha \in [M]$ such that

$$s_{L_\alpha}(\mathcal{F}) > \omega^\alpha.$$

Then $sup\{s_L(\mathcal{F}) : L \in [M]\} = \omega_1$ A contradiction (Theorem 3.9); hence there exists $L \in [\mathbb{N}]$ with $[L]^{<\omega} \subseteq \mathcal{F}$.

The above corollary implies Pták's classical theorem.

Pták's theorem ([P]) Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}$ and $0 < \varepsilon < 1$. If for every non-negative, real valued function f on $\mathbb{N}$ with finite support and such that $\sum_{n \in \mathbb{N}} f(n) = 1$ there exists $F \in \mathcal{F}$ such that $\sum_{n \in F} f(n) > \varepsilon$, then there exists $L \in [\mathbb{N}]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$.

Proof It follows from the previous corollary observing that for every $F \in [\mathbb{N}]^{<\omega}$, $1 \leq \alpha < \omega_1$ and $s \in \mathcal{A}_{\omega^\alpha}$

$$w_{\omega^\alpha}(F, s) = \sum_{n \in F} f^s_\alpha(n) \ ;$$

where the convex combinations f^s_α are defined recursively as follows:

(1) $f_0^{\{k\}}(n) = 1$ if $n = k$; and $f_0^{\{k\}}(n) = 0$ otherwise, for every $\{k\} \in \mathcal{A}_1$

(2)$f^s_{\alpha+1} = \frac{1}{k}\sum_{i=1}^k f^{s_i}_\alpha$, for every $s = s_1 \cup \ldots \cup_{s_k} \in \mathcal{A}_{\omega^{\alpha+1}}$.

(3)$f^s_\lambda = f^s_{\lambda_k}$, $k = min\ s$, for every $s \in \mathcal{A}_\lambda$ where λ is a non zero, countable limit ordinal.

Acknowledgement

I would like to thank Professor S. Negrepontis and A. Katavolos for their helpful comments and support during the preparation of this paper.

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