RAMSEY THEORY WITH MIXED TYPES OF SUBSTITUTION

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Abstract. Taking as our starting point the Farah-Hindman-McLeod partition theorem for located words over a finite alphabet, which we reprove, and defining a general notion S of substitution for variable located words over a finite alphabet, which includes either the Bergelson-Blass-Hindman type, or the Gowers type, or a mixture of these two types, we build for every notion S of substitution a full Ramsey theory,namely we obtain (1) a strong S-partition theorem for all the variable located words, (2) an S-partition theorem for all the k-tuples of variable located words, and in fact for all the Schreier families of order ξ, for every countable ordinal ξ, (3) an S-partition theorem for infinite sequences of variable located words, and (4) an Ellentuck type characterization of S-completely Ramsey partitions of the set of infinite sequences of located words over a finite alphabet.

Introduction

Gowers in [G], for the purpose of making the classical Carlson [C] and Furstenberg-Katznelson [FK] Ramsey theory more useful to the theory of Banach spaces proved a remarkable partition theorem (Theorem 5 in [G]). While it was not stated this way by Gowers, this theorem can be natually stated in terms of the notion of variable located words over a finite alphabet, introduced in the Bergelson-Blass-Hindman partition theory [BBH], involving a novel concept of substitution for variable located words over a totally ordered alphabet. Recently, a general partition theorem was proved by Farah, Hindman, McLeod in [FHM] (Theorem 3.13), which results in combining, for the first time, the Gowers [G] (Theorem 5), and the Bergelson-Blass-Hindman partition theorem for located words [BBH] (Theorem 4.1).

In the present work, we take as our starting point the Farah-Hindman-McLeod partition theorem; for completeness, we present a self-contained proof of their result for the special case we are interested in Theorem 1.2 below. More specifically we define (in Definition 2.1) a general notion S of substitution for variable located words over a finite alphabet Σ to be a suitable set of transformations $\{R_\alpha : \alpha \in \Sigma\}$, where R_α can be of the Bergelson-Blass-Hindman type, or of the Gowers type, or of a composition of the two types (Remark 2.2). With the help of the Farah-Hindman-McLeod result, we obtain

1991 *Mathematics Subject Classification.* Primary 05D10; 37Dxx.
Key words and phrases. partition theorems, substitution, located words, Schreier families.

for every notion S of substitution (1) a strong S-partition theorem for variable located words over a finite alphabet Σ (Theorem 2.3), and (2) an S-partition theorem for the k-tuples ($k \in \mathbb{N}$) of variable located words over a finite alphabet Σ, and in fact for all the Schreier families of order ξ, for every countable ordinal ξ, of variable located words over Σ (Theorem 3.5); in addition we establish (3) a Nash-Williams type S-partition theorem for infinite orderly sequences of variable located words over a finite alphabet (Theorem 4.15), and (4) an Ellentuck type characterization of the S-completely Ramsey partitions of the set of infinite orderly sequences of variable located words over a finite alphabet (in Theorem 5.7).

These theorems extend the Ramsey type theorems for variable located words over a finite alphabet proved by Gowers [G] (Theorem 5), Bergelson-Blass-Hindman [BBH] (Theorems 4.1, 5.1), Farah-Hindman-McLeod [FHM] (special cases of Theorem 3.13), and the partition theorem for infinite sequences of variable located words proved by Bergelson-Blass-Hindman [BBH] (Theorems 6.1). We note that the Gowers [G] and the Farah-Hindman-Mcleod [FHM] papers do not deal with partition theorems on infinite sequences of variable located words, of the Nash-Williams or Ellentuck type.

1. Partition theorems for located words over a finite alphabet

Let Σ be a finite alphabet. A *located word* over Σ is a function from a finite subset of $\mathbb{N} = \{1, 2, ...\}$ into the alphabet Σ ([BBH]). So, if ϑ is the function with domain the empty set, the set of all the located words over a non empty finite alphabet Σ is

$L(\Sigma) = \{w = w_{n_1} \dots w_{n_l} : l \in \mathbb{N}, n_1 < \dots < n_l \in \mathbb{N}, w_{n_1}, \dots, w_{n_l} \in \Sigma\} \cup \{\vartheta\}$, and

$L(\emptyset) = \{\vartheta\}$, in case $\Sigma = \emptyset$.

For a located word $w = w_{n_1} \dots w_{n_l} \in L(\Sigma) \setminus \{\vartheta\}$, we denote by $dom(w) = \{n_1, \dots, n_l\}$ the *domain* of w and we set $dom(\vartheta) = \emptyset$.

Let Σ be a finite alphabet (empty or non-empty) and $\upsilon \notin \Sigma$ an entity which is called a *variable*. The set of all *variable located words* over Σ with variable υ is defined to be

$L(\Sigma; \upsilon) = L(\Sigma \cup \{\upsilon\}) \setminus L(\Sigma)$.

We set

$L^0(\Sigma; \upsilon) = \{w = w_{n_1} \dots w_{n_l} \in L(\Sigma; \upsilon) : \Sigma \subseteq \{w_{n_1}, \dots, w_{n_l}\}\} \subseteq L(\Sigma; \upsilon)$.

We endow the set $L(\Sigma \cup \{\upsilon\})$ with a relation defining for $w, u \in L(\Sigma \cup \{\upsilon\})$

$$w < u \Leftrightarrow \text{either } \vartheta \in \{w, u\} \text{ or } w, u \neq \vartheta \text{ and } \max dom(w) < \min dom(u).$$

For two located words $w, u \in L(\Sigma \cup \{\upsilon\})$ such that $w < u$ we define the concatenating located word $w \star u \in L(\Sigma \cup \{\upsilon\})$ as follows:
if $w = w_{n_1} \dots w_{n_r}, u = u_{m_1} \dots u_{m_l} \in L(\Sigma \cup \{\upsilon\}) \setminus \{\vartheta\}$ we set

$$w \star u = w_{n_1} \dots w_{n_r} u_{m_1} \dots u_{m_l}, \text{ and}$$

$$w \star \vartheta = \vartheta \star w = w \text{ and } \vartheta \star \vartheta = \vartheta \text{ for every } w \in L(\Sigma \cup \{\upsilon\}).$$

Let Σ be a totally ordered finite alphabet with cardinality $k \in \mathbb{N} \cup \{0\}$ and a variable $\upsilon \notin \Sigma$. Then we define the function

$$S_{k+1} : L(\Sigma \cup \{\upsilon\}) \longrightarrow L(\Sigma) \text{ as follows:}$$

In case $\Sigma = \emptyset$, we set $S_1(w) = \vartheta$ for every $w \in L(\{\upsilon\})$.
In case $\Sigma = \{\alpha_1, \dots, \alpha_k\}$, $k \in \mathbb{N}$, we set $S_{k+1}(\vartheta) = \vartheta$, and for $w = w_{n_1} \dots w_{n_l} \in L(\Sigma \cup \{\upsilon\}) \setminus \{\vartheta\}$ we set $S_{k+1}(w) = \vartheta$ if $w_{n_i} = \alpha_1$ for every $1 \leq i \leq l$, and
$S_{k+1}(w) = u_{m_1} \dots u_{m_s}$ if $\{m_1 < \dots < m_s\} = \{n \in \{n_1, \dots, n_l\} : w_n \neq \alpha_1\} \neq \emptyset$, where, for $1 \leq i \leq s$, $u_{m_i} = \alpha_{j-1}$ if $w_{m_i} = \alpha_j$, $1 < j \leq k$, and $u_{m_i} = \alpha_k$ if $w_{m_i} = \upsilon$.
We remark that $dom(S_{k+1}(w)) \subseteq dom(w)$, $S_{k+1}(w \star u) = S_{k+1}(w) \star S_{k+1}(u)$ for every $w, u \in L(\Sigma \cup \{\upsilon\})$ with $w < u$ and that $S_{k+1}(L(\Sigma \cup \{\upsilon\})) = L(\Sigma)$.
We define the functions $S^i_{k+1} : L(\Sigma \cup \{\upsilon\}) \longrightarrow L(\Sigma \cup \{\upsilon\})$ for every $i \in \mathbb{N} \cup \{0\}$, by the rule: S^0_{k+1} to be the identity function on $L(\Sigma \cup \{\upsilon\})$, $S^1_{k+1} = S_{k+1}$ and $S^{i+1}_{k+1}(w) = S_{k+1}(S^i_{k+1}(w))$ for every $w \in L(\Sigma \cup \{\upsilon\})$. Observe that $S^i_{k+1}(w) = \vartheta$ for every $i \geq k+1$ and $w \in L(\Sigma \cup \{\upsilon\})$.

Also, for a finite alphabet Σ with cardinality $k \in \mathbb{N} \cup \{0\}$ and a variable $\upsilon \notin \Sigma$ we define the functions

$$T^{k+1}_q : L(\Sigma \cup \{\upsilon\}) \longrightarrow L(\Sigma \cup \{\upsilon\}), \text{ for every } 0 \leq q \leq k+1 \text{ as follows:}$$

In case $\Sigma = \emptyset$, we set $T^1_0(w) = \vartheta$ and $T^1_1(w) = w$ for every $w \in L(\upsilon)$.
In case $\Sigma = \{\alpha_1, \dots, \alpha_k\}$ for $k \in \mathbb{N}$, we set $T^{k+1}_q(\vartheta) = \vartheta$ for every $0 \leq q \leq k+1$, and for $w = w_{n_1} \dots w_{n_l} \in L(\Sigma \cup \{\upsilon\}) \setminus \{\vartheta\}$ we set
$T^{k+1}_{k+1}(w) = w$,
$T^{k+1}_0(w) = w_{m_1} \dots w_{m_s}$ if $\{m_1 < \dots < m_s\} = \{n \in \{n_1, \dots, n_l\} : w_n \neq \upsilon\} \neq \emptyset$, and $T^{k+1}_0(w) = \vartheta$ if $w_{n_i} = \upsilon$ for every $1 \leq i \leq l$, and for $1 \leq q \leq k$
$T^{k+1}_q(w) = u_{n_1} \dots u_{n_l}$, where, for $1 \leq i \leq l$, $u_{n_i} = \alpha_q$ if $w_{n_i} = \upsilon$ and $u_{n_i} = w_{n_i}$ if $w_{n_i} \in \Sigma$.
We remark that, for every $0 \leq q \leq k+1$, $dom(T^{k+1}_q(w)) \subseteq dom(w)$, $T^{k+1}_q(w \star u) = T^{k+1}_q(w) \star T^{k+1}_q(u)$ for every $w, u \in L(\Sigma \cup \{\upsilon\})$ with $w < u$, $T^{k+1}_q(w) = w$ for every $w \in L(\Sigma)$ and that $T^{k+1}_q(L(\Sigma \cup \{\upsilon\})) = L(\Sigma)$ if $0 \leq q \leq k$.

Definition 1.1. Let $k \in \mathbb{N}$, $\Sigma = \{\alpha_1, \ldots, \alpha_k\}$ be a non-empty totally ordered finite alphabet and a variable $\upsilon = \alpha_{k+1} \notin \Sigma$. For every $0 \leq p \leq k$ we consider the alphabets $\Sigma_0 = \emptyset$, $\Sigma_p = \{\alpha_1, \ldots, \alpha_p\}$ and the variables $\upsilon_p = \alpha_{p+1} \notin \Sigma_p$ respectively. According to the previously mentioned terminology, for every $0 \leq p \leq k$ are defined the functions

$$S_{p+1} : L(\Sigma_p \cup \{\upsilon_p\}) \longrightarrow L(\Sigma_p), \text{ and}$$
$$T_q^{p+1} : L(\Sigma_p \cup \{\upsilon_p\}) \rightarrow L(\Sigma_p) \text{ for every } 0 \leq q \leq p.$$

For $0 \leq p \leq k$, we define the family of functions from $L(\Sigma_p \cup \{\upsilon_p\})$ to $L(\Sigma_p)$,

$$\mathcal{F}_{p+1} = \{S_{p+1}\} \cup \{T_q^{p+1} : 0 \leq q \leq p\}.$$

Since $L(\Sigma_{p+1}) = L(\Sigma_p \cup \{\upsilon_p\})$ for every $0 \leq p \leq k$, by convolution, we can define for each $0 \leq p \leq k$ the family $\mathcal{U}_p$ of functions from $L(\Sigma \cup \{\upsilon\})$ to $L(\Sigma_p) \subseteq L(\Sigma)$ defining $\mathcal{U}_{k+1} = \{T_{k+1}^{k+1}\}$, where $T_{k+1}^{k+1} : L(\Sigma \cup \{\upsilon\}) \rightarrow L(\Sigma \cup \{\upsilon\})$ is the identity map, and for $0 \leq p \leq k$

$$\mathcal{U}_p = \{F \circ G : G \in \mathcal{U}_{p+1}, F \in \mathcal{F}_{p+1}\}.$$

With the previously mentioned definitions we can state a general partition theorem for located words over a finite totally ordered alphabet (Theorem 1.2 below), which unifies and extends the partition theorems for located words proved by Gowers in [G] and Bergelson-Blass-Hindman in [BBH] (see Remark 1.4). This theorem follows from the more general partition theorem for layered partial semigroups proved by Farah, Hindman, McLeod in [FHM] (Theorem 3.13). For completeness, we present here a self-contained proof of the special case of their theorem with which we are concerned in this paper.

Theorem 1.2 ([FHM])**.** *Let $k \in \mathbb{N}$, $\Sigma = \{\alpha_1, \ldots, \alpha_k\}$ be a finite alphabet and $\upsilon \notin \Sigma$ a variable. For every finite coloring $L(\Sigma \cup \{\upsilon\}) = A_1 \cup \cdots \cup A_r$ of $L(\Sigma \cup \{\upsilon\})$, there exist a sequence $(w_n)_{n \in \mathbb{N}}$ in $L^0(\Sigma; \upsilon)$ with $w_n < w_{n+1}$ for every $n \in \mathbb{N}$ and $1 \leq i_p \leq r$ for every $1 \leq p \leq k+1$ such that*

$$H_1(w_{n_1}) \star \ldots \star H_\lambda(w_{n_\lambda}) \in A_{i_{k+1}} \cap L^0(\Sigma; \upsilon)$$

for every $\lambda \in \mathbb{N}$, $n_1 < \cdots < n_\lambda \in \mathbb{N}$, $H_1, \ldots, H_\lambda \in \bigcup_{q=1}^{k+1} \mathcal{U}_q$ with $T_{k+1}^{k+1} \in \{H_1, \ldots, H_\lambda\}$, and, for $1 \leq p \leq k$,

$$H_1(w_{n_1}) \star \ldots \star H_\lambda(w_{n_\lambda}) \in A_{i_p} \cap L^0(\{\alpha_1, \ldots, \alpha_{p-1}\}; \alpha_p)$$

for every $\lambda \in \mathbb{N}$, $n_1 < \cdots < n_\lambda \in \mathbb{N}$, $H_1, \ldots, H_\lambda \in \bigcup_{q=1}^{p} \mathcal{U}_q$ with $\{H_1, \ldots, H_\lambda\} \cap \mathcal{U}_p \neq \emptyset$.

A particular case of Theorem 1.2 gives the following:

Corollary 1.3. *Let $k \in \mathbb{N}$, $\Sigma = \{\alpha_1, \dots, \alpha_k\}$ be a finite alphabet and $\upsilon \notin \Sigma$ a variable. For every finite coloring $L(\Sigma;\upsilon) = A_1 \cup \dots \cup A_r$ of $L(\Sigma;\upsilon)$ and every finite coloring $L(\Sigma) = C_1 \cup \dots \cup C_s$ of $L(\Sigma)$, there exist a sequence $(w_n)_{n\in\mathbb{N}}$ in $L(\Sigma;\upsilon)$ with $w_n < w_{n+1}$ for every $n \in \mathbb{N}$ and $1 \leq i_0 \leq r$, $1 \leq j_0 \leq s$ satisfing*

$$H_1(w_{n_1}) \star \ldots \star H_\lambda(w_{n_\lambda}) \in A_{i_0}$$

for every $\lambda \in \mathbb{N}$, $H_1, \dots, H_\lambda \in \{T_q^{k+1} : 0 \leq q \leq k+1\} \cup \{S_{k+1}^q : 0 \leq q \leq k\}$ with $T_{k+1}^{k+1} = S_{k+1}^0 \in \{H_1, \dots, H_\lambda\}$, $n_1 < \dots < n_\lambda \in \mathbb{N}$, and

$$H_1(w_{n_1}) \star \ldots \star H_\lambda(w_{n_\lambda}) \in C_{j_0}$$

for every $\lambda \in \mathbb{N}$, $H_1, \dots, H_\lambda \in \{T_q^{k+1} : 0 \leq q \leq k\} \cup \{S_{k+1}^q : 1 \leq q \leq k\}$ with $\{H_1, \dots, H_\lambda\} \cap \big(\{T_q^{k+1} : 0 \leq q \leq k\} \cup \{S_{k+1}\}\big) \neq \emptyset$, $n_1 < \dots < n_\lambda \in \mathbb{N}$.

Remark 1.4. (i) The particular case of the previous corollary, where $H_1, \dots, H_\lambda \in \{S_{k+1}^q : 0 \leq q \leq k\}$, is an equivalent reformulation, with the terminology of located words, of Gowers partition theorem, proved in [G] (Theorem 5).

(ii) The particular case of the previous corollary, where $H_1, \dots, H_\lambda \in \{T_q^{k+1} : 0 \leq q \leq k+1\}$, is a consequence of the partition theorem for located words proved by Bergelson,Blass and Hindman in [BBH] (Theorem 4.1).

We are proceeding now to a proof of Theorem 1.2. Firstly, we will refer some fundamental known results about the left compact semigroups and we will prove Proposition 1.6, which has central role in the proof of Theorem 1.2. Also, for completeness, we will mention some basic notions about ultrafilters.

Left compact semigroups. A non-empty, *left compact semigroup* is a semigroup $(X,+)$, $X \neq \emptyset$ endowed with a topology $\mathfrak{T}$ such that $(X,\mathfrak{T})$ is a compact Hausdorff space and the maps $f_y : X \longrightarrow X$ with $f_y(x) = x + y$ for $x \in X$ are continuous for every $y \in X$.

Let $(X,+)$ be a semigroup. An element x of X is called *idempotent* of $(X,+)$ if $x + x = x$. According to a fundamental result due to Ellis ([El]), every non-empty, left compact semigroup contains an idempotent. On the set of all idempotents of $(X,+)$ is defined a partial order $\leq$ by the rule

$$x_1 \leq x_2 \Longleftrightarrow x_1 + x_2 = x_2 + x_1 = x_1.$$

An idempotent x of $(X,+)$ is called *minimal for* X if every idempotent x_1 of X satisfing the relation $x_1 \leq x$ is equal to x.

In the following proposition are summarized some facts concerning to minimal idempotents (see [FK], [HiS]). We mention that a subset I of X is called *two-sided ideal* of $(X,+)$ if $X+I\subseteq I$ and $I+X\subseteq I$.

Proposition 1.5. *Let $(X,+)$ be a non-empty, left compact semigroup.*

(i) *X contains an idempotent x_1 minimal for X.*

(ii) *For every idempotent x of X there exists an idempotent x_1 of X which is minimal for X and $x_1\leq x$.*

(iii) *Every two-sided ideal of X contains all the minimal for X idempotents of X.*

(iv) *An idempotent x of X is minimal for X if and only if x is contained in the smallest two-sided ideal of X.*

(v) *If x is a minimal idempotent for X and x_1+x (resp.$x+x_1$) is an idempotent of X, for some $x_1\in X$, then x_1+x (resp. $x+x_1$) is a minimal idempotent for X.*

Now we will state and prove a result about minimal idempotents which has central role in the proof of Theorem 1.2. An analogous result for idempotents (not necessarily minimal) has be proved in [To] and also a similar result stated for layered partial semigroups has be proved in [FHM].

Proposition 1.6. *Let $(X,+)$ be a non-empty, left compact semigroup, $I\subseteq X$ be a closed two-sided ideal of X and $T:X\longrightarrow X$ be a continuous homomorphisms on X. If $T^k=T^{k+1}$ for some $k\in\mathbb{N}$, then for a given idempotent $x_0\in T^k(X)$ minimal for $T^k(X)$ there exists an idempotent $x\in I$ minimal for X such that*

(1) *$x\leq T(x)\leq\cdots\leq T^k(x)=x_0$, and*

(2) *$T^i(x)\in T^i(I)$ is a minimal idempotent for $T^i(X)$ for every $i\in\{1,\dots,k\}$.*

Proof. We will prove it by induction on k. Let $T=T^2$ and let an idempotent $x_0\in T(X)$ minimal for $T(X)$. Then $T(x_0)=x_0$. According to Proposition 1.5, there exists an idempotent $x\in I$ minimal for X with $x\leq x_0$. Then $T(x)\leq T(x_0)=x_0$. Since x_0 is minimal for $T(X)$, we have $x\leq T(x)=x_0$.

Assume that the result is true for some $k\geq 1$. Let $T^{k+1}=T^{k+2}$ and let an idempotent $x_0\in T^{k+1}(X)$ minimal for $T^{k+1}(X)$. Set $X_1=T(X)$, $I_1=T(I)$ and $T_1:X_1\longrightarrow X_1$ the restriction of T to X_1. By the induction hypothesis, since $T_1^k=T_1^{k+1}$, there exists an idempotent $y\in I_1$ minimal for X_1 such that $y\leq T_(y)\leq\cdots\leq T^k(y)=x_0$ and $T^i(y)$ is a minimal idempotent for $T^{i+1}(X)$ for every $i\in\{1,\dots,k\}$.

Let $Y=\{z\in I:T(z)=y\}$. Then Y and $Y+y$ are non-empty, left compact semigroups. Hence, $Y+y$ contains an idempotent z_0+y, with $z_0\in Y$, minimal for $Y+y$.

Set $x = y + (z_0 + y)$. Then $x \in I$ is an idempotent of X, $T(x) = y$ and $x \leq y$. Also, x is a minimal idempotent for X. Indeed, let x_1 be an idempotent for X with $x_1 < x$. According to Proposition 1.5, there exists an idempotent $x_2 \in I$ minimal for X with $x_2 \leq x_1 < x$. Then $T(x_2) \leq T(x) = y$. Since y is a minimal idempotent for X_1, we have that $T(x_2) = y$. Hence, $x_2 \in Y$ and consequently $x_2 = x_2 + y \in Y + y$. According to Proposition 1.5(v), x is a minimal idempotent for $Y + y$, since $z_0 + y \in Y + y$ is a minimal idempotent for $Y + y$ and $x = y + (z_0 + y)$ is an idempotent of $Y + y$ with $y = y + y \in Y + y$. Since $x_2 \leq x_1 < x$ and $x_2 \in Y + y$, we have $x_2 = x$ and consequently that $x_1 = x$. □

Ultrafilters. Let X be a non-empty set. An *ultrafilter* on the set X is a zero-one finite additive measure μ defined on all subsets of X. The set of all ultrafilters on the set X is denoted by βX. So, $\mu \in \beta X$ if and only if

(i) $\mu(A) \in \{0, 1\}$ for every $A \subseteq X$ and $\mu(X) = 1$, and
(ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ for every $A, B \subseteq X$ with $A \cap B = \emptyset$.

For $x \in X$ is defined the ultrafilter μ_x on X corresponging a set $A \subseteq X$ to $\mu_x(A) = 1$ if $x \in A$ and $\mu_x(A) = 0$ if $x \notin A$. The ultrafilters μ_x for $x \in X$ are called *principal ultrafilters* on X. So, μ is a non-principal ultrafilter on X if and only if $\mu(A) = 0$ for every finite subset A of X. It is easy to see that for $\mu \in \beta X$ and $A \subseteq X$ with $\mu(A) = 1$ we have $\mu(X \setminus A) = 0$, $\mu(B) = 1$ for every $B \subseteq X$ with $A \subseteq B$ and $\mu(A \cap B) = 1$ for every $B \subseteq X$ with $\mu(B) = 1$.

The set βX becomes a compact Hausdorff space if it be endowed with the topology $\mathfrak{T}$ which has basis the family $\{A^* : A \subseteq X\}$, where $A^* = \{\mu \in \beta X : \mu(A) = 1\}$. It is easy to see that $(A \cap B)^* = A^* \cap B^*$, $(A \cup B)^* = A^* \cup B^*$ and $(X \setminus A)^* = \beta X \setminus A^*$ for every $A, B \subseteq X$. We always consider the set βX endowed with the topology $\mathfrak{T}$.

Let a function $T : X \longrightarrow Y$. Then the function

$$\beta T : \beta X \longrightarrow \beta Y \text{ with } \beta T(\mu)(B) = \mu(T^{-1}(B)) \text{ for } \mu \in \beta X \text{ and } B \subseteq Y$$

is continuous.

If $(X, +)$ is a semigroup, then a binary operation $+$ is defined on βX corresponding to every $\mu_1, \mu_2 \in \beta X$ the ultrafilter $\mu_1 + \mu_2 \in \beta X$ given by

$$(\mu_1 + \mu_2)(A) = \mu_1(\{x \in X : \mu_2(\{y \in X : x + y \in A\}) = 1\}) \text{ for every } A \subseteq X.$$

With this operation the set βX becomes a semigroup and for every $\mu \in \beta X$ the function $T_\mu : \beta X \longrightarrow \beta X$ with $T_\mu(\mu_1) = \mu_1 + \mu$ is continuous.

Hence, if $(X, +)$ is a semigroup, then βX becomes a left compact semigroup.

Proof of Theorem 1.2. Let $k \in \mathbb{N}$, $\Sigma = \{\alpha_1, \ldots, \alpha_k\}$ be a finite alphabet and $\alpha_{k+1} = \upsilon \notin \Sigma$ be a variable. We set $X_{k+1} = L(\Sigma \cup \{\alpha_{k+1}\})$ the set of all the located words over $\Sigma \cup \{\alpha_{k+1}\}$, $V_{k+1} = L(\Sigma; \alpha_{k+1}) \subseteq X_{k+1}$ the set of all the variable located words over Σ with variable α_{k+1} and $V^0_{k+1} = L^0(\Sigma; \alpha_{k+1}) \subseteq V_{k+1}$. We endow the set X_{k+1} with an operation $+$ defining for $w = w_{n_1} \ldots w_{n_r}, u = u_{m_1} \ldots u_{m_l} \in X_{k+1}$ the located word

$$w + u = v_{k_1} \ldots v_{k_s} \in X_{k+1},$$

where $\{k_1, \ldots, k_s\} = \{n_1, \ldots, n_r\} \cup \{m_1, \ldots, m_l\}$ and, for $1 \leq i \leq s$, $v_i = w_i$ if $i \notin \{m_1, \ldots, m_l\}$, $v_i = u_i$ if $i \notin \{n_1, \ldots, n_r\}$, $v_i = \alpha_{\max\{p,q\}}$ if $w_i = \alpha_p$ and $u_i = \alpha_q$ for some $p, q \in \{1, 2, \ldots, k+1\}$.
Observe that $(X_{k+1}, +)$ is a semigroup and $w + u = w \star u$ for every $w, u \in X_{k+1}$ with $w < u$.

Since $(X_{k+1}, +)$ is a semigroup, βX_{k+1} has the structure of a left compact semigroup as described above. For every $A \subseteq X_{k+1}$ and $w \in X_{k+1}$ we set

$$A_w = \{u \in A : w < u\} \text{ and}$$

$$\theta A = \bigcap \{(A_w)^* : w \in X_{k+1}\}.$$

where $(A_w)^* = \{\mu \in \beta X_{k+1} : \mu(A_w) = 1\}$.
Claim 1 If A is a non-empty subset of X_{k+1} and satisfies
(i) $w + u \in A$ for every $w, u \in A$ with $w < u$ and
(ii) for every $n \in \mathbb{N}$ there exists $u \in A$ with $n < \min dom(u)$,
then $\theta A \subseteq A^*$ is a non-empty left compact subsemigroup of βX_{k+1} and contains non-principal ultrafilters on X_{k+1}.

Indeed, for every $w \in X_{k+1}$ the set $(A_w)^* = \beta X_{k+1} \setminus (X_{k+1} \setminus A_w)^*$ is a compact subset of βX_{k+1} , so θA is a compact subset of βX_{k+1}. The set A satisfies property (ii), so for every $w \in X_{k+1}$, we have $A_w \neq \emptyset$ and consequently $(A_w)^* \neq \emptyset$, since $\mu_u \in (A_w)^*$ for $u \in A_w$. Also, according to property (ii), the family $\{(A_w)^* : w \in X_{k+1}\}$ has the finite intersection property and consequently $\theta A \neq \emptyset$. Since A satisfies property (i), $(\theta A, +)$ is a semigroup. Indeed, for $\mu_1, \mu_2 \in \theta A$ and $w \in X_{k+1}$
$\mu_1 \star \mu_2(A_w) = \mu_1(\{u_1 \in A_w : \mu_2(\{u_2 \in A_{w+u_1} : u_1 + u_2 \in A_w\}) = 1\}) =$
$= \mu_1(\{u_1 \in A_w : \mu_2(A_{w+u_1}) = 1\}) = \mu_1(A_w) = 1.$
Hence, θA is a non-empty left compact subsemigroup of βX_{k+1}.

Let $1 \leq p \leq k+1$. We denote by $X_p = L(\{\alpha_1, \ldots, \alpha_p\}) \subseteq X_{k+1}$ the set of all the located words over the alphabet $\{\alpha_1, \ldots, \alpha_p\}$, by $V_p = L(\{\alpha_1, \ldots, \alpha_{p-1}\}; \alpha_p) \subseteq X_p$ the set of all the variable located words over the alphabet $\{\alpha_1, \dots, \alpha_{p-1}\}$ with variable α_p and we set $V_p^0 = L^0(\{\alpha_1, \dots, \alpha_{p-1}\}; \alpha_p) \subseteq V_p$. Finally, let $X_0 = \{\vartheta\}$.

According to the Claim 1, $\theta X_p, \theta V_p, \theta V_p^0$ are non-empty left compact subsemigroups of βX_{k+1} such that $\theta V_p^0 \subseteq \theta V_p \subseteq \theta X_p$ for every $1 \leq p \leq k+1$. Moreover, θV_p^0 is a two sided ideal of θX_p, for every $1 \leq p \leq k+1$. Indeed, for $1 \leq p \leq k+1$, $\mu_1 \in \theta V_p^0$, $\mu_2 \in \theta X_p$ and $w \in X_{k+1}$ we have
$\mu_1 \star \mu_2((V_p^0)_w) = \mu_1(\{u_1 \in (V_p^0)_w : \mu_2(\{u_2 \in (X_p)_{w+u_1} : u_1 + u_2 \in (V_p^0)_w\}) = 1\}) =$
$= \mu_1(\{u_1 \in (V_p^0)_w : \mu_2((X_p)_{w+u_1}) = 1\}) = \mu_1((V_p^0)_w) = 1 = \mu_2 \star \mu_1((V_p^0)_w)$.
Notice that $\theta X_0 = \{\mu_\vartheta\} \subseteq \beta X_{k+1}$ and that $\theta X_p \subseteq \theta X_{p+1} \subseteq \theta X_{k+1}$ for every $0 \leq p \leq k$.

According to Definition 1.1, we have for every $1 \leq p \leq k+1$ the functions

$$S_p : X_p \longrightarrow X_{p-1}, \text{ and } T_q^p : X_p \to X_{p-1} \text{ for every } 0 \leq q < p.$$

Let $T_p^p = S_0 : X_p \longrightarrow X_p$ be the identity map.

Let, $1 \leq p \leq k+1$, $\beta S_p : \beta X_p \longrightarrow \beta X_p$ with $\beta S_p(\mu)(A) = \mu((S_p)^{-1}(A))$ for every $\mu \in \beta X_p$ and $A \subseteq X_p$ and let $\mathtt{S}_p : \theta X_p \longrightarrow \theta X_p$ the restriction of βS_p to θX_p. Then $\mathtt{S}_p$ is the restriction of $\mathtt{S}_{k+1}$ to θX_p and is a continuous homomorphism onto θX_{p-1}. Also, $\mathtt{S}_{k+1}(\theta X_p) = \mathtt{S}_p(X_p) = \theta X_{p-1}$, $\mathtt{S}_{k+1}(\theta V_p^0) = \theta V_{p-1}^0$ for every $1 \leq p \leq k+1$. Indeed, for $1 \leq p \leq k+1$, $\mu_1, \mu_2 \in \theta X_p$ and $A \subseteq X_p$
$\mathtt{S}_p(\mu_1 \star \mu_2)(A) = \mu_1(\{u_1 \in X_p : \mu_2(\{u_2 \in (X_p)_{u_1} : S_p(u_1 + u_2) \in A\}) = 1\}) =$
$= \mu_1(\{u_1 \in X_p : \mu_2(\{u_2 \in (X_p)_{u_1} : S_p(u_1) + S_p(u_2) \in A\}) = 1\}) = \mathtt{S}_p(\mu_1) \star \mathtt{S}_p(\mu_2)(A)$.
Also, $\beta S_{k+1}(\theta X_p) \subseteq \theta X_{p-1}$, since for $\mu \in \theta X_p$ and $w \in X_{k+1}$ we have
$\beta S_{k+1}(\mu)((X_{p-1})_w) = \mu(\{u \in (X_p)_w : S_{k+1}(u) \in (X_{p-1})_w\}) = \mu((X_p)_w) = 1$, and
$\theta X_{p-1} \subseteq \beta S_{k+1}(\theta X_p)$, since for $\mu \in \theta X_{p-1}$, the family
$\mathcal{F} = \{(S_{k+1})^{-1}(A) \subseteq X_{k+1} : A \subseteq X_k, \mu(A) = 1\} \cup \{(X_p)_w : w \in X_k\}$,
has the finite intersection property. Indeed, for $A \subseteq X_k$ with $\mu(A) = 1$ and $w \in X_k$, we have $\mu(A \cap (X_{p-1})_w) = 1$ and for $u \in A \cap (X_{p-1})_w$ there exists $v \in (X_p)_w$ with $S_{k+1}(v) = u$, so $v \in (S_{k+1})^{-1}(A) \cap (X_p)_w$. Hence, there exists $\mu_1 \in \theta X_{k+1}$ such that $\mu_1(B) = 1$ for every $B \in \mathcal{F}$. Then $\mu_1 \in \theta X_p$ and $\mu = \beta S_{k+1}(\mu_1)$.
Analogously, can be proved that $\beta S_{k+1}(\theta V_p^0) = \theta V_{p-1}^0$ for every $1 \leq p \leq k+1$.

Observe that $\mathtt{S}_{k+1}^{k+1}(\mu) = \mu_\vartheta = \mathtt{S}_{k+1}^{k+2}(\mu)$ for all $\mu \in \theta X_{k+1}$. So, according to Proposition 1.6, there exists an idempotent $\mu_{k+1} \in \theta V_{k+1}^0$ minimal for θX_{k+1} such that

(1) $\mu_{k+1} \leq \mathtt{S}_{k+1}(\mu_{k+1}) \leq \cdots \leq \mathtt{S}_{k+1}^{k+1}(\mu_{k+1}) = \mu_\vartheta$, and
(2) $\mathtt{S}_{k+1}^p(\mu_{k+1}) \in \mathtt{S}^p(\theta V_{k+1}^0) = \theta V_{k+1-p}^0$ is a minimal idempotent for $\mathtt{S}^p(\theta X_{k+1}) = \theta X_{k+1-p}$ for all $p \in \{1, \dots, k+1\}$.

Let the functions $\beta T_q^p : \beta X_p \longrightarrow \beta X_p$ and let $\mathtt{T}_q^p : \theta X_p \longrightarrow \theta X_{p-1}$ the restriction of βT_q^p to θX_p for all $1 \leq p \leq k+1$, $0 \leq q \leq p$. Then $\mathtt{T}_q^p$ are continuous homomorphism onto θX_{p-1} and $\beta T_q^p(\mu) = \mu$ for every $\mu \in \theta X_{p-1}$. Indeed, for $\mu_1, \mu_2 \in \theta X_p$, $\mu \in \theta X_{p-1}$, $A \subseteq X_p$ and $w \in X_{k+1}$ we have:

(i) $\mathtt{T}_q^p(\mu_1 \star \mu_2)(A) = \mu_1(\{u_1 \in X_p : \mu_2(\{u_2 \in (X_p)_{u_1} : T_q^p(u_1+u_2) \in A\}) = 1\}) =$
$= \mu_1(\{u_1 \in X_p : \mu_2(\{u_2 \in (X_p)_{u_1} : T_q^p(u_1) + T_q^p(u_2) \in A\}) = 1\}) =$
$= \mathtt{T}_q^p(\mu_1) \star \mathtt{T}_q^p(\mu_2)(A)$,

(ii) $\beta T_q^p(\mu_1)((X_{p-1})_w) = \mu_1(\{u \in (X_p)_w : T_q^p(u) \in (X_{p-1})_w\}) = \mu_1((X_p)_w) = 1$, and

(iii) $\beta T_q^p(\mu)(A) = \mu(\{u \in X_{p-1} : T_q^p(u) = u \in A\}) = \mu(A \cap X_{p-1}) = \mu(A)$.

As we have already proved, there exists an idempotent $\mu_{k+1} \in \theta V_{k+1}^0$ minimal for θX_{k+1} such that $\mathtt{S}_{k+1}^{k+1-p}(\mu_{k+1}) \in \theta V_p^0$ is a minimal idempotent for θX_p and $\mathtt{S}_{k+1}^{p-1}(\mu_{k+1}) \leq \mathtt{S}_{k+1}^{p}(\mu_{k+1})$ for all $p \in \{1, \dots, k+1\}$. We will prove that

(3) $\mathtt{T}_q^p\big(\mathtt{S}_{k+1}^{k+1-p}(\mu_{k+1})\big) = \mathtt{S}_{k+1}^{k+2-p}(\mu_{k+1})$ for every $p \in \{1, \dots, k+1\}$, $0 \leq q < p$.

Indeed, let $p \in \{1, \dots, k+1\}$. Since $\mathtt{T}_q^p$ are continuous homomorphisms onto θX_{p-1} for every $0 \leq q < p$, we have that $\mathtt{T}_q^p\big(\mathtt{S}_{k+1}^{k+1-p}(\mu_{k+1})\big) \leq \mathtt{T}_q^p\big(\mathtt{S}_{k+1}^{k+2-p}(\mu_{k+1})\big)$ for every $0 \leq q \leq p$. But $\mathtt{S}_{k+1}^{k+2-p}(\mu_{k+1}) \in \theta V_{p-1}^0 \subseteq \theta X_{p-1}$, hence $\mathtt{T}_q^p\big(\mathtt{S}_{k+1}^{k+2-p}(\mu_{k+1})\big) = \mathtt{S}_{k+1}^{k+2-p}(\mu_{k+1})$. Now, since $\mathtt{S}_{k+1}^{k+2-p}(\mu_{k+1})$ is a minimal idempotent for θX_{p-1}, we have that $\mathtt{T}_q^p\big(\mathtt{S}_{k+1}^{k+1-p}(\mu_{k+1})\big) = \mathtt{S}_{k+1}^{k+2-p}(\mu_{k+1})$ for every $0 \leq q < p$.

In conclution, if $\mathcal{F}_p^* = \{\mathtt{S}_p\} \cup \{\mathtt{T}_q^p : 0 \leq q < p\}$ for every $p \in \{1, \dots, k+1\}$ and $\mathcal{U}_{k+1}^* = \{\mathtt{T}_{k+1}^{k+1}\}$, $\mathcal{U}_p^* = \{\mathtt{F} \circ \mathtt{G} : \mathtt{G} \in \mathcal{U}_{p+1}^*, \mathtt{F} \in \mathcal{F}_{p+1}^*\}$ for every $p \in \{1, \dots, k\}$, then there exist idempotents $\mu_1, \mu_2, \dots, \mu_{k+1}$ of θX_{k+1} such that $\mu_p \in \theta V_p^0$ is a minimal idempotent for θX_p for every $p \in \{1, \dots, k+1\}$ satisfing:

(1) $\mu_{p_1} + \mu_{p_2} = \mu_{p_2} + \mu_{p_1} = \mu_{\max\{p_1, p_2\}}$ for every $1 \leq p_1, p_2 \leq k+1$, and

(2) $\mu_p = \mathtt{H}(\mu_{k+1})$ for every $\mathtt{H} \in \mathcal{U}_p^*$, $1 \leq p \leq k+1$.

Let $\mathcal{F}_p = \{S_p\} \cup \{T_q^p : 0 \leq q < p\}$ for every $p \in \{1, \dots, k+1\}$ and $\mathcal{U}_{k+1} = \{T_{k+1}^{k+1}\}$, $\mathcal{U}_p = \{F \circ G : G \in \mathcal{U}_{p+1}, F \in \mathcal{F}_{p+1}\}$ for every $p \in \{1, \dots, k\}$.

Claim 2 We will construct, by induction on n, the required sequence $(w_n)_{n \in \mathbb{N}}$ in V_{k+1}^0. Since, $X_{k+1} = A_1 \cup \dots \cup A_r$, there exist $1 \leq i_p \leq r$ such that $\mu_p(A_{i_p}) = 1$ for every $1 \leq p \leq k+1$. Then $\mu_p(A_{i_p} \cap V_p^0) = 1$. Starting with $w_0 \in V_{k+1}^0$ and $B_{p,1} = A_{i_p} \cap V_p^0$ for every $1 \leq p \leq k+1$, can be constructed inductively an increasing sequence $w_1 < w_2 < \cdots$ in V_{k+1}^0 and $k+1$ decreasing sequences $A_{i_p} \cap V_p^0 \supseteq B_{p,1} \supseteq B_{p,2} \supseteq \cdots$ for $1 \leq p \leq k+1$, such that for every $n \in \mathbb{N}$ to hold:

(i) $\mu_p(B_{p,n}) = 1$ for every $1 \leq p \leq k+1$,

(ii) $w_n \in B_{k+1,n}$ and $H(w_n) \in B_{p,n}$ for every $1 \leq p \leq k$, $H \in \mathcal{U}_p$,

(iii) $B_{k+1,n+1} = \{u \in (B_{k+1,n})_{w_n} : H_1(w_n) + H_2(u) \in B_{k+1,n}$ for $H_1, H_2 \in \bigcup_{q=1}^{k+1} \mathcal{U}_q$, $\{H_1, H_2\} \cap \mathcal{U}_{k+1} \neq \emptyset\}$, and, for $1 \leq p \leq k$,
$B_{p,n+1} = \{u \in (B_{p,n})_{w_n} : H_1(w_n) + u \in B_{q,n}$ for all $k+1 \leq q \leq p+1, H_1 \in \mathcal{U}_q$, and $H_2(w_n) + u \in B_{p,n}$ for all $H_2 \in \bigcup_{q=1}^{p} \mathcal{U}_q\}$.

The proof of Claim 2 follows from the properties (1), (2) of the idempotent ultrafilters.

The sequence $(w_n)_{n\in\mathbb{N}}$ has the required properties. We will prove by induction on λ that

$$H_1(w_{n_1}) + \ldots + H_\lambda(w_{n_\lambda}) \in B_{k+1,n_1} \subseteq A_{i_{k+1}} \cap V_{k+1}^0$$

for every $\lambda \in \mathbb{N}$, $n_1 < \cdots < n_\lambda \in \mathbb{N}$, $H_1, \ldots, H_\lambda \in \bigcup_{q=1}^{k+1} \mathcal{U}_q$ with $\{H_1, \ldots, H_\lambda\} \cap \mathcal{U}_{k+1} \neq \emptyset$, and also that, for every $1 \leq p \leq k$,

$$H_1(w_{n_1}) + \ldots + H_\lambda(w_{n_\lambda}) \in B_{p,n_1} \subseteq A_{i_p} \cap V_p^0$$

for every $\lambda \in \mathbb{N}$, $n_1 < \cdots < n_\lambda \in \mathbb{N}$, $H_1, \ldots, H_\lambda \in \bigcup_{q=1}^{p} \mathcal{U}_q$ with $\{H_1, \ldots, H_\lambda\} \cap \mathcal{U}_p \neq \emptyset$.

Indeed, for $n_1 \in \mathbb{N}$, we have $T_{k+1}^{k+1}(w_{n_1}) = w_{n_1} \in B_{k+1,n_1}$ and $H_1(w_{n_1}) \in B_{p,n_1}$ for every $1 \leq p \leq k$, $H_1 \in \mathcal{U}_p$. Assume that the accertion holds for some $\lambda \geq 1$ and let $n_1 < \cdots < n_\lambda < n_{\lambda+1} \in \mathbb{N}$ and $H_1, \ldots, H_\lambda, H_{\lambda+1} \in \bigcup_{q=1}^{k+1} \mathcal{U}_q$.

Case 1. If there exists $1 \leq p \leq k+1$ such that $H_1, \ldots, H_\lambda, H_{\lambda+1} \in \bigcup_{q=1}^{p} \mathcal{U}_q$ and $\{H_2, \ldots, H_{\lambda+1}\} \cap \mathcal{U}_p \neq \emptyset$, then, according to the induction hypothesis, $u = H_2(w_{n_2}) + \ldots + H_{\lambda+1}(w_{n_{\lambda+1}}) \in B_{p,n_2} \subseteq B_{p,n_1+1}$. Hence, $H_1(w_{n_1}) + u = H_1(w_{n_1}) + \ldots + H_{\lambda+1}(w_{n_{\lambda+1}}) \in B_{p,n_1}$.

Case 2. If there exists $1 \leq p \leq k+1$ such that $H_1, \ldots, H_\lambda, H_{\lambda+1} \in \bigcup_{q=1}^{p} \mathcal{U}_q$ and $\{H_2, \ldots, H_{\lambda+1}\} \cap \mathcal{U}_p = \emptyset$, $H_1 \in \mathcal{U}_p$, then, let $1 \leq p_1 < p$ such that $H_2, \ldots, H_\lambda, H_{\lambda+1} \in \bigcup_{q=1}^{p_1} \mathcal{U}_q$ and $\{H_2, \ldots, H_{\lambda+1}\} \cap \mathcal{U}_{p_1} \neq \emptyset$. According to the induction hypothesis, $u = H_2(w_{n_2}) + \ldots + H_{\lambda+1}(w_{n_{\lambda+1}}) \in B_{p_1,n_2} \subseteq B_{p_1,n_1+1}$. Hence, $H_1(w_{n_1}) + u = H_1(w_{n_1}) + \ldots + H_{\lambda+1}(w_{n_{\lambda+1}}) \in B_{p,n_1}$.

This finishes the proof. $\square$

2. Sets of substitutions for variable located words over a finite alphabet

In this section we introduce the notion of sets of substitutions for variable located words over a finite alphabet, in order to state and prove refined partition theorems for variable located words (Theorem 2.3 below). These refined partition theorems can be the starting points for proving Ramsey type partition theorems, corresponding to each countable ordinal, for variable located words, as we do in Section 3, and also Nash-Williams type partition theorems for infinite sequences of variable located words, as we do in Section 4.

Definition 2.1. Let $\Sigma = \{\alpha_1, ..., \alpha_k\}$, $k \in \mathbb{N}$ be a finite non-empty alphabet and $\upsilon \notin \Sigma$ a variable. We define a set of functions $\{R_1, \dots, R_{k+1}\}$ to be a *set of substitutions* for $L(\Sigma \cup \{\upsilon\})$ if, the functions

$$R_i : L(\Sigma \cup \{\upsilon\}) \longrightarrow L(\Sigma \cup \{\upsilon\}), \text{ for every } 1 \leq i \leq k+1,$$

satisfy the following four properties:

(1) $R_i(w \star u) = R_i(w) \star R_i(u)$ for every $w, u \in L(\Sigma \cup \{\upsilon\})$ with $w < u$,

(2) R_{k+1} is the identity function on $L(\Sigma \cup \{\upsilon\})$,

(3) if $n \in \mathbb{N}$ and $w \in L(\Sigma \cup \{\upsilon\})$ with $w : \{n\} \longrightarrow \{\upsilon\}$, then $R_i(w) : \{n\} \longrightarrow \{\alpha_i\}$ for every $1 \leq i \leq k$, and

(4) for every finite coloring $L(\Sigma; \upsilon) = A_1 \cup \dots \cup A_r$ of $L(\Sigma; \upsilon)$ there exist a sequence $(w_n)_{n \in \mathbb{N}}$ in $L(\Sigma; \upsilon)$ with $w_n < w_{n+1}$ for every $n \in \mathbb{N}$ and $1 \leq i_0 \leq r$ such that

$$R_{i_1}(w_{n_1}) \star \ldots \star R_{i_\lambda}(w_{n_\lambda}) \in A_{i_0}$$

for every $n_1 < \dots < n_\lambda \in \mathbb{N}$ and $i_1, \dots, i_\lambda \subseteq \{1, 2, \dots, k+1\}$ with $k+1 \in \{i_1, \dots, i_\lambda\}$.

If $\Sigma = \emptyset$, then the set of *substitutions* for $L(\{\upsilon\})$ is the set $E(\emptyset) = \{R_1\}$, where R_1 is the identity function on $L(\{\upsilon\})$.

Remark 2.2. Let $\Sigma = \{\alpha_1, ..., \alpha_k\}$, $k \in \mathbb{N}$ be an alphabet and $\upsilon \notin \Sigma$ a variable.

(i) According to Theorem 1.2, all the sets

$$\{H_1, \dots, H_{k+1}\} \subseteq \bigcup_{q=1}^{k+1} \mathcal{U}_q$$

such that, if $n \in \mathbb{N}$ and $w \in L(\Sigma \cup \{\upsilon\})$ with $w : \{n\} \longrightarrow \{\upsilon\}$, then $R_i(w) : \{n\} \longrightarrow \{\alpha_i\}$ for every $1 \leq i \leq k$, and $H_{k+1} = T_{k+1}^{k+1}$.

(ii) The set

$$\{(S_{k+1})^{k-i+1} : 1 \leq i \leq k+1\}$$

is a set of substitutions for $L(\Sigma \cup \{\upsilon\})$ (Gowers substitutions).

(iii) The set

$$\{T_p^{k+1} : 1 \leq i \leq k+1\},$$

is a set of substitutions for $L(\Sigma \cup \{\upsilon\})$ (Bergelson,Blass,Hindman substitutions).

(iv) Let $m \in \mathbb{N}$, $1 \leq m < k$ and $\{n_1, \dots, n_m\} \subseteq \{1, 2, \dots, k\}$. Then the set

$$\{T_i^{k+1} : i \in \{n_1, \dots, n_m, k+1\}\} \cup \{(S_{k+1})^{k+1-j} : 1 \leq j \leq n, j \notin \{n_1, \dots, n_m\}\}$$

is a set of substitutions for $L(\Sigma\cup\{\upsilon\})$, according to Corollary 1.3.

(v) The sets of the form

$$\{(S_{k+1})^{\epsilon_1}\circ T^{k+1}_{p_1},\dots,(S_{k+1})^{\epsilon_k}\circ T^{k+1}_{p_k},(S_{k+1})^{0}\circ T^{k+1}_{k+1}\},$$

where $0\leq p_1,\dots,p_k\leq k$, $0\leq \epsilon_1,\dots,\epsilon_k\leq k-1$ and, if $n\in\mathbb{N}$ and $w\in L(\Sigma\cup\{\upsilon\})$ with $w:\{n\}\longrightarrow\{\upsilon\}$, then $(S_{k+1})^{\epsilon_j}\circ T^{k+1}_{p_j}(w):\{n\}\longrightarrow\{\alpha_i\}$ for every $1\leq j\leq k$, are sets of substitutions for $L(\Sigma\cup\{\upsilon\})$.

Let Σ be a finite alphabet and $\upsilon\notin\Sigma$. We denote by $L^{\infty}(\Sigma;\upsilon)$ (resp. $L^{<\infty}(\Sigma;\upsilon)$) the family of all infinite (resp. finite) *orderly sequences of variable located words* over the alphabet Σ; thus

$$L^{<\infty}(\Sigma;\upsilon)=\{\mathbf{w}=(w_1,\dots,w_l): l\in\mathbb{N},\ w_1<\dots<w_l\in L(\Sigma;\upsilon)\}\cup\{\emptyset\},\text{ and}$$
$$L^{\infty}(\Sigma;\upsilon)=\{\vec{w}=(w_n)_{n\in\mathbb{N}}: w_n\in L(\Sigma;\upsilon)\text{ and } w_n<w_{n+1}\ \forall\ n\in\mathbb{N}\}.$$

By substitution and concatenation of the words of a given orderly sequence of variable located words we can extract new words and sequences.

Extractions of an orderly sequence of variable located words. Let Σ be a finite alphabet with cardinality $k\in\mathbb{N}\cup\{0\}$, $\upsilon\notin\Sigma$ and $\{R_1,\dots,R_{k+1}\}$ be a set of substitutions for $L(\Sigma\cup\{\upsilon\})$. For a given infinite orderly sequence $\vec{w}=(w_n)_{n\in\mathbb{N}}\in L^{\infty}(\Sigma;\upsilon)$ of variable located words over Σ are defined the set of *extracted variable located words* and the sets of *extracted finite and infinite sequences* of variable located words of $\vec{w}$ as follows:

$$EL(\vec{w})=\{u=R_{i_1}\big(w_{n_1}\big)\star\dots\star R_{i_\lambda}\big(w_{n_\lambda}\big)\in L(\Sigma;\upsilon):\lambda\in\mathbb{N}, n_1<\dots<n_\lambda\in\mathbb{N},$$
$$1\leq i_1,\dots,i_\lambda\leq k+1\text{ and } k+1\in\{i_1,\dots,i_\lambda\}\};$$

$EL^{<\infty}(\vec{w})=\{\mathbf{u}=(u_1,\dots,u_l)\in L^{<\infty}(\Sigma\cup\{\upsilon\}): l\in\mathbb{N}, u_1,\dots,u_l\in EL(\vec{w})\}\cup\{\emptyset\}$; and
$EL^{\infty}(\vec{w})=\{\vec{u}=(u_n)_{n\in\mathbb{N}}\in L^{\infty}(\Sigma;\upsilon): u_n\in EL(\vec{w})\text{ for every } n\in\mathbb{N}\}$.
We write $\vec{u}\prec\vec{w}$ if and only if $\vec{u}\in EL^{\infty}(\vec{w})$ if and only if $EL(\vec{u})\subseteq EL(\vec{w})$. Notice that $\vec{w}\prec\vec{e}$ for every $\vec{w}\in L^{<\infty}(\Sigma;\upsilon)$, where $\vec{e}=(e_n)_{n\in\mathbb{N}}$ with $e_n=\upsilon$ for every $n\in\mathbb{N}$.

Using the notion of extractions for a given set of substitutions for $L(\Sigma\cup\{\upsilon\})$, stroger partition theorems for located words can be proved, according to the following theorem.

Theorem 2.3. *Let Σ be a finite ordered alphabet of cardinality $k\in\mathbb{N}\cup\{0\}$,$\upsilon\notin\Sigma$ be a variable and $\{R_1,\dots,R_{k+1}\}$ be a set of substitutions for $L(\Sigma\cup\{\upsilon\})$. For every finite coloring $L(\Sigma;\upsilon)=A_1\cup\dots\cup A_r$ of $L(\Sigma;\upsilon)$ and every infinite orderly sequence $\vec{w}\in L^{\infty}(\Sigma;\upsilon)$ of variable located words over Σ there exist an extraction $\vec{u}\prec\vec{w}$ of $\vec{w}$ and $1\leq i_0\leq r$ satisfing $EL(\vec{u})\subseteq A_{i_0}$.*

Proof. There exist an one to one and onto correspondence between the set $EL(\vec{w})$ of the extracted variable located words of $\vec{w}$, according to $\{R_1, \dots, R_{k+1}\}$ and the set $L(\Sigma; \upsilon)$, which in case $\Sigma = \{\alpha_1, ..., \alpha_k\}$ for $k \in \mathbb{N}$ is given by the function

$$\phi : L(\Sigma; \upsilon) \to EL(\vec{w}) \text{ with}$$

$\phi(t_{n_1} \dots t_{n_\lambda}) = R_{i_1}(w_{n_1}) \star \dots \star R_{i_\lambda}(w_{n_\lambda})$, where $i_j = k+1$ if $t_{n_j} = \upsilon$ and $i_j = p$ if $t_{n_j} = \alpha_p$, $1 \leq p \leq k$, for every $1 \leq j \leq \lambda$.
In case $\Sigma = \emptyset$, for $t_{n_1} \dots t_{n_\lambda} \in L(\Sigma; \upsilon)$ we set $\phi(t_{n_1} \dots t_{n_\lambda}) = w_{n_1} \star \dots \star w_{n_\lambda}$.

According to Definition 2.1 (4), there exist an infinite sequence $\vec{t} = (t_n)_{n \in \mathbb{N}}$ in $L(\Sigma; \upsilon)$ with $t_n < t_{n+1}$ for every $n \in \mathbb{N}$ and $1 \leq i_0 \leq r$ satisfing $EL(\vec{t}) \subseteq (\phi)^{-1}(A_{i_0})$.

Set $u_n = \phi(t_n) \in EL(\vec{w})$ for every $n \in \mathbb{N}$ and $\vec{u} = (u_n)_{n \in \mathbb{N}}$. Then $\vec{u} \prec \vec{w}$ and $EL(\vec{u}) \subseteq \phi(EL(\vec{t}))$. Hence, $EL(\vec{u}) \subseteq A_{i_0}$. □

Remark 2.4. (i) We can obtain refined partition theorems for variable located words, appling Theorem 2.3 for the sets of substitutions referred to Remark 2.2(i).

(ii) A partition theorem stonger than Gowers's partition theorem proved in [G] (Theorem 5) can be proved appling Theorem 2.3 for the set of substitutions referred to Remark 2.2(iii).

(iii) Bergelson, Blass and Hindman in [BBH] (Corollary 4.3) proved a result analogous to Theorem 2.3 using the set of substitutions referred to Remark 2.2(iv).

3. Partition theorems for finite sequences of variable located words over a finite alphabet

Given a set of substitutions for the variable located words over a finite alphabet Σ, we can prove Ramsey type partition theorems of every countable order ξ (Theorem 3.5) for the variable located words over Σ, extenting Theorem 2.3, corresponting to case $\xi = 1$, to every countable order ξ.

Applying Theorem 3.5 for concrete sets of substitutions we get corresponding partition theorems for finite sequences of variable located words over a finite alphabet Σ of every countable order. In particular, we can get an extension of Gowers partition theorem (Theorem 5 in [G]) to every countable ordinal ξ, and an analogous extension to every countable order ξ of Theorems 4.1 and 5.1 in [BBH] of Bergelson, Blass, Hindman, corresponding to finite ordinals $\xi < \omega$. Theorem 3.5 for $\Sigma = \emptyset$ has been proved in [FN](Theorem 2.6).

In order to state Theorem 3.5 we will need the definition of the system $(L^\xi(\Sigma;\upsilon))_{\xi<\omega_1}$ of Schreier families of finite orderly sequences of variable located words over a finite alphabet Σ. Instrumental for this definition is the recursive system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ of thin Schreier families of finite ordered sets of natural numbers given below, where (in case 3(iii)) the Cantor normal form of ordinals (cf. [KM], [L]) is employed.

We denote by $[X]^{<\omega}$ the set of all finite subsets and by $[X]^{<\omega}_{>0}$ the set of all non-empty, finite subsets of a set X. For $s_1, s_2 \in [\mathbb{N}]^{<\omega}_{>0}$ we write $s_1 < s_2$ if $\max s_1 < \min s_2$.

Definition 3.1 (The Schreier system, [F1, Def. 7], [F2, Def. 1.5] [F3, Def. 1.4])**.** For every non-zero, countable, limit ordinal λ choose and fix a strictly increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ of successor ordinals smaller than λ with $\sup_n \lambda_n = \lambda$. The system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ is defined recursively as follows:

(1) $\mathcal{A}_0 = \{\emptyset\}$ and $\mathcal{A}_1 = \{\{n\} : n \in \mathbb{N}\}$;

(2) $\mathcal{A}_{\zeta+1} = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s = \{n\} \cup s_1$, where $n \in \mathbb{N}$, $\{n\} < s_1$ and $s_1 \in \mathcal{A}_\zeta\}$;

(3i) $\mathcal{A}_{\omega^{\beta+1}} = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s = \bigcup_{i=1}^n s_i$, where $n = \min s_1$, $s_1 < \cdots < s_n$ and $s_1, \ldots, s_n \in \mathcal{A}_{\omega^\beta}\}$;

(3ii) for a non-zero, countable limit ordinal λ,
$\mathcal{A}_{\omega^\lambda} = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s \in \mathcal{A}_{\omega^{\lambda_n}}$ with $n = \min s\}$; and

(3iii) for a limit ordinal ξ such that $\omega^\alpha < \xi < \omega^{\alpha+1}$ for some $0 < \alpha < \omega_1$, if $\xi = \omega^\alpha p + \sum_{i=1}^m \omega^{a_i} p_i$, where $m \in \mathbb{N}$ with $m \geq 0$, $p, p_1, \ldots, p_m$ are natural numbers with $p, p_1, \ldots, p_m \geq 1$ (so that either $p > 1$, or $p = 1$ and $m \geq 1$) and $a, a_1, \ldots, a_m$ are ordinals with $a > a_1 > \cdots a_m > 0$,
$\mathcal{A}_\xi = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s = s_0 \cup (\bigcup_{i=1}^m s_i)$ with $s_m < \cdots < s_1 < s_0$, $s_0 = s_1^0 \cup \cdots \cup s_p^0$ with $s_1^0 < \cdots < s_p^0 \in \mathcal{A}_{\omega^a}$, and $s_i = s_1^i \cup \cdots \cup s_{p_i}^i$ with $s_1^i < \cdots < s_{p_i}^i \in \mathcal{A}_{\omega^{a_i}}$ $\forall\ 1 \leq i \leq m\}$.

Definition 3.2 (The Schreier systems $(L^\xi(\Sigma;\upsilon))_{\xi<\omega_1}$)**.** Let Σ be a finite alphabet and $\upsilon \notin \Sigma$. We define the families $L^\xi(\Sigma;\upsilon)$ for every countable ordinal ξ as follows:

$L^0(\Sigma;\upsilon) = \{\emptyset\}$; and

for every countable ordinal $\xi \geq 1$,

$L^\xi(\Sigma;\upsilon) = \{\mathbf{w} = (w_1, \ldots, w_k) \in L^{<\infty}(\Sigma;\upsilon) : \{\min dom(w_1), \ldots, \min dom(w_k)\} \in \mathcal{A}_\xi\}$.

Remark 3.3. (i) $L^\xi(\Sigma;\upsilon) \subseteq L^{<\infty}(\Sigma;\upsilon)$ and $\emptyset \notin L^\xi(\Sigma)$ for every $\xi \geq 1$.

(ii) For $k \in \mathbb{N}$
$L^k(\Sigma;\upsilon) = \{(w_1, \ldots, w_k) : w_1 < \cdots < w_k \in L(\Sigma;\upsilon)\}$.

(iii) $L^\omega(\Sigma;\upsilon) = \{(w_1, \ldots, w_n) \in L^{<\infty}(\Sigma;\upsilon) : n \in \mathbb{N}$, and $\min dom(w_1) = n\}$.

The following proposition justifies the recursiveness of the system $(L^{\xi}(\Sigma;\upsilon))_{\xi<\omega_1}$. For a family $\mathcal{F}\subseteq L^{<\infty}(\Sigma\cup\{\upsilon\})$ and a located word $t\in L(\Sigma\cup\{\upsilon\})$, we set

$\mathcal{F}(t)=\{\mathbf{w}\in L^{<\infty}(\Sigma\cup\{\upsilon\})$: either $\mathbf{w}=(w_1,\dots,w_l)\neq\emptyset$ and $(t,w_1,w_2,\dots,w_l)\in\mathcal{F}$

or $\mathbf{w}=\emptyset$ and $(s)\in\mathcal{F}\}$,

$\mathcal{F}-t=\{\mathbf{w}\in\mathcal{F}$: either $\mathbf{w}=(w_1,\dots,w_l)\neq\emptyset$ and $t<w_1$, or $\mathbf{w}=\emptyset\}$.

Proposition 3.4. *For every countable ordinal $\xi\geq 1$, there exists a concrete sequence (ξ_n) of countable ordinals with $\xi_n<\xi$ such that for every finite alphabet Σ, $t\in L(\Sigma;\upsilon)$, with $\min dom(t)=n$,*

$$L^{\xi}(\Sigma;\upsilon)(t)=L^{\xi_n}(\Sigma;\upsilon)\cap(L^{<\infty}(\Sigma;\upsilon)-t).$$

Moreover, $\xi_n=\zeta$ for every $n\in\mathbb{N}$ if $\xi=\zeta+1$, and (ξ_n) is a strictly increasing sequence with $\sup_n\xi_n=\xi$ if ξ is a limit ordinal.

Proof. It follows from Proposition 1.6 in [F3], according to which for every countable ordinal $\xi>0$ there exists a concrete sequence (ξ_n) of countable ordinals, with $\xi_n<\xi$, such that $\mathcal{A}_{\xi}(n)=\mathcal{A}_{\xi_n}\cap[\{n+1,n+2,\dots\}]^{<\omega}$ for every $n\in\mathbb{N}$, where,

$\mathcal{A}_{\xi}(n)=\{s\in[\mathbb{N}]^{<\omega}: s\in[\mathbb{N}]^{<\omega}_{>0}, n<\min s$ and $\{n\}\cup s\in\mathcal{A}_{\xi}$ or $s=\emptyset$ and $\{n\}\in\mathcal{A}_{\xi}\}$.

Moreover, $\xi_n=\zeta$ for every $n\in\mathbb{N}$ if $\xi=\zeta+1$, and (ξ_n) is a strictly increasing sequence with $\sup_n\xi_n=\xi$ if ξ is a limit ordinal □

The principal result of this Section is the following:

Theorem 3.5 (Exteded Ramsey type partition theorem for located words). *Let $\xi\geq 1$ be a countable ordinal, Σ be a finite ordered alphabet of cardinality $k\in\mathbb{N}\cup\{0\}$,$\upsilon\notin\Sigma$ be a variable and $\{R_1,\dots,R_{k+1}\}$ be a set of substitutions for $L(\Sigma\cup\{\upsilon\})$. For every family $\mathcal{F}\subseteq L^{<\infty}(\Sigma;\upsilon)$ of finite orderly sequences of variable located words over Σ and every infinite orderly sequence $\vec{w}\in L^{\infty}(\Sigma;\upsilon)$ of variable located words over Σ there exists an extraction $\vec{u}\prec\vec{w}$ of $\vec{w}$ over Σ such that*

either $L^{\xi}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{u})\subseteq\mathcal{F}$, or $L^{\xi}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{u})\subseteq L^{<\infty}(\Sigma;\upsilon)\setminus\mathcal{F}$.

For the proof of Theorem 3.5 we will make use of a diagonal argument, contained in the following Lemma 3.6.

Notation. Let Σ be a finite ordered alphabet of cardinality $k\in\mathbb{N}\cup\{0\}$ and $\upsilon\notin\Sigma$. For $\vec{w}=(w_n)_{n\in\mathbb{N}}\in L^{\infty}(\Sigma;\upsilon)$, $s\in L(\Sigma;\upsilon)$ and $\mathbf{s}=(s_1,\dots,s_l)\in L^{<\infty}(\Sigma;\upsilon)$, we set

$\vec{w}-s=(w_n)_{n\geq l}\in L^{\infty}(\Sigma;\upsilon)$, where $l=\min\{n\in\mathbb{N}: s<w_n\}$,

$\vec{w}-\mathbf{s}=\vec{w}-s_l$.

for a given set $\{R_1, \dots, R_{k+1}\}$ of substitutions for $L(\Sigma \cup \{\upsilon\})$ we define the set of extracted variable located words of $\mathbf{s}$ as follows:

$$EL(\mathbf{s}) = \{u = R_{i_1}(s_{n_1}) \star \dots \star R_{i_\lambda}(s_{n_\lambda}) \in L(\Sigma;\upsilon) : 1 \leq n_1 < \dots < n_\lambda \leq l \text{ and } 1 \leq i_1, \dots, i_m \leq k+1\}.$$

Lemma 3.6. *Let Σ be a finite ordered alphabet of cardinality $k \in \mathbb{N} \cup \{0\}$,$\upsilon \notin \Sigma$ be a variable, $\{R_1, \dots, R_{k+1}\}$ be a set of substitutions for $L(\Sigma \cup \{\upsilon\})$, $\vec{w} = (w_n)_{n \in \mathbb{N}} \in L^\infty(\Sigma;\upsilon)$ an infinite orderly sequence of variable located words over Σ and $\Pi = \{(w, \vec{s}) : w \in L(\Sigma;\upsilon),\ \vec{s} = (s_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma;\upsilon)$ with $\vec{s} \prec \vec{w}$ and $w < s_n \forall\ n \in \mathbb{N}\}$. If a subset $\mathcal{R}$ of Π satisfies*

(i) *for every $(w, \vec{s}) \in \Pi$, there exists $(w, \vec{s}_1) \in \mathcal{R}$ with $\vec{s}_1 \prec \vec{s}$; and*

(ii) *for every $(w, \vec{s}) \in \mathcal{R}$ and $\vec{s}_1 \prec \vec{s}$, we have $(w, \vec{s}_1) \in \mathcal{R}$,*

then there exists $\vec{u} \prec \vec{w}$, such that $(w, \vec{s}) \in \mathcal{R}$ for all $w \in EL(\vec{u})$ and $\vec{s} \prec \vec{u} - w$.

Proof. Let $u_0 = w_1$. According to condition (i), there exists $\vec{s}_1 = (s_n^1)_{n\in\mathbb{N}} \in L^\infty(\Sigma;\upsilon)$ with $\vec{s}_1 \prec \vec{w} - u_0$ such that $(u_0, \vec{s}_1) \in \mathcal{R}$. Let $u_1 = s_1^1$. Of course, $u_0 < u_1$ and $u_0, u_1 \in EL(\vec{w})$. We assume now that there have been constructed $\vec{s}_1, \dots, \vec{s}_n \in L^\infty(\Sigma;\upsilon)$ and $u_0, u_1, \dots, u_n \in EL(\vec{w})$, with $\vec{s}_n \prec \dots \prec \vec{s}_1 \prec \vec{w}$, $u_0 < u_1 < \dots < u_n$ and $(s, \vec{s}_i) \in \mathcal{R}$ for all $1 \leq i \leq n$, $s \in EL((u_0, \dots, u_{i-1}))$.

We will construct $\vec{s}_{n+1}$ and u_{n+1}. Let $\{t_1, \dots, t_k\} = EL((u_0, \dots, u_n))$. According to condition (i), there exist $\vec{s}^{\,1}_{n+1}, \dots, \vec{s}^{\,k}_{n+1} \in L^\infty(\Sigma;\upsilon)$ such that $\vec{s}^{\,k}_{n+1} \prec \dots \prec \vec{s}^{\,1}_{n+1} \prec \vec{s}_n - u_n$ and $(t_i, \vec{s}^{\,i}_{n+1}) \in \mathcal{R}$ for every $1 \leq i \leq k$. Set $\vec{s}_{n+1} = \vec{s}^{\,k}_{n+1}$. If $\vec{s}_{n+1} = (s_n^{n+1})_{n\in\mathbb{N}}$, set $u_{n+1} = s_1^{n+1}$. Of course $u_n < u_{n+1}$, $u_{n+1} \in EL(\vec{w})$ and, according to condition (ii), $(t_i, \vec{s}_{n+1}) \in \mathcal{R}$ for all $1 \leq i \leq k$.

Set $\vec{u} = (u_0, u_1, u_2, \dots) \in L^\infty(\Sigma;\upsilon)$. Then $\vec{u} \prec \vec{w}$, since $u_0 < u_1 < \dots \in EL(\vec{w})$. Let $w \in EL(\vec{u})$ and $\vec{s} \prec \vec{u} - w$. Set $n_0 = \min\{n \in \mathbb{N} : w \in EL((u_0, u_1, \dots, u_n))\}$. Since $w \in EL((u_0, u_1, \dots, u_{n_0}))$, we have $(w, \vec{s}_{n_0+1}) \in \mathcal{R}$. Then, according to (ii), we have that $(w, \vec{u} - u_{n_0}) \in \mathcal{R}$, since $\vec{u} - u_{n_0} \prec \vec{s}_{n_0+1}$, and also that $(w, \vec{s}) \in \mathcal{R}$, since $\vec{s} \prec \vec{u} - u_{n_0} = \vec{u} - w$. $\square$

Proof of Theorem 3.5. For $\xi = 1$ the theorem holds, according to Theorem 2.3. Let $\xi > 1$. Assume that the theorem is valid for every $\zeta < \xi$. Let $\mathcal{F} \subseteq L^{<\infty}(\Sigma;\upsilon)$ and $\vec{w} \in L^\infty(\Sigma;\upsilon)$. If $t \in L(\Sigma;\upsilon)$ with $\min dom(t) = n$ and $\vec{s} = (s_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma;\upsilon)$ with $\vec{s} \prec \vec{w}$, then, according to Proposition 3.4, there exists $\xi_n < \xi$ such that

$$L^\xi(\Sigma;\upsilon)(t) = L^{\xi_n}(\Sigma;\upsilon) \cap (L^{<\infty}(\Sigma;\upsilon) - t).$$

Using the induction hypothesis, there exists $\vec{s}_1 \in L^\infty(\Sigma; \upsilon)$ with $\vec{s}_1 \prec \vec{s}$ such that

either $L^{\xi_n}(\Sigma; \upsilon) \cap EL^{<\infty}(\vec{s}_1) \subseteq \mathcal{F}(t)$, or $L^{\xi_n}(\Sigma; \upsilon) \cap EL^{<\omega}(\vec{s}_1) \subseteq L^{<\infty}(\Sigma; \upsilon) \setminus \mathcal{F}(t)$.

Set $\vec{s}_t = \vec{s}_1 - t$. Then $\vec{s}_t \prec \vec{s}_1 \prec \vec{s} \prec \vec{w}$, and

either $L^\xi(\Sigma; \upsilon)(t) \cap EL^{<\infty}(\vec{s}_t) \subseteq \mathcal{F}(t)$, or $L^\xi(\Sigma; \upsilon)(t) \cap EL^{<\omega}(\vec{s}_t) \subseteq L^{<\infty}(\Sigma; \upsilon) \setminus \mathcal{F}(t)$.

Let $\mathcal{R} = \{(t, \vec{s}) : t \in L(\Sigma; \upsilon),\ \vec{s} = (s_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma; \upsilon)$ with $\vec{s} \prec \vec{w},\ t < s_n \forall\ n \in \mathbb{N}$, and

either $L^\xi(\Sigma; \upsilon)(t) \cap EL^{<\infty}(\vec{s}) \subseteq \mathcal{F}(t)$, or $L^\xi(\Sigma; \upsilon)(t) \cap EL^{<\infty}(\vec{s}) \subseteq L^{<\infty}(\Sigma; \upsilon) \setminus \mathcal{F}(t)\}$.

The family $\mathcal{R}$ satisfies the conditions (i) (by the above arguments) and (ii) (obviously) of Lemma 3.6. Hence there exists $\vec{u}_1 \prec \vec{w}$ such that $(t, \vec{s}) \in \mathcal{R}$ for all $t \in EL(\vec{u}_1)$ and $\vec{s} \prec \vec{u}_1 - t$.

Let $\mathcal{F}_1 = \{t \in EL(\vec{u}_1) : L^\xi(\Sigma; \upsilon)(t) \cap EL^{<\infty}(\vec{u}_1 - t) \subseteq \mathcal{F}(t)\}$.

We use the induction hypothesis for $\xi = 1$ (Theorem 2.3). Then there exists a variable extraction $\vec{u} \prec \vec{u}_1$ of $\vec{u}_1$ such that

either $EL(\vec{u}) \subseteq \mathcal{F}_1$, or $EL(\vec{u}) \subseteq L(\Sigma; \upsilon) \setminus \mathcal{F}_1$.

Since $\vec{u} \prec \vec{u}_1$ we have that $EL(\vec{u}) \subseteq EL(\vec{u}_1)$, and, consequently, that $(t, \vec{u} - t) \in \mathcal{R}$ for all $t \in EL(\vec{u})$. Thus

either $L^\xi(\Sigma; \upsilon)(t) \cap EL^{<\infty}(\vec{u} - t) \subseteq \mathcal{F}(t)$ for all $t \in EL(\vec{u})$,

or $L^\xi(\Sigma; \upsilon)(t) \cap EL^{<\infty}(\vec{u} - t) \subseteq L^{<\infty}(\Sigma; \upsilon) \setminus \mathcal{F}(t)$ for all $t \in EL(\vec{u})$.

Hence,

either $L^\xi(\Sigma; \upsilon) \cap EL^{<\infty}(\vec{u}) \subseteq \mathcal{F}$, or $L^\xi(\Sigma; \upsilon) \cap EL^{<\infty}(\vec{u}) \subseteq L^{<\infty}(\Sigma; \upsilon) \setminus \mathcal{F}$. □

The particular case of Theorem 3.5, where ξ is a finite ordinal, has the following statement:

Corollary 3.7 (Ramsey type partition theorem for variable located words). *Let $k \in \mathbb{N}$, Σ be a finite ordered alphabet of cardinality $k \in \mathbb{N} \cup \{0\}$, $\upsilon \notin \Sigma$ be a variable, $\{R_1, \ldots, R_{k+1}\}$ be a set of substitutions for $L(\Sigma \cup \{\upsilon\})$ and $\vec{w} \in L^\infty(\Sigma; \upsilon)$ an infinite orderly sequence of variable located words over Σ. For every finite coloring $L^k(\Sigma; \upsilon) = A_1 \cup \cdots \cup A_r$ of $L^k(\Sigma; \upsilon)$ there exist an extraction $\vec{u} \prec \vec{w}$ of $\vec{w}$ over Σ and $1 \leq i_0 \leq r$ such that*

$$\{(t_1, \ldots, t_k) \in L^{<\infty}(\Sigma; \upsilon) : t_1, \ldots, t_k \in EL(\vec{u})\} \subseteq A_{i_0}.$$

4. Partition theorems for sequences of variable located words

The main result of this Section is Theorem 4.13 which strengthens Theorem 3.5 in case the partition family is a tree. Specificaly, given a partition family $\mathcal{F} \subseteq L^{<\infty}(\Sigma; \upsilon)$ of finite orderly sequences of variable located words over a finite alphabet Σ, a set of substitutions for $L(\Sigma \cup \{\upsilon\})$, an infinite orderly sequence $\vec{w} \in L^\infty(\Sigma; \upsilon)$ of variable located words over Σ and $\xi < \omega_1$, Theorem 3.5 provides no information on how to decide whether the homogeneous family $L^\xi(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u})$ falls in $\mathcal{F}$ or in its complement, while Theorem 4.13 in case the partition family $\mathcal{F}$ is a tree provides a criterion, in terms of a Cantor-Bendixson type index of $\mathcal{F}$, according to which we can have such a decition.

As a corollary of Theorem 4.13 we have a partition theorem for infinite orderly sequences of variable located words (Corollary 4.15) for each set of substitutions.

Notation. A finite orderly sequence $\mathbf{w} = (w_1, \dots, w_l) \in L^{<\infty}(\Sigma \cup \{\upsilon\})$ is an *initial segment* of $\mathbf{u} = (u_1, \dots, u_k) \in L^{<\infty}(\Sigma \cup \{\upsilon\})$ iff $l \leq k$ and $w_i = u_i$ for every $i = 1, \dots, l$ and $\mathbf{w}$ is an initial segment of $\vec{u} = (u_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma \cup \{\upsilon\})$ if $w_i = u_i$ for all $i = 1, \dots, l$. In these cases we write $\mathbf{w} \propto \mathbf{u}$ and $\mathbf{w} \propto \vec{u}$, respectively, and we set $\mathbf{u} \setminus \mathbf{w} = (u_{l+1}, \dots, u_k)$ and $\vec{u} \setminus \mathbf{w} = (u_n)_{n>l}$.

Definition 4.1. A family $\mathcal{F} \subseteq L^{<\infty}(\Sigma;\upsilon)$ is *thin* if there are no elements $\mathbf{s}, \mathbf{t} \in \mathcal{F}$ with $\mathbf{s} \neq \mathbf{t}$ and $\mathbf{s} \propto \mathbf{t}$.

Proposition 4.2. *Every family $L^\xi(\Sigma;\upsilon)$, for $\xi < \omega_1$ is thin.*

Proof. It follows by induction on ξ. □

Proposition 4.3. *Let ξ be a nonzero countable ordinal number, Σ a finite alphabet and $\upsilon \notin \Sigma$. Then*

(i) every infinite orderly sequence $\vec{s} = (s_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma;\upsilon)$ of variable located words has canonical representation with respect to $L^\xi(\Sigma;\upsilon)$, which means that there exists a unique strictly increasing sequence $(m_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ so that $(s_1, \dots, s_{m_1}) \in L^\xi(\Sigma;\upsilon)$ and $(s_{m_{n-1}+1}, \dots, s_{m_n}) \in L^\xi(\Sigma;\upsilon)$ for every $n > 1$; and,

(ii) every nonempty finite orderly sequence $\mathbf{s} = (s_1, \dots, s_k) \in L^{<\infty}(\Sigma;\upsilon)$ has canonical representation with respect to $L^\xi(\Sigma;\upsilon)$, which means that either $\mathbf{s} \in (L^\xi(\Sigma;\upsilon))^* \setminus L^\xi(\Sigma;\upsilon)$ or there exist unique $n \in \mathbb{N}$, and $m_1, \dots, m_n \in \mathbb{N}$ with $m_1 < \dots < m_n \leq k$ so that either $(s_1, \dots, s_{m_1}), \dots, (s_{m_{n-1}+1}, \dots, s_{m_n}) \in L^\xi(\Sigma;\upsilon)$ and $m_n = k$, or $(s_1, \dots, s_{m_1}), \dots, (s_{m_{n-1}+1}, \dots, s_{m_n}) \in L^\xi(\Sigma;\upsilon)$, $(s_{m_n+1}, \dots, s_k) \in (L^\xi(\Sigma;\upsilon))^* \setminus L^\xi(\Sigma;\upsilon)$.

Proof. It follows from the fact that every nonempty increasing sequence (finite or infinite) in $\mathbb{N}$ has canonical representation with respect to $\mathcal{A}_\xi$ (cf. [F3], Theorem 1.14) and that the family $L^\xi(\Sigma;\upsilon)$ is thin (Proposition 4.2). □

Definition 4.4. Let Σ be a finite, non empty alphabet, $E(\Sigma)$ a set of substitutions for $L(\Sigma \cup \{\upsilon\})$ and $\mathcal{F} \subseteq L^{<\infty}(\Sigma;\upsilon)$.

(i) $\mathcal{F}^* = \{\mathbf{t} \in L^{<\infty}(\Sigma;\upsilon) : \mathbf{t} \propto \mathbf{s} \text{ for some } \mathbf{s} \in \mathcal{F}\} \cup \{\emptyset\}$.

(ii) $\mathcal{F}$ is a *tree* if $\mathcal{F}^* = \mathcal{F}$.

(iii) $\mathcal{F}_* = \{\mathbf{t} \in L^{<\infty}(\Sigma;\upsilon) : \mathbf{t} \subseteq EL(\mathbf{s}) \text{ for some } \mathbf{s} \in \mathcal{F}\} \cup \{\emptyset\}$.

(iv) $\mathcal{F}$ is *hereditary* if $\mathcal{F}_* = \mathcal{F}$.

Now, using Proposition 4.3, we will give an alternative description of the second horn of the dichotomy, proved in Theorem 3.5, in case the partition family is a tree.

Proposition 4.5. *Let $\xi \geq 1$ be a countable ordinal, Σ a finite alphabet, $\upsilon \notin \Sigma$, $E(\Sigma)$ a set of substitutions for $L(\Sigma \cup \{\upsilon\})$, $\mathcal{F} \subseteq L^{<\infty}(\Sigma;\upsilon)$ be a tree and $\vec{u} \in L^{\infty}(\Sigma;\upsilon)$. Then*
$L^{\xi}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}) \subseteq L^{<\infty}(\Sigma;\upsilon) \setminus \mathcal{F}$ if and only if
$\mathcal{F} \cap EL^{<\infty}(\vec{u}) \subseteq (L^{\xi}(\Sigma;\upsilon))^* \setminus L^{\xi}(\Sigma;\upsilon)$.

Proof. Let $L^{\xi}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}) \subseteq L^{<\infty}(\Sigma;\upsilon) \setminus \mathcal{F}$ and $\mathbf{s} = (s_1,\ldots,s_k) \in \mathcal{F} \cap EL^{<\infty}(\vec{u})$. Then $\mathbf{s}$ has canonical representation with respect to $L^{\xi}(\Sigma;\upsilon)$ (Proposition 4.3), hence either $\mathbf{s} \in (L^{\xi}(\Sigma;\upsilon))^* \setminus L^{\xi}(\Sigma;\upsilon)$, as required, or there exists $\mathbf{s}_1 \in L^{\xi}(\Sigma;\upsilon)$ such that $\mathbf{s}_1 \propto \mathbf{s}$. The second case is impossible. Indeed, since $\mathcal{F}$ is a tree and $\mathbf{s} \in \mathcal{F} \cap EL^{<\infty}(\vec{u})$, we have $\mathbf{s}_1 \in \mathcal{F} \cap EL^{<\infty}(\vec{u}) \cap L^{\xi}(\Sigma;\upsilon)$; a contradiction to our assumption. Hence, $\mathcal{F} \cap EL^{<\infty}(\vec{u}) \subseteq (L^{\xi}(\Sigma;\upsilon))^* \setminus L^{\xi}(\Sigma;\upsilon)$. □

Definition 4.6. Let Σ be a finite alphabet and $\upsilon \notin \Sigma$. We set $D = \{(n,\alpha) : n \in \mathbb{N}, \alpha \in \Sigma \cup \{\upsilon\}\}$. Note that D is a countable set. Let $[D]^{<\omega}$ be the set of all finite subsets of D. Identifying every $\mathbf{s} \in L^{<\infty}(\Sigma;\upsilon)$ and every $\vec{s} \in L^{\infty}(\Sigma;\upsilon))$ with their characteristic functions $x_{\mathbf{s}} \in \{0,1\}^{[D]^{<\omega}}$ and $x_{\vec{s}} \in \{0,1\}^{[D]^{<\omega}}$ respectively, we topologize the sets $L^{<\infty}(\Sigma;\upsilon)$, $L^{\infty}(\Sigma;\upsilon)$ by the topology of pointwise convergence (equivalently by the product topology of $\{0,1\}^{[D]^{<\omega}}$). So, if $\sigma(\mathbf{s}) = \{s_1,\ldots,s_k\}$ for every $\mathbf{s} = (s_1,\ldots,s_k) \in L^{<\infty}(\Sigma;\upsilon)$,$\sigma(\vec{s}) = \{s_n : n \in \mathbb{N}\}$ for every $\vec{s} = (s_n)_{n\in\mathbb{N}} \in L^{\infty}(\Sigma;\upsilon)$ and $\sigma(\emptyset) = \emptyset$, then a family $\mathcal{F} \subseteq L^{<\infty}(\Sigma;\upsilon)$ is *pointwise closed* iff the family $\{x_{\sigma(\mathbf{s})} : \mathbf{s} \in \mathcal{F}\}$ is closed in $\{0,1\}^{[D]^{<\omega}}$ with the topology of pointwise convergence and a family $\mathcal{U} \subseteq L^{\infty}(\Sigma;\upsilon)$ is *pointwise closed* iff $\{x_{\sigma(\vec{s})} : \vec{s} \in \mathcal{U}\}$ is pointwise closed in $\{0,1\}^{[D]^{<\omega}}$.

Proposition 4.7. *Let Σ be a finite alphabet and $\upsilon \notin \Sigma$.*

(i) *If $\mathcal{F} \subseteq L^{<\infty}(\Sigma;\upsilon)$ is a tree, then $\mathcal{F}$ is pointwise closed if and only if there does not exist an infinite sequence $(\mathbf{s}_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $\mathbf{s}_n \propto \mathbf{s}_{n+1}$ and $\mathbf{s}_n \neq \mathbf{s}_{n+1}$ for all $n \in \mathbb{N}$.*

(ii) *If $\mathcal{F} \subseteq L^{<\infty}(\Sigma;\upsilon)$ is hereditary, then $\mathcal{F}$ is pointwise closed if and only if there does not exist $\vec{s} \in L^{\infty}(\Sigma;\upsilon)$ such that $EL^{<\infty}(\vec{s}) \subseteq \mathcal{F}$.*

(iii) *The hereditary family $(L^{\xi}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{s}))_*$ is pointwise closed for every countable ordinal ξ and $\vec{s} \in L^{\infty}(\Sigma;\upsilon)$.*

Proof. This follows directly from the definitions (for details cf. [FN], Proposition 3.11). ☐

Definition 4.8. Let Σ be a finite alphabet and $\upsilon \notin \Sigma$, $E(\Sigma)$ be a set of substitutions for $L(\Sigma \cup \{\upsilon\})$, $\mathcal{F} \subseteq L^{<\infty}(\Sigma;\upsilon)$ be a hereditary and pointwise closed family, and $\vec{s} \in L^{\infty}(\Sigma;\upsilon)$. For every $\xi < \omega_1$ we define the families $(\mathcal{F})^{\xi}_{\vec{s}}$ inductively as follows:

$A_{\mathbf{s}} = \{t \in EL(\vec{s}) : (s_1,\dots,s_k,t) \notin \mathcal{F}\}$ for every $\mathbf{s} = (s_1,\dots,s_k) \in \mathcal{F} \cap EL^{<\omega}(\vec{s})$ and

$A_{\emptyset} = \{t \in EL(\vec{s}) : (t) \notin \mathcal{F}\}$.

We set

$$(\mathcal{F})^{1}_{\vec{s}} = \{\mathbf{s} \in \mathcal{F} \cap EL^{<\omega}(\vec{s}) \cup \{\emptyset\} : A_{\mathbf{s}} \text{ does not contain an infinite orderly sequence } \}.$$

It is easy to verify that $(\mathcal{F})^{1}_{\vec{s}}$ is hereditary, hence pointwise closed (Proposition 4.7). So, we can define for every $\xi > 1$ the ξ-derivatives of $\mathcal{F}$ recursively as follows:

$(\mathcal{F})^{\zeta+1}_{\vec{s}} = ((\mathcal{F})^{\zeta}_{\vec{s}})^{1}_{\vec{s}}$ for all $\zeta < \omega_1$, and

$(\mathcal{F})^{\xi}_{\vec{s}} = \bigcap_{\beta<\xi} (\mathcal{F})^{\beta}_{\vec{s}}$ for ξ a limit ordinal.

The *strong Cantor-Bendixson index* $sO_{\vec{s}}(\mathcal{F})$ of $\mathcal{F}$ on $\vec{s}$ is the smallest countable ordinal ξ such that $(\mathcal{F})^{\xi}_{\vec{s}} = \emptyset$.

Remark 4.9. Let $\mathcal{F}_1, \mathcal{R}_1, \subseteq L^{<\infty}(\Sigma;\upsilon)$ be hereditary and pointwise closed families, $E(\Sigma)$ be a set of substitutions for $L(\Sigma \cup \{\upsilon\})$ and $\vec{s} \in L^{\infty}(\Sigma;\upsilon)$.

(i) $sO_{\vec{s}}(\mathcal{F}_1)$ is a countable successor ordinals less than or equal to the "usual" Cantor-Bendixson index $O(\mathcal{F}_1)$ of $\mathcal{F}_1$ into $\{0,1\}^{[D]^{<\omega}}$ (cf. [KM]).

(ii) $sO_{\vec{s}}(\mathcal{F}_1 \cap EL^{<\infty}(\vec{s})) = sO_{\vec{s}}(\mathcal{F}_1)$.

(iii) $sO_{\vec{s}}(\mathcal{F}_1) \leq sO_{\vec{s}}(\mathcal{R}_1)$ if $\mathcal{F}_1 \subseteq \mathcal{R}_1$.

(iv) If $\mathbf{s} \in (\mathcal{F}_1)^{\xi}_{\vec{s}}$ and $\vec{s}_1 \prec \vec{s}$, then $\mathbf{s}_1 \in (\mathcal{F}_1)^{\xi}_{\vec{s}_1}$ for every $\mathbf{s}_1 \in EL^{<\infty}(\vec{s}_1)$ with $\sigma(\mathbf{s}_1) = \sigma(\mathbf{s}) \cap EL(\vec{s}_1)$, since $EL(\vec{s}_1) \subseteq EL(\vec{s})$.

(v) If $\vec{s}_1 \prec \vec{s}$, then $sO_{\vec{s}_1}(\mathcal{F}_1) \geq sO_{\vec{s}}(\mathcal{F}_1)$, according to (iv).

(vi) If $\sigma(\vec{s}_1) \setminus \sigma(\vec{s})$ is a finite set, then $sO_{\vec{s}_1}(\mathcal{F}_1) \geq sO_{\vec{s}}(\mathcal{F}_1)$.

Proposition 4.10. *Let Σ be a finite alphabet, $\upsilon \notin \Sigma$, $E(\Sigma)$ a set of substitutions for $L(\Sigma \cup \{\upsilon\})$, $\vec{s} \in L^{\infty}(\Sigma;\upsilon)$ and $\xi < \omega_1$ be an ordinal.*

If $\vec{s}_1 \prec \vec{s}$, then $sO_{\vec{s}_1}\Big((L^{\xi}(\Sigma;\upsilon) \cap EL^{<\omega}(\vec{s}))_*\Big) = \xi + 1$.

Proof. Let $\vec{s}_1 \prec \vec{s}$. For every $s \in EL(\vec{s})$ with $\min dom(s) = n$ we have, according to Proposition 3.4, that

$$(L^{\xi}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{s}))(s) = L^{\xi_n}(\Sigma;\upsilon)) \cap EL^{<\infty}(\vec{s} - s) \text{ for some } \xi_n < \xi.$$

The family $(L^{\xi}(\Sigma;\upsilon)\cap EL^{<\omega}(\vec{s}))_*$ is hereditary and pointwise closed (Proposition 4.7). We will prove by induction that $\Big((L^{\xi}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{s}))_*\Big)^{\xi}_{\vec{s}_1}=\{\emptyset\}$ for every $\xi<\omega_1$. Of course, $(L^{1}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{s}))_*=\{(s):s\in EL(\vec{s})\}\cup\{\emptyset\}$. Thus we have that $\Big((L^{1}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{s}))_*\Big)^{1}_{\vec{s}_1}=\{\emptyset\}$.

Let $\xi>1$ and assume that $\Big((L^{\varsigma}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{s}))_*\Big)^{\zeta}_{\vec{s}_1}=\{\emptyset\}$ for every $\zeta<\xi$ and $\vec{s}_1\prec\vec{s}$. Hence, for every $s\in EL(\vec{s}_1)$ with $\min s=n$ and $\vec{s}_1\prec\vec{s}$ we have that

$$\Big((L^{\xi}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{s}))(s)_*\Big)^{\xi_n}_{\vec{s}_1}=\Big((L^{\xi_n}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{s}-s))_*\Big)^{\xi_n}_{\vec{s}_1}=\{\emptyset\}.$$

This gives that $(s)\in\Big((L^{\xi}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{s}))_*\Big)^{\xi_n}_{\vec{s}_1}$. So, $\emptyset\in\Big((L^{\xi}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{s}))_*\Big)^{\xi}_{\vec{s}_1}$, since if $\xi=\zeta+1$, then $(s)\in\Big((L^{\xi}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{s}))_*\Big)^{\zeta}_{\vec{s}_1}$ for every $s\in EL(\vec{s}_1)$ and if ξ is a limit ordinal, then $\emptyset\in\Big((L^{\xi}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{s}))_*\Big)^{\xi_n}_{\vec{s}_1}$ for every $n\in\mathbb{N}$ and since $\sup\xi_n=\xi$.

If $\{\emptyset\}\neq\Big((L^{\xi}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{s}))_*\Big)^{\xi}_{\vec{s}_1}$ for $\vec{s}_1\prec\vec{s}$, then there exist $\vec{s}_2\prec\vec{s}_1$ and $s\in EL(\vec{s}_2)$ such that $\Big((L^{\xi}(\Sigma;\upsilon)\cap EL^{<\omega}(\vec{s}))(s)_*\Big)^{\xi}_{\vec{s}_2}=\Big((L^{\xi_n}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{s}-s))_*\Big)^{\xi}_{\vec{s}_1}\neq\emptyset$ (see Lemma 2.8 in [F4]). This is a contradiction to the induction hypothesis. Hence, $\Big((L^{\xi}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{s}))_*\Big)^{\xi}_{\vec{s}_1}=\{\emptyset\}$ and $sO_{\vec{s}_1}((L^{\xi}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{s}))_*)=\xi+1$ for every $\xi<\omega_1$. □

Corollary 4.11. *Let ξ_1,ξ_2 be countable ordinals with $\xi_1<\xi_2$ and $\vec{w}\in L^{\infty}(\Sigma;\upsilon)$. Then there exist $\vec{u_1}\prec\vec{w}$ such that $(L^{\xi_1}(\Sigma;\upsilon))_*\cap EL^{<\infty}(\vec{u_1})\subseteq(L^{\xi_2}(\Sigma;\upsilon))^*\setminus L^{\xi_2}(\Sigma;\upsilon)$.*

Proof. Of course the family $(L^{\xi_1}(\Sigma;\upsilon))_*\subseteq L^{<\infty}(\Sigma;\upsilon)$ is a tree. According to Theorem 3.5 and Proposition 4.5 there exists $\vec{u_1}\prec\vec{w}$ such that:
either $L^{\xi_2}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{u_1})\subseteq(L^{\xi_1}(\Sigma;\upsilon))_*$,
or $(L^{\xi_1}(\Sigma;\upsilon))_*\cap EL^{<\infty}(\vec{u_1})\subseteq(L^{\xi_2}(\Sigma;\upsilon))^*\setminus L^{\xi_2}(\Sigma;\upsilon)$.
The first alternative is impossible, according to Proposition 4.10. Indeed, $\xi_2+1=sO_{\vec{u_1}}((L^{\xi_2}(\Sigma;\upsilon)\cap EL^{<\infty}(\vec{u_1}))_*)\leq sO_{\vec{u}}((L^{\xi_1}(\Sigma;\upsilon))_*)=\xi_1+1$, a contradiction. □

Definition 4.12. Let $\mathcal{F}\subseteq L^{<\infty}(\Sigma;\upsilon)$ be a family of finite orderly sequences of variable located words over a finite alphabet Σ and $E(\Sigma)$ a set of substitutions for $L(\Sigma\cup\{\upsilon\})$. The family $\mathcal{F}_h=\{\mathbf{s}\in\mathcal{F}:EL(\mathbf{s})\subseteq\mathcal{F}\}\cup\{\emptyset\}$ is the largest subfamily of $\mathcal{F}\cup\{\emptyset\}$ which is hereditary.

The following theorem extends Theorem 3.5 in case the partition family is a tree.

Theorem 4.13. *Let Σ be a finite alphabet, $\upsilon \notin \Sigma$, $E(\Sigma)$ a set of substitutions for $L(\Sigma \cup \{\upsilon\})$, $\mathcal{F} \subseteq L^{<\infty}(\Sigma;\upsilon)$ a tree of finite orderly sequences of variable located words over Σ, and $\vec{w} \in L^{\infty}(\Sigma;\upsilon)$ an infinite orderely sequence of variable located words over Σ. Then we have the following cases:*

[Case 1] *The family $\mathcal{F}_h \cap EL^{<\infty}(\vec{w})$ is not pointwise closed.*

Then, there exists $\vec{u} \prec \vec{w}$ such that $EL^{<\infty}(\vec{u}) \subseteq \mathcal{F}$.

[Case 2] *The family $\mathcal{F}_h \cap EL^{<\infty}(\vec{w})$ is pointwise closed.*

Then, setting $\zeta^{\mathcal{F}}_{\vec{w}} = \sup\{sO_{\vec{u}}(\mathcal{F}_h) : \vec{u} \prec \vec{w}\}$, which is a countable ordinal, the following subcases obtain:

2(i) *If $\xi + 1 < \zeta^{\mathcal{F}}_{\vec{w}}$, then there exists $\vec{u} \prec \vec{w}$ such that*

$$L^{\xi}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}) \subseteq \mathcal{F} \ ;$$

2(ii) *if $\xi + 1 > \xi > \zeta^{\mathcal{F}}_{\vec{w}}$, then for every $\vec{w}_1 \prec \vec{w}$ there exists $\vec{u} \prec \vec{w}_1$ such that $L^{\xi}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}) \subseteq L^{<\infty}(\Sigma;\upsilon) \setminus \mathcal{F}$; (equivalently $\mathcal{F} \cap EL^{<\infty}(\vec{u}) \subseteq (L^{\xi}(\Sigma;\upsilon))^{*} \setminus L^{\xi}(\Sigma;\upsilon)$) ; and*

2(iii) *if $\xi + 1 = \zeta^{\mathcal{F}}_{\vec{w}}$ or $\xi = \zeta^{\mathcal{F}}_{\vec{w}}$, then there exists $\vec{u} \prec \vec{w}$ such that either $L^{\xi}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}) \subseteq \mathcal{F}$ or $L^{\xi}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}) \subseteq L^{<\infty}(\Sigma;\upsilon) \setminus \mathcal{F}$.*

Proof. [Case 1] If the hereditary family $\mathcal{F}_h \cap EL^{<\infty}(\vec{w})$ is not pointwise closed, then, there exists $\vec{u} \in L^{\infty}(\Sigma;\upsilon)$ such that $EL^{<\infty}(\vec{u}) \subseteq \mathcal{F}_h \cap EL^{<\infty}(\vec{w}) \subseteq \mathcal{F}$, according to Proposition 4.7. Of course, $\vec{u} \prec \vec{w}$.

[Case 2] If the hereditary family $\mathcal{F}_h \cap EL^{<\infty}(\vec{w})$ is pointwise closed, then $\zeta^{\mathcal{F}}_{\vec{w}}$ is a countable ordinal, since the "usual" Cantor-Bendixson index $O(\mathcal{F}_h)$ of $\mathcal{F}_h$ into $\{0,1\}^{[D]^{<\omega}}$ is countable (Remark 4.9(i)) and also $sO_{\vec{u}}(\mathcal{F}_h) \leq O(\mathcal{F}_h)$ for every $\vec{u} \prec \vec{w}$.

2(i) Let $\xi + 1 < \zeta^{\mathcal{F}}_{\vec{w}}$. Then there exists $\vec{u}_1 \prec \vec{w}$ such that $\xi + 1 < sO_{\vec{u}_1}(\mathcal{F}_h)$. According to Theorem 3.5 and Proposition 4.5, there exists $\vec{u} \prec \vec{u}_1$ such that

either $L^{\xi}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}) \subseteq \mathcal{F}_h \subseteq \mathcal{F}$,

or $\mathcal{F}_h \cap EL^{<\infty}(\vec{u}) \subseteq (L^{\xi}(\Sigma;\upsilon))^{*} \setminus L^{\xi}(\Sigma;\upsilon) \subseteq (L^{\xi}(\Sigma;\upsilon))^{*} \subseteq (L^{\xi}(\Sigma;\upsilon))_{*}$.

The second alternative is impossible. Indeed, if $\mathcal{F}_h \cap EL^{<\infty}(\vec{u}) \subseteq (L^{\xi}(\Sigma;\upsilon))_{*}$, then, according to Remark 4.9 and Proposition 4.10,

$sO_{\vec{u}_1}(\mathcal{F}_h) \leq sO_{\vec{u}}(\mathcal{F}_h) = sO_{\vec{u}}(\mathcal{F}_h \cap EL^{<\infty}(\vec{u})) \leq sO_{\vec{u}}((L^{\xi}(\Sigma;\upsilon))_{*}) = \xi + 1$;

a contradiction. Hence, $L^{\xi}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}) \subseteq \mathcal{F}$.

2(ii) Let $\xi + 1 > \xi > \zeta^{\mathcal{F}}_{\vec{w}}$ and $\vec{w}_1 \prec \vec{w}$. According to Theorem 3.5, there exists $\vec{u_1} \prec \vec{w}_1$ such that

either $L^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}_1) \subseteq \mathcal{F}_h$, or $L^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u_1}) \subseteq L^{<\infty}(\Sigma;\upsilon) \setminus \mathcal{F}_h$.
The first alternative is impossible. Indeed, if $L^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}_1) \subseteq \mathcal{F}_h$, then, according to Remark 4.9 and Proposition 4.10, we have that

$\zeta^{\mathcal{F}}_{\vec{w}} + 1 = sO_{\vec{u}_1}((L^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}_1))_*) \leq sO_{\vec{u}_1}(\mathcal{F}_h) \leq \zeta^{\mathcal{F}}_{\vec{w}}$;

a contradiction. Hence,

$$L^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}_1) \subseteq L^{<\infty}(\Sigma;\upsilon) \setminus \mathcal{F}_h \ . \tag{1}$$

According to Theorem 3.5, there exists $\vec{u} \prec \vec{u}_1$ such that

either $L^{\xi}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}) \subseteq \mathcal{F}$, or $L^{\xi}(\Sigma;\upsilon) \cap EL^{<\omega}(\vec{u}) \subseteq L^{<\infty}(\Sigma;\upsilon) \setminus \mathcal{F}$.

We claim that the first alternative does not hold. Indeed, if $L^{\xi}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}) \subseteq \mathcal{F}$, then $(L^{\xi}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}))^* \subseteq \mathcal{F}^* = \mathcal{F}$. Using the canonical representation of every infinite orderly sequence of variable located words with respect to $L^{\xi}(\Sigma;\upsilon)$ (Proposition 4.3) it is easy to check that

$(L^{\xi}(\Sigma;\upsilon))^* \cap EL^{<\infty}(\vec{u}) = (L^{\xi}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}))^*$.

Hence, $(L^{\xi}(\Sigma;\upsilon))^* \cap EL^{<\infty}(\vec{u}) \subseteq \mathcal{F}$.

Since $\xi > \zeta^{\mathcal{F}}_{\vec{w}}$, according to Corollary 4.11, there exists $\vec{t} \prec \vec{u}$ such that

$(L^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma;\upsilon))_* \cap EL^{<\infty}(\vec{t}) \subseteq (L^{\xi}(\Sigma;\upsilon))^* \cap EL^{<\infty}(\vec{u}) \subseteq \mathcal{F}$.

So, $(L^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma;\upsilon))_* \cap EL^{<\infty}(\vec{t}) \subseteq \mathcal{F}_h$. This is a contradiction to the relation (1). Hence, $L^{\xi}(\Sigma;\upsilon) \cap EL^{<\infty}(\vec{u}) \subseteq L^{<\infty}(\Sigma;\upsilon) \setminus \mathcal{F}$ and $\mathcal{F} \cap EL^{<\infty}(\vec{u}) \subseteq (L^{\xi}(\Sigma;\upsilon))^* \setminus L^{\xi}(\Sigma;\upsilon)$.

2(iii) In the cases $\zeta^{\mathcal{F}}_{\vec{w}} = \xi + 1$ or $\zeta^{\mathcal{F}}_{\vec{w}} = \xi$, we use Theorem 3.5. □

The following immediate corollary to Theorem 4.13 is more useful for applications.

Corollary 4.14. *Let $\mathcal{F} \subseteq L^{<\omega}(\Sigma;\upsilon)$ which is a tree, $E(\Sigma)$ a set of substitutions for $L(\Sigma \cup \{\upsilon\})$, and let $\vec{w} \in L^{\infty}(\Sigma;\upsilon)$. Then*

(i) *either there exists $\vec{u} \prec \vec{w}$ such that $EL^{<\infty}(\vec{u}) \subseteq \mathcal{F}$,*

(ii) *or for every countable ordinal $\xi > \zeta^{\mathcal{F}}_{\vec{w}}$ there exists $\vec{u} \prec \vec{w}_1$, such that for every $\vec{u}_1 \prec \vec{u}$ the unique initial segment of $\vec{u}_1$ which is an element of $L^{\xi}(\Sigma;\upsilon)$ belongs to $L^{<\infty}(\Sigma;\upsilon) \setminus \mathcal{F}$.*

Theorem 4.13 implies the following theorem, which provides us with a partition theorem for infinite orderly sequences of variable located words for each set of substitutions.

Theorem 4.15 (Partition theorem for infinite orderly sequences of located words)**.** *Let Σ be a finite alphabet, $\upsilon \notin \Sigma$, $E(\Sigma)$ a set of substitutions for $L(\Sigma \cup \{\upsilon\})$. If $\mathcal{U} \subseteq L^{\infty}(\Sigma;\upsilon)$ is a pointwise closed family of infinite orderly sequences of variable located words and $\vec{w} \in L^{\infty}(\Sigma;\upsilon)$, then there exists $\vec{u} \prec \vec{w}$ such that* *either* $EL^\infty(\vec{u}) \subseteq \mathcal{U}$, *or* $EL^\infty(\vec{u}) \subseteq L^\infty(\Sigma;\upsilon) \setminus \mathcal{U}$.

Proof. Let $\mathcal{F}_\mathcal{U} = \{\mathbf{s} \in L^{<\infty}(\Sigma;\upsilon)$: there exists $\vec{s} \in \mathcal{U}$ such that $\mathbf{s} \propto \vec{s}\}$. Since the family $\mathcal{F}_\mathcal{U}$ is a tree, we can use Corollary 4.14. We have the following two cases:
[Case 1] There exists $\vec{u} \prec \vec{w}$ such that $EL^{<\infty}(\vec{u}) \subseteq \mathcal{F}_\mathcal{U}$. Then, $EL^\infty(\vec{u}) \subseteq \mathcal{U}$. Indeed, if $\vec{s} = (s_n)_{n\in\mathbb{N}} \in EL^\infty(\vec{u})$, then $(s_1,\dots,s_n) \in \mathcal{F}_\mathcal{U}$ for every $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$ there exists $\vec{s}_n \in \mathcal{U}$ such that $(s_1,\dots,s_n) \propto \vec{s}_n$. Since $\mathcal{U}$ is pointwise closed, we have that $\vec{s} \in \mathcal{U}$ and consequently that $EL^\infty(\vec{u}) \subseteq \mathcal{U}$.
[Case 2] There exists $\vec{u} \prec \vec{w}$ such that for every $\vec{u}_1 \prec \vec{u}$ there exists an initial segment of $\vec{u}_1$ which belongs to $L^{<\infty}(\Sigma;\upsilon) \setminus \mathcal{F}_\mathcal{U}$. Hence, $EL^\infty(\vec{s}) \subseteq L^\infty(\Sigma;\upsilon) \setminus \mathcal{U}$. ☐

Theorem 4.15 for the traditional mode of substitution, proved by a different approach by Bergelson, Blass and Hindman in [BBH] (Theorem 6.1), while Theorem 4.15 in the particular case where $\Sigma = \emptyset$ proved by an analogous approach in [FN] (Corollary 4.10).

5. The characterization of Ramsey partitions of the set of infinite sequences of variable located words

As a consequence of Theorem 4.13 we will state and prove, in Theorem 5.2 below, a stronger partition theorem than Theorem 4.15 for infinite orderly sequences of variable located words over a finite alphabet according to a given set of substitutions, involving an Ellentuck topology $\mathfrak{T}_E$ on the space $L^\omega(\Sigma;\upsilon)$ depended of the given set of substitutions.

A simple consequence of Theorem 5.2 (together with Corollary 5.5) is the characterization of completely Ramsey partitions of $L^\omega(\Sigma;\upsilon)$ for a given set of substitutions in terms of the Baire property in the relating topology $\mathfrak{T}_E$.

We define below the topology $\mathfrak{T}_E$ on $L^\omega(\Sigma;\upsilon)$ for a given set of substitutions for $L(\Sigma \cup \{\upsilon\})$, an analogue of the Ellentuck topology on $\mathbb{N}$ defined in [E].

Definition 5.1. Let Σ be a finite alphabet, $\upsilon \notin \Sigma$ and $E(\Sigma)$ a set of substitutions for $L(\Sigma \cup \{\upsilon\})$. We define the topology $\mathfrak{T}_E$ on $L^\omega(\Sigma;\upsilon)$ as the topology with basic open sets of the form:

$$[\mathbf{s}, \vec{s}] = \{\vec{t} \in L^\omega(\Sigma;\upsilon) : \mathbf{s} \propto \vec{t} \quad \text{and} \quad \vec{t} - \mathbf{s} \prec \vec{s}\}\ ,$$

where $\mathbf{s} \in L^{<\omega}(\Sigma;\upsilon)$ and $\vec{s} \in L^\omega(\Sigma;\upsilon)$.

The topology $\mathfrak{T}_E$ is stronger than the relative topology of $L^\omega(\Sigma;\upsilon)$ with respect to the pointwise convergence topology of $\{0,1\}^{[D]^{<\omega}}$, which has basic open sets of the form $[\mathbf{s}, \vec{e}] = \{\vec{t} \in L^\omega(\Sigma;\upsilon) : \mathbf{s} \propto \vec{t}\}$ where $\vec{e} = (e_n)_{n\in\mathbb{N}}$ with $e_n = \upsilon$ for every $n \in \mathbb{N}$.

We denote by $\hat{\mathcal{U}}$ and $\mathcal{U}^{\diamondsuit}$ the closure and the interior respectively of a family $\mathcal{U} \subseteq L^{\omega}(\Sigma;\upsilon)$ in the topology $\mathfrak{T}_E$. Then it is easy to see that

$$\hat{\mathcal{U}} = \{\vec{s} \in L^{\omega}(\Sigma;\upsilon) : [\mathbf{s},\vec{s}] \cap \mathcal{U} \neq \emptyset \ \text{ for every } \ \mathbf{s} \propto \vec{s}\} \ ; \quad \text{and}$$
$$\mathcal{U}^{\diamondsuit} = \{\vec{s} \in L^{\omega}(\Sigma;\upsilon) : \ \text{ there exists } \mathbf{s} \propto \vec{s} \ \text{ such that } \ [\mathbf{s},\vec{s}] \subseteq \mathcal{U}\} \ .$$

If $\mathbf{s} = (s_1,\ldots,s_k) \in L^{<\omega}(\Sigma;\upsilon)$ and $\mathbf{t} = (t_1,\ldots,t_l) \in L^{<\omega}(\Sigma;\upsilon)$ with $s_k < t_1$, then we set $\mathbf{s} \odot \mathbf{t} = (s_1,\ldots,s_k,t_1,\ldots,t_k) \in L^{<\omega}(\Sigma;\upsilon)$.

Theorem 5.2. *Let Σ be a finite alphabet, $\upsilon \notin \Sigma$, $E(\Sigma)$ a set of substitutions for $L(\Sigma \cup \{\upsilon\})$, $\mathcal{U} \subseteq L^{\omega}(\Sigma;\upsilon)$, $\mathbf{s} \in L^{<\omega}(\Sigma;\upsilon)$ and $\vec{w} \in L^{\omega}(\Sigma;\upsilon)$. Then*

either there exists $\vec{u} \prec \vec{w}$ such that $[\mathbf{s},\vec{u}] \subseteq \hat{\mathcal{U}}$,
or there exists a countable ordinal $\xi_0 = \zeta^{\mathcal{U}}_{(\mathbf{s},\vec{w})}$ such that for every $\xi > \xi_0$ there exists $\vec{u} \prec \vec{w} - \mathbf{s}$ with $[\mathbf{s} \odot \mathbf{t},\vec{u}] \subseteq L^{\omega}(\Sigma;\upsilon) \setminus \mathcal{U}$ for every $\mathbf{t} \in L^{\xi}(\Sigma;\upsilon) \cap EL^{<\omega}(\vec{u})$.

We will give the proof after the following lemma which is analogous to Lemma 3.6.

Lemma 5.3. *Let Σ be a finite alphabet, $\upsilon \notin \Sigma$, $E(\Sigma)$ a set of substitutions for $L(\Sigma \cup \{\upsilon\})$, $\mathcal{R} \subseteq \{[\mathbf{s},\vec{s}] : \mathbf{s} \in L^{<\omega}(\Sigma;\upsilon)$ and $\vec{s} \in L^{\omega}(\Sigma;\upsilon)\}$ with the properties:*
(i) for every $(\mathbf{s},\vec{s}) \in L^{<\omega}(\Sigma;\upsilon) \times L^{\omega}(\Sigma;\upsilon)$ there exists $\vec{s}_1 \prec \vec{s}$ such that $[\mathbf{s},\vec{s}_1] \in \mathcal{R}$; and
(ii) for every $[\mathbf{s},\vec{s}] \in \mathcal{R}$ and $\vec{s}_1 \prec \vec{s}$ we have $[\mathbf{s},\vec{s}_1] \in \mathcal{R}$.
Then, for every $(\mathbf{s},\vec{w}) \in L^{<\omega}(\Sigma;\upsilon) \times L^{\omega}(\Sigma;\upsilon)$ there exists $\vec{u} \in [\mathbf{s},\vec{w}]$ such that $[\mathbf{s}\odot\mathbf{t},\vec{t}] \in \mathcal{R}$ for every $\mathbf{t} \in EL^{<\omega}(\vec{u} - \mathbf{s})$ and $\vec{t} \prec \vec{u} - \mathbf{s}$.

Proof. Let $\mathbf{s} = (s_1,\ldots,s_k) \in L^{<\omega}(\Sigma;\upsilon)$ and $\vec{w} \in L^{\omega}(\Sigma;\upsilon)$. We can assume that $\vec{w} - \mathbf{s} = \vec{w}$. According to the assumption (i), there exists $\vec{s}_1 \prec \vec{w}$ such that $[\mathbf{s},\vec{s}_1] \in \mathcal{R}$. Assume that $\vec{s}_n \prec \cdots \prec \vec{s}_1 \in L^{\omega}(\Sigma;\upsilon)$ have been constructed and $\vec{s}_n = (s^n_i)_{i\in\mathbb{N}}$ for every $n \in \mathbb{N}$.

Set $\{\mathbf{t}_1,\ldots,\mathbf{t}_r\} = VEL^{<\omega}((s^1_1,\ldots,s^n_n))$. According to (i), there exist $\vec{s}^1_{n+1} \prec \vec{s}_n - s^n_n$ such that $[\mathbf{s} \odot \mathbf{t}_1, \vec{s}^1_{n+1}] \in \mathcal{R}$, $\vec{s}^2_{n+1} \prec \vec{s}^1_{n+1}$ such that $[\mathbf{s} \odot \mathbf{t}_2, \vec{s}^2_{n+1}] \in \mathcal{R}$, and finally $\vec{s}^r_{n+1} \prec \vec{s}^{r-1}_{n+1} \prec \vec{s}_n - s^n_n$ such that $[\mathbf{s} \odot \mathbf{t}_r, \vec{s}^r_{n+1}] \in \mathcal{R}$. Set $\vec{s}_{n+1} = \vec{s}^r_{n+1} = (s^{n+1}_i)_{i\in\mathbb{N}}$. Then, according to (ii), $[\mathbf{s} \odot \mathbf{t}_i, \vec{s}_{n+1}] \in \mathcal{R}$ for every $1 \leq i \leq r$.

Set $\vec{u} = (s_1,\ldots,s_k,s^1_1,s^2_2,\ldots) \in L^{\omega}(\Sigma;\upsilon)$. Then $\vec{u} \in [\mathbf{s},\vec{w}]$. Let $\mathbf{t} \in EL^{<\omega}(\vec{u} - \mathbf{s})$ with $\mathbf{t} \neq \emptyset$. If $n_0 = \min\{n \in \mathbb{N} : \mathbf{t} \in VEL^{<\omega}((s^1_1,\ldots,s^n_n))\}$, then $[\mathbf{s} \odot \mathbf{t}, \vec{s}_{n_0+1}] \in \mathcal{R}$. According to assumption (ii), $[\mathbf{s} \odot \mathbf{t}, \vec{u} - s^{n_0}_{n_0}] \in \mathcal{R}$. Hence, $[\mathbf{s} \odot \mathbf{t}, \vec{u}] = [\mathbf{s} \odot \mathbf{t}, \vec{u} - s^{n_0}_{n_0}] \in \mathcal{R}$. If $\mathbf{t} = \emptyset$, then $[\mathbf{s},\vec{s}_1] \in \mathcal{R}$, hence $[\mathbf{s},\vec{u}] \in \mathcal{R}$. $\square$

Proof of Theorem 5.2. Let $\mathcal{U} \subseteq L^{\omega}(\Sigma;\upsilon)$, $\mathbf{s} \in L^{<\omega}(\Sigma;\upsilon)$ and $\vec{w} \in L^{\omega}(\Sigma;\upsilon)$. Set

$$\mathcal{R}_{\mathcal{U}} = \{[\mathbf{s},\vec{s}] : (\mathbf{s},\vec{s}) \in L^{<\omega}(\Sigma;\upsilon) \times L^{\omega}(\Sigma;\upsilon) \text{ and } \\ \text{either } [\mathbf{s},\vec{s}] \cap \mathcal{U} = \emptyset \text{ or } [\mathbf{s},\vec{s}_1] \cap \mathcal{U} \neq \emptyset \text{ for every } \vec{s}_1 \prec \vec{s}\} .$$

It is easy to check that $\mathcal{R}_{\mathcal{U}}$ satisfies the assumptions (i) and (ii) of Lemma 5.3, hence, there exists $\vec{w}_1 \in [\mathbf{s},\vec{w}]$ such that $[\mathbf{s} \odot \mathbf{t}, \vec{w}_1] \in \mathcal{R}_{\mathcal{U}}$ for every $\mathbf{t} \in EL^{<\omega}(\vec{w}_1 - \mathbf{s})$. Set

$$\mathcal{F} = \{\mathbf{t} \in EL^{<\omega}(\vec{w}_1 - \mathbf{s}) : [\mathbf{s} \odot \mathbf{t}, \vec{w}_2] \cap \mathcal{U} \neq \emptyset \text{ for every } \vec{w}_2 \prec \vec{w}_1\} .$$

The family $\mathcal{F}$ is a tree. Indeed, let $\mathbf{t} \in \mathcal{F}$ and $\mathbf{t}_1 \propto \mathbf{t}$. Then $[\mathbf{s} \odot \mathbf{t}_1, \vec{w}_1] \in \mathcal{R}_{\mathcal{U}}$, since $\mathbf{t}_1 \in EL^{<\omega}(\vec{w}_1 - \mathbf{s})$. So either $[\mathbf{s} \odot \mathbf{t}_1, \vec{w}_1] \cap \mathcal{U} = \emptyset$ which is impossible, since $[\mathbf{s} \odot \mathbf{t}, \vec{w}_1] \cap \mathcal{U} \neq \emptyset$, or $[\mathbf{s} \odot \mathbf{t}_1, \vec{w}_2] \cap \mathcal{U} \neq \emptyset$ for every $\vec{w}_2 \prec \vec{w}_1$. Hence, $\mathbf{t}_1 \in \mathcal{F}$.

We use Theorem 4.13 for $\mathcal{F}$ and $\vec{w}_1 - \mathbf{s}$. We have the following cases:

[Case 1] There exists $\vec{u} \prec \vec{w}_1 - \mathbf{s} \prec \vec{w}$ such that $EL^{<\omega}(\vec{u}) \subseteq \mathcal{F}$. This gives that $[\mathbf{s} \odot \mathbf{t}, \vec{u}_1] \cap \mathcal{U} \neq \emptyset$ for every $\mathbf{t} \in EL^{<\omega}(\vec{u})$ and $\vec{u}_1 \prec \vec{u}$, which implies that $[\mathbf{s},\vec{u}] \subseteq \hat{\mathcal{U}}$.

[Case 2] There exists a countable ordinal $\xi_0 = \zeta^{\mathcal{U}}_{(\mathbf{s},\vec{w})}$ such that for every $\xi > \xi_0$ there exists $\vec{u} \prec \vec{w}_1 - \mathbf{s} \prec \vec{w} - \mathbf{s}$ with $L^{\xi}(\Sigma;\upsilon) \cap EL^{<\omega}(\vec{u}) \subseteq L^{<\omega}(\Sigma;\upsilon) \setminus \mathcal{F}$. Then $[\mathbf{s} * \mathbf{t}, \vec{u}] \subseteq L^{\omega}(\Sigma;\upsilon) \setminus \mathcal{U}$ for every $\mathbf{t} \in L^{\xi}(\Sigma;\upsilon) \cap EL^{<\omega}(\vec{u})$. □

Applying Theorem 5.2 to partitions $\mathcal{U}$ of $L^{\omega}(\Sigma;\upsilon)$ that are closed or meager in a topology $\mathfrak{T}_E$ relating to a set of substitutions for $L(\Sigma \cup \{\upsilon\})$, we consider the following consequences.

Corollary 5.4. *Let Σ be a finite alphabet, $\upsilon \notin \Sigma$, $E(\Sigma)$ a set of substitutions for $L(\Sigma \cup \{\upsilon\})$, $\mathcal{U}$ be a closed, in the relating topology $\mathfrak{T}_E$, subset of $L^{\omega}(\Sigma;\upsilon)$, $\mathbf{s} \in L^{<\omega}(\Sigma;\upsilon)$ and $\vec{w} \in L^{\omega}(\Sigma;\upsilon)$. Then*

either there exists $\vec{u} \prec \vec{w}$ such that $[\mathbf{s},\vec{u}] \subseteq \mathcal{U}$,

or there exists a countable ordinal $\xi_0 = \zeta^{\mathcal{U}}_{(\mathbf{s},\vec{w})}$, such that for every $\xi > \xi_0$ there exists $\vec{u} \prec \vec{w} - \mathbf{s}$ such that $[\mathbf{s} \odot \mathbf{t}, \vec{u}] \subseteq L^{\omega}(\Sigma;\upsilon) \setminus \mathcal{U}$ for every $\mathbf{t} \in L^{\xi}(\Sigma;\upsilon) \cap EL^{<\omega}(\vec{u})$.

Corollary 5.5. *Let Σ be a finite alphabet, $\upsilon \notin \Sigma$, $E(\Sigma)$ a set of substitutions for $L(\Sigma \cup \{\upsilon\})$, $\mathcal{U}$ be a subset of $L^{\omega}(\Sigma;\upsilon)$ meager in the relating topology $\mathfrak{T}_E$, $\mathbf{s} \in L^{<\omega}(\Sigma;\upsilon)$ and $\vec{w} \in L^{\omega}(\Sigma;\upsilon)$. Then, there exists a countable ordinal ξ_0 such that for every $\xi > \xi_0$ there exists $\vec{u} \prec \vec{w} - \mathbf{s}$ such that $[\mathbf{s} \odot \mathbf{t}, \vec{u}] \subseteq L^{\omega}(\Sigma;\upsilon) \setminus \mathcal{U}$ for every $\mathbf{t} \in L^{\xi}(\Sigma;\upsilon) \cap VEL^{<\omega}(\vec{u})$.*

Proof. We use Theorem 5.2 for $\mathcal{U}$. We will prove that the first alternative is impossible. Indeed, let $\vec{u} \prec \vec{w}$ such that $[\mathbf{s},\vec{u}] \subseteq \mathcal{U}$. If $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ with $(\hat{\mathcal{U}_n})^{\diamond} = \emptyset$ for every $n \in \mathbb{N}$, then we set

$$\mathcal{R} = \{[\mathbf{t}, \vec{t}] : \mathbf{t} \in L^{<\omega}(\Sigma; \upsilon),\ \vec{t} \in L^{\omega}(\Sigma; \upsilon) \text{ and } [\mathbf{t}, \vec{t}] \cap \mathcal{U}_k = \emptyset \text{ for every } k \in \mathbb{N} \text{ with } k \leq |\mathbf{t}|\} ;$$

where $|\mathbf{t}|$ denotes the cardinality of the set $\sigma(\mathbf{t})$.

The family $\mathcal{R}$ satisfies the conditions (i) and (ii) of Lemma 5.3. Indeed, according to Theorem 5.2, for every $\mathbf{t} \in L^{<\omega}(\Sigma; \upsilon)$, $\vec{t} \in L^{\omega}(\Sigma; \upsilon)$ and $k \in \mathbb{N}$ there exists $\vec{t_1} \prec \vec{t}$ such that $[\mathbf{t}, \vec{t_1}] \cap \mathcal{U}_k = \emptyset$, as it is impossible $[\mathbf{t}, \vec{t_1}] \subseteq \hat{\mathcal{U}}_k$. Thus $\mathcal{R}$ satisfies (i) and obviously satisfies (ii). Hence, there exists $\vec{u}_1 \in [\mathbf{s}, \vec{u}]$ such that $[\mathbf{s} \odot \mathbf{t}, \vec{u}_1] \in \mathcal{R}$ for every $\mathbf{t} \in EL^{<\omega}(\vec{u}_1 - \mathbf{s})$. Then, $[\mathbf{s}, \vec{u}_1] \cap \mathcal{U} = \emptyset$. Indeed, let $\vec{u}_2 \in [\mathbf{s}, \vec{u}_1] \cap \mathcal{U}$. Then, $\vec{u}_2 \in [\mathbf{s}, \vec{u}_1] \cap \mathcal{U}_k$ for some $k \in \mathbb{N}$. Hence, there exists $\mathbf{t} \in EL^{<\omega}(\vec{u}_1 - \mathbf{s})$ with $\mathbf{s} \odot \mathbf{t} \propto \vec{u}_2$, $k \leq |\mathbf{s} \odot \mathbf{t}|$ and $[\mathbf{s} \odot \mathbf{t}, \vec{u}_1] \cap \mathcal{U}_k \neq \emptyset$. Then, $[\mathbf{s} \odot \mathbf{t}, \vec{u}_1] \notin \mathcal{R}$. A contradiction, since $\mathbf{t} \in EL^{<\omega}(\vec{u}_1 - \mathbf{s})$. Thus, $[\mathbf{s}, \vec{u}_1] \cap \mathcal{U} = \emptyset$ and consequently $\vec{u}_1 \notin \hat{\mathcal{U}}$. But $\vec{u}_1 \in [\mathbf{s}, \vec{u}] \subseteq \hat{\mathcal{U}}$, a contradiction. Hence, the second alternative of Theorem 5.2 holds for $\mathcal{U}$. □

Definition 5.6. Let Σ be a finite alphabet, $\upsilon \notin \Sigma$, $E(\Sigma)$ a set of substitutions for $L(\Sigma \cup \{\upsilon\})$. A family $\mathcal{U} \subseteq L^{\omega}(\Sigma; \upsilon)$ of infinite orderly sequences of variable located words is called *completely Ramsey for* $E(\Sigma)$ if for every $\mathbf{s} \in L^{<\omega}(\Sigma; \upsilon)$ and every $\vec{w} \in L^{\omega}(\Sigma; \upsilon)$ there exists $\vec{u} \prec \vec{w}$ such that

$$\text{either } [\mathbf{s}, \vec{u}] \subseteq \mathcal{U} \quad \text{or} \quad [\mathbf{s}, \vec{u}] \subseteq L^{\omega}(\Sigma; \upsilon) \setminus \mathcal{U} .$$

A further consequence of Theorem 5.2 is the characterization of completely Ramsey families of infinite orderly sequences of variable located words for a given set of substitutions.

Corollary 5.7. *Let Σ be a finite alphabet, $\upsilon \notin \Sigma$, $E(\Sigma)$ a set of substitutions for $L(\Sigma \cup \{\upsilon\})$. A family $\mathcal{U} \subseteq L^{\omega}(\Sigma; \upsilon)$ is completely Ramsey for $E(\Sigma)$ if and only if $\mathcal{U}$ has the Baire property in the relating to $E(\Sigma)$ topology $\mathfrak{T}_E$.*

Proof. Let $\mathcal{U} \subseteq L^{\omega}(\Sigma; \upsilon)$ have the Baire property in the topology $\mathfrak{T}_E$. Then $\mathcal{U} = \mathcal{B} \triangle \mathcal{C} = (\mathcal{B} \cup \mathcal{C}^c) \cup (\mathcal{C} \cap \mathcal{B}^c)$, where $\mathcal{B} \subseteq L^{\omega}(\Sigma; \upsilon)$ is $\mathfrak{T}_E$-closed and $\mathcal{C} \subseteq L^{\omega}(\Sigma; \upsilon)$ is $\mathfrak{T}_E$-meager ($\mathcal{C}^c = L^{\omega}(\Sigma; \upsilon) \setminus \mathcal{C}$). Let $\mathbf{s} \in L^{<\omega}(\Sigma; \upsilon)$ and $\vec{w} \in L^{\omega}(\Sigma; \upsilon)$. According to Corollary 5.4 and Proposition 4.3, there exists $\vec{u}_1 \prec \vec{w}$ such that $[\mathbf{s}, \vec{u}_1] \subseteq \mathcal{C}^c$ and according to Corollary 5.5 there exists $\vec{u} \prec \vec{u}_1$ such that

either $[\mathbf{s}, \vec{u}] \subseteq \mathcal{B} \cap [\mathbf{s}, \vec{u}_1] \subseteq \mathcal{B} \cap \mathcal{C}^c \subseteq \mathcal{U}$ or $[\mathbf{s}, \vec{u}] \subseteq \mathcal{B}^c \cap [\mathbf{s}, \vec{u}_1] \subseteq \mathcal{B}^c \cap \mathcal{C}^c \subseteq \mathcal{U}^c$.

Hence, $\mathcal{U}$ is completely Ramsey for $E(\Sigma)$.

On the other hand, if $\mathcal{U}$ is completely Ramsey for $E(\Sigma)$, then $\mathcal{U} = \mathcal{U}^\diamondsuit \cup (\mathcal{U} \setminus \mathcal{U}^\diamondsuit)$ and $\mathcal{U} \setminus \mathcal{U}^\diamondsuit$ is a meager set in $\mathfrak{T}_E$. Hence $\mathcal{U}$ has the Baire property in the topology $\mathfrak{T}_E$. $\square$

Acknowledgments

Both authors acknowledge partial support from the Kapodistrias research grant of Athens University.

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