AN EXTENDED RAMSEY PRINCIPLE FOR BANACH SPACES

V. FARMAKI

Abstract. We prove an extended Ramsey type partition principle to every countable order for every Banach space X with a Schauder base, exploiting the recursive thin Schreier system $(\mathcal{B}^\xi(X))_{\xi<\omega_1}$ of families of finite block bases of X; and we prove that Gowers's and Wagner's Nash-Williams type partition theorems for Banach spaces are concequences, in strengthened form, of this non-symmetric Ramsey principle.

Introduction

We prove, in Theorem 1.4 below, employing Schreier type ([S]) families, an extended to every countable ordinal Ramsey type ([R]) partition principle for Banach spaces. This is a partition principle for each countable order for tree-less partitions in the class $\Sigma^{<\omega}(X)$ of all finite normalized block bases of a Banach space X with a Schauder basis. The vehicle for stating and proving this theorem is the recursive thin Schreier system $(\mathcal{B}^\xi(X))_{\xi<\omega_1}$ of families of finite normalized block bases of X (Definition 1.2). We remark that in case $\xi < \omega$ is a finite ordinal, the family $\mathcal{B}^\xi(X)$ consists of all the normalized block bases with ξ blocks. Moreover, for partitions that are trees, we establish, in Theorem 2.5, a criterion, exploiting the notion of the ordinal index of a tree given in [Bou], for deciding which horn of the dichotomy proved in Theorem 1.4 actually holds for each countable ordinal.

In the context of Banach spaces the search for a symmetric Ramsey principle (as realized by Gowers in [G1], [G2]) rendered impossible by the appearance of the phenomenon of distortion (Milman [M], Odell-Schlumprecht [OS]), and the only prospect is for a non-symmetrical extended Ramsey principle strong enough to imply Gowers partition theorem ([G2, Theorem 2.1]), which has the character of a Nash-Williams type ([NW])theorem in Banach space theory. In fact, both Gowers's partition theorem and Wagner's strengthened "quantified" version in [W, Corollary 10] are derived, in strengthened form, from our extended Ramsey-type partition principle (Theorem 1.4).

1991 *Mathematics Subject Classification.* Primary 05D10; 46B15.
Key words and phrases. Gowers's dichotomy, Ramsey combinatorics, Schreier families.

In the case of the Banach space $X = c_0$, can be proved, following essentially the inductive procedure of Theorem 1.4, an extended (to every countable ordinal ξ) symmetric Ramsey type partition theorem for c_0, where the initial step $\xi = 1$ in induction is played by a result of Gowers proved in [G1]. That such a symmetric principle exists is due to the absence of distortion in $X = c_0$; the work of Milman ([M]) and Odell-Schlumprecht ([OS]) shows that such a symmetric Ramsey principle holds essentially only for Banach spaces containing c_0.

Notation. Let $(X, \| \ \|)$ be an infinite dimensional Banach space with a normalized Schauder basis $(e_n)_{n\in\mathbb{N}}$ (for every $x \in X$there is a unique sequence of scalars $(\lambda_n)_{n\in\mathbb{N}}$ such that $x = \sum_{n=1}^{\infty} \lambda_n e_n$). A *block* of $(e_n)_{n\in\mathbb{N}}$ is a finite linear combination of elements of $(e_n)_{n\in\mathbb{N}}$. The support of a non-zero block $s = \sum_{i=1}^{k} \lambda_i e_{n_i}$, written supp$(s)$, is the subset $(n_1 < \cdots < n_k)$ of natural numbers if $\lambda_i \neq 0$ for all $1 \leq i \leq k$. We write $s < t$ for two non-zero blocks of $(e_n)_{n\in\mathbb{N}}$ if $\max \operatorname{supp} s < \min \operatorname{supp} t$. A *block basis* of X is a (finite or infinite) sequence of non-zero blocks (s_n) with $s_1 < s_2 < \cdots$. It is a Schauder basis for the closed linear subspace of X genereted by $\{s_n : n \in \mathbb{N}\}$.

An infinite dimensional closed subspace Y of X with Schauder basis a block basis $(y_n)_{n\in\mathbb{N}}$ of X is called a *block subspace* of X and we will use the notation $Y < X$ to express it. Of course, if $Y < X$ and $Z < Y$ then $Z < X$. For a block subspace Y of X we set

$\Sigma(Y) = \{s \in Y : \|s\| = 1 \text{ and } s = \sum_{i=1}^{k} \lambda_i e_i \text{ for some } k \in \mathbb{N} \text{ and scalars } \lambda_1, \ldots, \lambda_k\}$;

$\Sigma^{<\omega}(Y) = \{(s_1, \ldots, s_k) : k \in \mathbb{N}, \text{ and } s_1 < \cdots < s_k \in \Sigma(Y)\}$; and

$\Sigma^{\omega}(Y) = \{(s_n)_{n\in\mathbb{N}} : (s_n)_{n\in\mathbb{N}} \subseteq \Sigma(Y) \text{ and } s_1 < s_2 < \cdots\}$.

Let $\Phi \subseteq \Sigma^{<\omega}(X)$, $s \in \Sigma(X)$, $Y < X$ and $\boldsymbol{\delta} = (\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$. We set

$\Phi_{\boldsymbol{\delta}} = \{(s_1, \ldots, s_k) \in \Sigma^{<\omega}(X) :$ there exists $(t_1, \ldots, t_k) \in \Phi$ such that $\|s_i - t_i\| < \delta_i$ and $\operatorname{supp} s_i = \operatorname{supp} t_i$ for all $1 \leq i \leq k\}$;

$\Phi_{-\boldsymbol{\delta}} = \{(s_1, \ldots, s_k) \in \Phi : (t_1, \ldots, t_k) \in \Phi$ for every $(t_1, \ldots, t_k) \in \Sigma^{<\omega}(X)$ such that $\|s_i - t_i\| < \delta_i$ and $\operatorname{supp} s_i = \operatorname{supp} t_i$ for all $1 \leq i \leq k\}$;

$\Phi(s) = \{(s_1, \ldots, s_k) \in \Sigma^{<\omega}(X) : (s, s_1, \ldots, s_k) \in \Phi\}$; and

$\Phi - s = \{(s_1, \ldots, s_k) \in \Phi : s < s_1 < \cdots < s_k\}$.

It is easy to prove that $(\Phi_{-\boldsymbol{\delta}})_{\boldsymbol{\delta}} \subseteq \Phi \subseteq (\Phi_{\boldsymbol{\delta}})_{-\boldsymbol{\delta}}$, $(\Phi^c)_{\boldsymbol{\delta}} = (\Phi_{-\boldsymbol{\delta}})^c$, where $\Phi^c = \Sigma^{<\omega}(X) \setminus \Phi$.

Let Y be the subspace of X generated by a block basis $(y_n)_{n\in\mathbb{N}}$ of X and $s \in \Sigma(X)$. We denote by $Y - s$ the block subspace of Y generated by $\{y_n : n \in \mathbb{N} \text{ and } s < y_n\}$. We write $s < Y$ if $Y = Y - s$.

Let M be an infinite subset of $\mathbb{N}$. We denote by $[M]^{<\omega}_{>0}$ the set of all non-empty, finite subsets of M, $[M]^{<\omega}$ the set of all finite subsets of M and $[M]$ the set of all the infinite subsets of M, considering them as strictly increasing sequences. For a family $\mathcal{A}$ of finite subsets of $\mathbb{N}$ and $n \in \mathbb{N}$ we set

$\mathcal{A}(n) = \{s \in [\mathbb{N}]^{<\omega} : \{n\} < s \text{ and } \{n\} \cup s \in \mathcal{A}\}$;
$\mathcal{A}^* = \{s \in [\mathbb{N}]^{<\omega} : s \text{ is an initial segment of some } t \in \mathcal{A}\}$; and
$\mathcal{A}_* = \{s \in [\mathbb{N}]^{<\omega} : s \text{ is a subset of some } t \in \mathcal{A}\}$.

To save writing let us assume from now on that all the Banach spaces and subspaces are infinite dimensional and that all the Banach spaces with a (Schauder) basis are endowed with a normalized Schauder basis.

1. A Ramsey type dichotomy for each countable ordinal

In this section we shall state and prove our fundamental result Theorem 1.4. This is a Ramsey type partition theorem for each countable ordinal stated for a Banach space X with a Schauder basis. The vehicle for stating and proving this theorem is the recursive thin Schreier system $(\mathcal{B}^\xi(X))_{\xi<\omega_1}$ of families of finite block bases of X.

For the definition of the families $\mathcal{B}^\xi(X)$ for $\xi < \omega_1$ we use the thin Schreier families $\mathcal{A}_\xi$ of finite sets of natural numbers defined initially in [F2] and completelly in [F3].

Definition 1.1 (The recursive thin Schreier systems $(\mathcal{A}_\xi)_{\xi<\omega_1}$, [F3, Def. 1.4])**.** For every non-zero, countable, limit ordinal λ choose and fix a strictly increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ of successor ordinals with $\sup_n \lambda_n = \lambda$. The system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ is defined recursively as follows:

(1) $\mathcal{A}_0 = \{\emptyset\}$ and $\mathcal{A}_1 = \{\{n\} : n \in \mathbb{N}\}$;
(2) $\mathcal{A}_{\zeta+1} = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s = \{n\} \cup s_1, \text{ where } n \in \mathbb{N}, \{n\} < s_1 \text{ and } s_1 \in \mathcal{A}_\zeta\}$;
(3i) $\mathcal{A}_{\omega^{\beta+1}} = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s = \bigcup_{i=1}^n s_i, \text{ where } n = \min s_1,\ s_1 < \cdots < s_n \text{ and } s_1, \ldots, s_n \in \mathcal{A}_{\omega^\beta}\}$;
(3ii) for a non-zero, countable limit ordinal λ,
$\mathcal{A}_{\omega^\lambda} = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s \in \mathcal{A}_{\omega^{\lambda_n}} \text{ with } n = \min s\}$; and
(3iii) for a limit ordinal ξ such that $\omega^\alpha < \xi < \omega^{\alpha+1}$ for some $0 < \alpha < \omega_1$, if $\xi = \omega^\alpha.p + \sum_{i=1}^m \omega^{a_i}.p_i$, where $m \in \mathbb{N}$ with $m \geq 0$, $p, p_1, \ldots, p_m$ are natural numbers with $p, p_1, \ldots, p_m \geq 1$ (so that either $p > 1$, or $p = 1$ and $m \geq 1$) and $a, a_1, \ldots, a_m$ are ordinals with $a > a_1 > \cdots a_m > 0$,
$\mathcal{A}_\xi = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s = s_0 \cup (\bigcup_{i=1}^m s_i) \text{ with } s_m < \cdots < s_1 < s_0,\ s_0 = s_1^0 \cup \cdots \cup s_p^0$

with $s_1^0 < \cdots < s_p^0 \in \mathcal{A}_{\omega^a}$, and $s_i = s_1^i \cup \cdots \cup s_{p_i}^i$ with $s_1^i < \cdots < s_{p_i}^i \in \mathcal{A}_{\omega^{a_i}}$ $\forall\, 1 \leq i \leq m\}$.

Definition 1.2 (Recursive thin Schreier systems $(\mathcal{B}^\xi(X))_{\xi<\omega_1}$ in a Banach space X)**.** Let X be a Banach space with a basis $(e_n)_{n\in\mathbb{N}}$ and $(\mathcal{A}_\xi)_{\xi<\omega_1}$ a recursive thin Schreier system. For every $0 < \xi < \omega_1$ we define the family $\mathcal{B}^\xi(X)$ of finite block bases of X as follows:

$\mathcal{B}^1(X) = \{(s) : s \in \Sigma(X)\}$; and

$\mathcal{B}^\xi(X) = \{(s_1, \ldots, s_k) \in \Sigma^{<\omega}(X) : (\min\operatorname{supp} s_1, \ldots, \min\operatorname{supp} s_k) \in \mathcal{A}_\xi\}$ for $\xi > 1$.

The following proposition justifies the term "recursive" of the systems $(\mathcal{B}^\xi)_{\xi<\omega_1}$.

Proposition 1.3. *Let X be a Banach space with a basis and $(\mathcal{B}^\xi(X))_{\xi<\omega_1}$ a recursive thin Schreier system in X. For every countable ordinal $\xi \geq 1$ there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ of countable ordinals with $\xi_n < \xi$, such that for every $s \in \Sigma(X)$ with* $\min\operatorname{supp} s = n$

$$\mathcal{B}^\xi(X)(s) = \mathcal{B}^{\xi_n}(X) \cap (\Sigma^{<\omega}(X) - s)\ .$$

Moreover, $\xi_n = \zeta$ for every $n \in \mathbb{N}$ if $\xi = \zeta + 1$ and $(\xi_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence with $\sup_n \xi_n = \xi$ *if ξ is a limit ordinal.*

Proof. According to Theorem 1.6 in [F3], for every countable ordinal $\xi > 0$ there exists a countable ordinal $\xi_n < \xi$ such that $\mathcal{A}_\xi(n) = \mathcal{A}_{\xi_n} \cap [\{n+1, n+2, \ldots\}]^{<\omega}$ for every $n \in \mathbb{N}$. Moreover, $\xi_n = \zeta$ for every $n \in \mathbb{N}$ if $\xi = \zeta + 1$ and (ξ_n) is a strictly increasing sequence with $\sup_n \xi_n = \xi$ if ξ is a limit ordinal.
Hence, $\mathcal{B}^\xi(X)(s) = \mathcal{B}^{\xi_n}(X) \cap (\Sigma^{<\omega}(X) - s)$ for every $s \in \Sigma(X)$ with $\min\operatorname{supp} s = n$. □

Gowers in [G2] defined the following game between two players S and P. Let Y be a block subspace of X and $\Psi \subseteq \Sigma^{<\omega}(X)$ a family of finite block bases of X, then the game $\Psi[Y]$ is defined as follows: in the n^{th} move of the game, at first S chooses a block subspace Z_n of Y, and then player P chooses some vector $s_n \in Z_n$. The aim of P is to construct a sequence $(s_1, \ldots, s_k) \in \Psi$ in some move k. A winning strategy for P, in this game $\Psi[Y]$, is a function φ (which, for any finite block basis $s_1, \ldots, s_n$ and any block subspace Z of Y gives a vector $s = \varphi(s_1, \ldots, s_n, Z) \in Z$) such that given any sequence $Z_1, Z_2, \ldots$ of block subspaces of Y, there exists $k \in \mathbb{N}$ such that the sequence $(s_1, \ldots, s_k)$ defined inductively by $s_1 = \varphi(\emptyset, Z_1)$, $s_i = \varphi(s_1, \ldots, s_{i-1}, Z_i)$ for $i \leq k$ belongs to Ψ.

Theorem 1.4. *Let X be a Banach space with a basis and $\Phi \subseteq \Sigma^{<\omega}(X)$ be a family of finite block bases of X. For every countable ordinal ξ, every block subspace Y of X and $\boldsymbol{\delta} = (\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ there exists a block subspace Y_0 of Y such that:*

either (i) $\mathcal{B}^\xi(X)\cap\Sigma^{<\omega}(Y_0)\subseteq\Phi$;

or (ii) *the player P has a winning strategy for the game* $\big(\mathcal{B}^\xi(X)\cap(\Phi^c)_{\boldsymbol\delta}\big)[Y_0]$.

In the proof of this theorem we will make use of a diagonal argument contained in the following lemma, which is a modification of Corollary 2.3 in [G2].

Notation. For a block subspace Y of X we define $\Pi_Y=\{(s,Z): s\in\Sigma(Y),\, Z<Y-s\}$. If $\mathcal{G}\subseteq\Pi_Y$ and $\delta>0$, then

$$\mathcal{G}_\delta=\{(s,Z)\in\Pi : \text{there exists } s'\in\Sigma(Y) \text{ such that } \operatorname{supp} s'=\operatorname{supp} s,\, \|s-s'\|<\delta,\,(s',Z)\in\mathcal{G}\}\ .$$

Lemma 1.5. *Let Y be a block subspace of X, $\delta>0$ and $\mathcal{G}\subseteq\Pi_Y$ with the properties:*

(1) *for every $(s,Z)\in\Pi_Y$ there exists $(s,Z_1)\in\mathcal{G}$ with $Z_1<Z$; and*

(2) *if $(s,Z_1)\in\mathcal{G}$ and $Z_2<Z_1$, then $(s,Z_2)\in\mathcal{G}$.*

Then, there exists $Z<Y$ such that $(s,Z_1)\in\mathcal{G}_\delta$ for every $s\in\Sigma(Z)$ and $Z_1<Z-s$.

Proof. Let $Y_0=Y$ and $y_1\in\Sigma(Y)$. We assume that $Y_{n-1}<\cdots<Y_1<Y_0$ and $y_n\in Y_{n-1},\dots,y_1\in Y_0$ with $(y_1,\dots,y_n)\in\Sigma^{<\omega}(X)$ have been chosen. If $[y_1,\dots,y_n]$ is the subspace of Y generated by the vectors $y_1,\dots,y_n$, then there exist $m_n\in\mathbb{N}$ and vectors $s_1^n,\dots,s_{m_n}^n$ in the unit sphere of $[y_1,\dots,y_n]$ so that for every s in the unit sphere of $[y_1,\dots,y_n]$ there exists $s_i^n\in\{s_1^n,\dots,s_{m_n}^n\}$ with $\operatorname{supp} s_i^n=\operatorname{supp} s$ and $\|s_i^n-s\|<\delta$. By the property (1) of $\mathcal{G}$, there exist block subspaces $Z_{m_n}^n<\cdots<Z_1^n<Y_{n-1}$ so that $(s_i^n,Z_i^n)\in\mathcal{G}$ for all $1\le i\le m_n$. Set $Y_n=Z_{m_n}$ and choose $y_{n+1}\in\Sigma(Y_n)$ with $y_{n+1}>y_n$.

The block subspace Z of Y with basis $(y_n)_{n\in\mathbb{N}}$ satisfies the conclusion. Indeed, let $s\in\Sigma(Z)$ and let n be the minimum natural number such that s be in the unit sphere of $[y_1,\dots,y_n]$. Let $Z_1<Z-s<Y_n$. Then there exist $1\le i\le m_n$ and $s_i^n\in\Sigma(Y)$ with $\operatorname{supp} s_i^n=\operatorname{supp} s$, $\|s_i^n-s\|<\delta$ and $(s_i^n,Y_n)\in\mathcal{G}$. According to property (2) $(s_i^n,Z_1)\in\mathcal{G}$. This gives that $(s,Z_1)\in\mathcal{G}_\delta$. □

Exploiting the recursive nature of the Schreier system $(\mathcal{B}^\xi(X))_{\xi<\omega_1}$ we will prove by induction the following theorem, which is equivalent to the main Theorem 1.4.

Theorem 1.6. *Let X be a Banach space with a basis and $\Phi\subseteq\Sigma^{<\omega}(X)$. For every countable ordinal ξ, every block subspace Y of X and $\boldsymbol\delta=(\delta_n)_{n\in\mathbb{N}}\subseteq\mathbb{R}^+$, there exists a block subspace Y_0 of Y such that:*

either $\mathcal{B}^\xi(X)\cap\Sigma^{<\omega}(Y_0)\subseteq\Phi_{\boldsymbol\delta}$;

or *the player P has a winning strategy for the game* $\big(\mathcal{B}^\xi(X)\cap(\Phi^c)_{\boldsymbol\delta}\big)[Y_0]$.

Proof. Let $\xi = 1$. Then, either there exists $Y_0 < Y$ such that $\mathcal{B}^1(Y_0) \subseteq \Phi$; or for every $Z_1 < Y$ we have $\mathcal{B}^1(Z_1) \cap \Phi^c \neq \emptyset$.

Let $\xi > 1$. We assume that the theorem is valid for every $\zeta < \xi$. Let $s \in \Sigma(Y)$ with $\min \operatorname{supp} s = n$ and $Z_0 < Y - s$. According to Proposition 1.3, there exists $\xi_n < \xi$ such that $\mathcal{B}^\xi(X)(s) = \mathcal{B}^{\xi_n}(X) \cap (\Sigma^{<\omega}(X) - s)$. Using the induction hypothesis for $\Phi(s)$, Z_0, ξ_n and $\boldsymbol{\delta}^1 = (\delta_2, \delta_3, \ldots)$, we have that there exists $Z < Z_0$ such that

either $\mathcal{B}^{\xi_n}(X) \cap \Sigma^{<\omega}(Z) \subseteq (\Phi(s))_{\boldsymbol{\delta}^1}$;

or the player P has a winning strategy for the game $\big(\mathcal{B}^{\xi_n}(X) \cap ((\Phi(s))^c)_{\boldsymbol{\delta}^1}\big)[Z]$.

Then there exists $Z < Z_0$ such that

either (i) $\mathcal{B}^\xi(X)(s) \cap \Sigma^{<\omega}(Z) \subseteq (\Phi(s))_{\boldsymbol{\delta}^1}$;

or (ii) the player P has a winning strategy for the game $\big(\mathcal{B}^\xi(X)(s) \cap ((\Phi(s))^c)_{\boldsymbol{\delta}^1}\big)[Z]$.

Let $\mathcal{G} = \{(s, Z) : s \in \Sigma(Y),\ Z < Y,\ s < Z \text{ and } (s, Z) \text{ satisfies either (i) or (ii)}\}$. The family $\mathcal{G}$ satisfies the condition (1) of Lemma 1.5 (by the above argument) and also the condition (2) (obviously), hence there exists $Y_1 < Y$ such that $(s, Z) \in \mathcal{G}_{\delta_1}$ for every $s \in \Sigma(Y_1)$ and $Z < Y_1 - s$. Let

$$\mathcal{G}^1 = \{(s, Z) \in \mathcal{G} : (s, Z) \ \text{ satisfies (i)}\} \text{ and}$$
$$\mathcal{G}^2 = \{(s, Z) \in \mathcal{G} : (s, Z) \ \text{ satisfies (ii)}\}\ .$$

Of course $\mathcal{G} = \mathcal{G}^1 \cup \mathcal{G}^2$, and $\mathcal{G}_{\delta_1} = (\mathcal{G}^1)_{\delta_1} \cup (\mathcal{G}^2)_{\delta_1}$.

Set $\Psi = \{s \in \Sigma(Y_1) : (s, Y_1 - s) \in (\mathcal{G}^1)_{\delta_1}\}$. Then, either there exists $Y_0 < Y_1$ such that $\Sigma(Y_0) \subseteq \Psi$ or $\Sigma(Z) \cap \Psi^c \neq \emptyset$ for every $Z < Y_1$. Thus, there exists $Y_0 < Y_1 < Y$ so that

either $(s, Y_1 - s) \in (\mathcal{G}^1)_{\delta_1}$ for every $s \in \Sigma(Y_0)$;

or for every $Z < Y_0$ there exists $s \in \Sigma(Z)$ such that $(s, Y_1 - s) \in (\mathcal{G}^2)_{\delta_1}$.

Let $(s, Y_1 - s) \in (\mathcal{G}^1)_{\delta_1}$ for every $s \in \Sigma(Y_0)$ and let $(s, s_1, \ldots, s_k) \in \mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Y_0)$. Then $s \in \Sigma(Y_0)$ and $(s_1, \ldots, s_k) \in \mathcal{B}^\xi(X)(s) \cap \Sigma^{<\omega}(Y_0)$. Since $(s, Y_1 - s) \in (\mathcal{G}^1)_{\delta_1}$, there exists $s' \in \Sigma(Y)$ such that $\operatorname{supp} s' = \operatorname{supp} s$, $\|s - s'\| < \delta_1$ and $(s', Y_1 - s') \in \mathcal{G}^1$. Hence, $\mathcal{B}^\xi(X)(s') \cap \Sigma^{<\omega}(Y_1 - s') \subseteq (\Phi(s'))_{\boldsymbol{\delta}^1}$. Since $(s', s_1, \ldots, s_k) \in \mathcal{B}^\xi(X)$ and $Y_0 < Y_1$ we have $(s_1, \ldots, s_k) \in \mathcal{B}^\xi(X)(s') \cap \Sigma^{<\omega}(Y_1 - s') \subseteq (\Phi(s'))_{\boldsymbol{\delta}^1}$. Then there exists $(t_1, \ldots t_k) \in \Phi(s')$ with $\operatorname{supp} s_i = \operatorname{supp} t_i$ and $\|t_i - s_i\| < \delta_{i+1}$ for every $1 \leq i \leq k$. So, $(s', t_1, \ldots, t_k) \in \Phi$ and consequently $(s, s_1, \ldots, s_k) \in \Phi_{\boldsymbol{\delta}}$. This proves that $\mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Y_0) \subseteq \Phi_{\boldsymbol{\delta}}$ if $(s, Y_1 - s) \in (\mathcal{G}^1)_{\delta_1}$ for every $s \in \Sigma(Y_0)$.

Now, let for every $Z < Y_0$ there exists $s \in \Sigma(Z)$ such that $(s, Y_1 - s) \in (\mathcal{G}^2)_{\delta_1}$ and let $(Z_n)_{n \in \mathbb{N}}$ be a sequence of block subspaces of Y_0. Then there exists $s_1 \in \Sigma(Z_1)$ such that $(s_1, Y_1 - s) \in (\mathcal{G}^2)_{\delta_1}$. Set $\varphi(\emptyset, Z_1) = s_1$. Since $(s_1, Y_1 - s) \in (\mathcal{G}^2)_{\delta_1}$, there exists $s \in \Sigma(Y)$

such that $\operatorname{supp} s_1 = \operatorname{supp} s$, $\|s_1 - s\| < \delta_1$ and $(s, Y_1 - s) \in \mathcal{G}^2$. Hence, since $Y_0 - s < Y_1 - s$, the player P has a winning strategy φ' for the game $\big(\mathcal{B}^\xi(X)(s) \cap ((\Phi(s))^c)_{\boldsymbol{\delta}^1}\big)[Y_0 - s]$. So, the player P continuing the game with the strategy φ' has a winning strategy φ (i.e. $\varphi(\emptyset, Z_1) = s_1$, $\varphi(s_1, Z) = \varphi'(\emptyset, Z)$, and $\varphi(s_1, \dots, s_{i-1}, Z) = \varphi'(s_2, \dots, s_{i-1}, Z)$ for all $2 < i < k$) for the game $\big(\mathcal{B}^\xi(X) \cap (\Phi^c)_{\boldsymbol{\delta}}\big)[Y_0]$. Indeed, if $(s_2, \dots, s_k) \in \mathcal{B}^\xi(X)(s) \cap ((\Phi(s))^c)_{\boldsymbol{\delta}^1}$, then, $(s, s_2, \dots, s_k) \in \mathcal{B}^\xi(X)$ and consequently $(s_1, s_2, \dots, s_k) \in \mathcal{B}^\xi(X)$ and also, since there exists $(t_2, \dots, t_k) \in (\Phi(s))^c$ with $\operatorname{supp} s_i = \operatorname{supp} t_i$ and $\|s_i - t_i\| < \delta_i$ for every $2 \le i \le k$, we have, $(s, t_2, \dots, t_k) \in \Phi^c$ and consequently $(s_1, s_2, \dots, s_k) \in (\Phi^c)_{\boldsymbol{\delta}}$.

This finishes the proof. □

Proof of Theorem 1.4. Let $\boldsymbol{\delta}^1 = (\delta_n/2)_{n \in \mathbb{N}}$ and $\Phi^1 = \Phi_{-\boldsymbol{\delta}^1}$. Applying Theorem 1.6 to Φ^1 (with $\boldsymbol{\delta}$ replaced by $\boldsymbol{\delta}^1$), we have the existence of a block subspace Y_0 of Y such that

either $\mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Y_0) \subseteq (\Phi^1)_{\boldsymbol{\delta}^1} = (\Phi_{-\boldsymbol{\delta}^1})_{\boldsymbol{\delta}^1} \subseteq \Phi$;

or the player P has a winning for the game $\big(\mathcal{B}^\xi(X) \cap (\Phi^c)_{\boldsymbol{\delta}}\big)[Y_0]$,

since $((\Phi^1)^c)_{\boldsymbol{\delta}^1} = ((\Phi_{-\boldsymbol{\delta}^1})^c)_{\boldsymbol{\delta}^1} \subseteq ((\Phi^c)_{\boldsymbol{\delta}^1})_{\boldsymbol{\delta}_1} \subseteq (\Phi^c)_{\boldsymbol{\delta}}$.

This finishes the proof. □

2. Strengthened Gowers's partition theorem

The main result of this section is Theorem 2.5. This theorem, on the one hand constitutes a strengthened form of Theorem 1.4, in case we restrict ourselves, not to arbitrary, but only to partitions which are trees, characterizing which horn of the dichotomy actually holds for each countable ordinal, and on the other hand is a strengthened form of Gowers's partition theorem ([G2, Theorem 2.1]) standing and functioning as a Nash-Williams theorem ([NW]) in Banach spaces. In the proof of our theorem we mainly use Theorem 1.4 and also we exploit the notion of the ordinal index of a tree given in [Bou].

Definition 2.1. Let $\Phi \subseteq \Sigma^{<\omega}(X)$.

(i) Φ is a **tree** if $\Phi^* = \Phi$, where

$\Phi^* = \{(s_1, \dots, s_k) \in \Sigma^{<\omega}(X) : \exists (t_1, \dots, t_m) \in \Phi \text{ with } k \le m \text{ and } s_i = t_i \text{ for all } i \le k\}$;

(ii) Φ is **hereditary** if $\Phi_* = \Phi$, where

$\Phi_* = \{(s_1, \dots, s_k) \in \Sigma^{<\omega}(X) : \exists (t_1, \dots, t_m) \in \Phi \text{ with } \{s_1, \dots, s_k\} \subseteq [t_1, \dots, t_m]\}$;

(iii) Φ_h (resp. Φ_t) is the largest subfamily of Φ which is hereditary (resp. a tree) and

$\Phi_h = \{(s_1, \dots, s_k) \in \Phi\colon \Sigma^{<\omega}([s_1, \dots, s_k]) \subseteq \Phi\}$;

$\Phi_t = \{(s_1, \dots, s_k) \in \Phi : (s_1, \dots, s_m) \in \Phi \text{ for every } m \le k\}$;

(iv) $\bar{\Phi} = \{(s_1, \dots, s_k) \in \Sigma^{<\omega}(X) : (s_1, \dots, s_k) \in \Phi_{\boldsymbol{\delta}} \text{ for every } \boldsymbol{\delta} = (\delta_i)_{i=1}^k \subseteq \mathbb{R}^+\}$.

We will define now a metric d on $\Sigma(X)$ stronger than the norm, which also makes $\Sigma(X)$ a Polish (separable, complete) space. This metric d has been defined and used firstly in [F1].

Definition 2.2. Let X be a Banach space with a basis. We define a metric d on $\Sigma(X)$ as follows: If $s, t \in \Sigma(X)$, then we set

$$d(s,t) = 1 \quad \text{if} \quad \operatorname{supp} s \neq \operatorname{supp} t \; ; \quad \text{and}$$
$$d(s,t) = \|s-t\| \quad \text{if} \quad \operatorname{supp} s = \operatorname{supp} t \; .$$

Having the metric d on $\Sigma(X)$, we say a tree $\Phi \subseteq \Sigma^{<\omega}(X)$ $\mathfrak{T}_d$**-closed** if and only if for every sequence $((s_1^n,\ldots,s_k^n))_{n\in\mathbb{N}} \subseteq \Phi$ such that there exist $s_1,\ldots,s_k \in \Sigma(X)$ with $d(s_i^n, s_i) \underset{n}{\to} 0$ for every $i = 1,\ldots,k$, we have $(s_1,\ldots,s_k) \in \Phi$, equivalently if $\bar{\Phi} = \Phi$.

A tree $\Phi \subseteq \Sigma^{<\omega}(X)$ is **well founded** if there is no sequence $(s_n)_{n\in\mathbb{N}}$ in $\Sigma(X)$ satisfying $(s_1,\ldots,s_n) \in \Phi$ for every $n \in \mathbb{N}$. A particular version of the Kunen-Martin theorem (see [Bou]) gives that a $\mathfrak{T}_d$-closed tree Φ is well founded if and only if it has countable ordinal index ($o(\Phi) < \omega_1$). The index $o(\Phi)$ of Φ is the smallest ordinal ξ so that $\Phi^\xi = \emptyset$, where
$\Phi^0 = \Phi$,
$\Phi^{\xi+1} = \bigcup_{n=1}^\infty \{(s_1,\ldots,s_n) \in \Phi^\xi : (s_1,\ldots,s_n,s) \in \Phi^\xi \text{ for some } s \in \Sigma(X)\}$ for $0 < \xi < \omega_1$,
$\Phi^\xi = \bigcap_{\beta<\xi} \Phi^\beta$ for a limit ordinal $0 < \xi < \omega_1$.
Observe that if Φ is hereditary, then Φ^ξ is hereditary for every $0 < \xi < \omega_1$.

In the proof of our main theorem (Theorem 2.5) we will use the following propositions:

Proposition 2.3. *Let X be a Banach space with a basis. For every countable ordinal ξ and every block subspace Y of X, the hereditary family $(\mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Y))_*$ is $\mathfrak{T}_d$-closed and has ordinal index $o((\mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Y))_*)$ equal to ξ.*

Proof. Let $k \in \mathbb{N}$, $s_1,\ldots,s_k \in \Sigma(X)$ and $(s_1^n,\ldots,s_k^n) \in (\mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Y))_*$ for every $n \in \mathbb{N}$ such that $\lim_n d(s_i^n, s_i) = 0$ for all $i \leq k$. Then for every $0 < \varepsilon_1,\ldots,\varepsilon_k < 1$ there exists $n_0 \in \mathbb{N}$ such that $\operatorname{supp} s_i^n = \operatorname{supp} s_i$, and $\|s_i^n - s_i\| < \varepsilon_i$ for every $n \geq n_0$ and $i \leq k$. Hence, $(s_1,\ldots,s_k) \in (\mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Y))_*$. So, $(\mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Y))_*$ is $\mathfrak{T}_d$-closed.

Since $(\mathcal{B}^1(X) \cap \Sigma^{<\omega}(Y))_* = \{(s) : s \in \Sigma(Y)\}$, we have that $o((\mathcal{B}^1(X) \cap \Sigma^{<\omega}(Y))_*) = 1$. Assume that $o((\mathcal{B}^\zeta(X) \cap \Sigma^{<\omega}(Y))_*) = \zeta$ for every $\zeta < \xi$. Then $o((\mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Y))_*) = \xi$. Indeed, according to Proposition 1.3 and the inductive hypothesis, for every $s \in \Sigma(Y)$ with $\min \operatorname{supp} s = n$ we have

$$o(((\mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Y))(s))_*) = o((\mathcal{B}^{\xi_n}(X) \cap \Sigma^{<\omega}(Y - s))_*) = \xi_n < \xi,$$

where $\xi_n = \zeta$ for every $n \in \mathbb{N}$ if $\xi = \zeta + 1$ and $(\xi_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence with $\sup_n \xi_n = \xi$ if ξ is a limit ordinal. By induction on $\zeta < \omega_1$ we have

$$(s, s_1, \ldots, s_k) \in ((\mathcal{B}^{\xi}(X) \cap \Sigma^{<\omega}(Y))_*)^{\zeta} \ \text{ if } \ (s_1, \ldots, s_k) \in (((\mathcal{B}^{\xi}(X) \cap \Sigma^{<\omega}(Y))(s))_*)^{\zeta} \ .$$

Let $s \in \Sigma(Y)$ with $\min \operatorname{supp} s = n$. Since $(((\mathcal{B}^{\xi}(X)\cap\Sigma^{<\omega}(Y))(s))_*)^{\zeta} \neq \emptyset$ for every $\zeta < \xi_n$ and $((\mathcal{B}^{\xi}(X) \cap \Sigma^{<\omega}(Y))_*)^{\zeta}$ is a hereditary family, we have $(s) \in ((\mathcal{B}^{\xi}(X) \cap \Sigma^{<\omega}(Y))_*)^{\xi_n}$ and consequently that $o((\mathcal{B}^{\xi}(X) \cap \Sigma^{<\omega}(Y))_*) \geq \xi$.

If $o((\mathcal{B}^{\xi}(X)\cap\Sigma^{<\omega}(Y))_*) > \xi$, then there exists $s \in \Sigma(Y)$ with $\min \operatorname{supp} s = n$ such that $(s) \in ((\mathcal{B}^{\xi}(X) \cap \Sigma^{<\omega}(Y))_*)^{\xi} \neq \emptyset$. Then $(s) \in ((\mathcal{B}^{\xi}(X) \cap \Sigma^{<\omega}(Y))_*)^{\xi_n+1}$ and consequently $(s, t) \in ((\mathcal{B}^{\xi}(X) \cap \Sigma^{<\omega}(Y))_*)^{\xi_n}$ for some $t \in \Sigma(Y)$. By induction on $\zeta < \omega_1$ we have

$$(t) \in ((\mathcal{B}^{\xi_k}(X) \cap \Sigma^{<\omega}(Y))_*)^{\zeta} \text{ for some } k \leq n, \text{ if } (s, t) \in ((\mathcal{B}^{\xi}(X) \cap \Sigma^{<\omega}(Y))_*)^{\zeta}$$

So, $((\mathcal{B}^{\xi_k}(X) \cap \Sigma^{<\omega}(Y))_*)^{\xi_n} \neq \emptyset$ for some $k \leq n$. A contradiction to the induction hypothesis, since $\xi_k \leq \xi_n$. Hence, $o((\mathcal{B}^{\xi}(X) \cap \Sigma^{<\omega}(Y))_*) = \xi$. □

Proposition 2.4. *Let $\Phi \subseteq \Sigma^{<\omega}(X)$ be a tree and $\xi_1 < \xi_2$ two countable ordinals. If there exists $Y_1 < X$ such that $\mathcal{B}^{\xi_2}(X) \cap \Sigma^{<\omega}(Y_1) \subseteq \Phi$, then there exists $Y_2 < Y_1$ such that $(\mathcal{B}^{\xi_1}(X) \cap \Sigma^{<\omega}(Y_2))_* \subseteq \Phi$.*

Proof. Let $(y_i)_{i\in\mathbb{N}}$ be a normalized block basis of Y_1, and $M = (n_i)_{i\in\mathbb{N}}$, where $n_i = \min \operatorname{supp} y_i$. According to Corollary 2.3 in [F3], there exists an infinite subset $L = (n_{l_i})_{i\in\mathbb{N}}$ of M such that $(\mathcal{A}_{\xi_1})_* \cap [L]^{<\omega} \subseteq (\mathcal{A}_{\xi_2})^* \setminus \mathcal{A}_{\xi_2}$. Let Y_2 be the subspace of Y_1 with basis $(y_{l_i})_{i\in\mathbb{N}}$. For every $(s_1, \ldots, s_k) \in (\mathcal{B}^{\xi_1}(X) \cap \Sigma^{<\omega}(Y_1))_*$ we have that $(\min \operatorname{supp} s_1, \ldots, \min \operatorname{supp} s_k) \in (\mathcal{A}_{\xi_1})_* \cap [L]^{<\omega} \subseteq (\mathcal{A}_{\xi_2})^* \setminus \mathcal{A}_{\xi_2}$. According to Theorem 1.14 in [F3], there exists $(m_1, \ldots, m_p) \in [L]^{<\omega} \cap \mathcal{A}_{\xi_2}$ with $k < p$, $m_j = \min \operatorname{supp} s_j$ for $j \leq k$ and $\max \operatorname{supp} s_k < m_{k+1}$. If $m_j = n_{l_{i_j}}$ for every $k < j \leq p$, we set $s_j = y_{l_{i_j}}$ for every $k < j \leq p$. Then $(s_1, \ldots, s_p) \in \mathcal{B}^{\xi_2}(X) \cap \Sigma^{<\omega}(Y_2)$ and consequently $(s_1, \ldots, s_k) \in (\mathcal{B}^{\xi_2}(X) \cap \Sigma^{<\omega}(Y_1))^* \subseteq \Phi$. Hence $(\mathcal{B}^{\xi_1}(X) \cap \Sigma^{<\omega}(Y_2))_* \subseteq (\mathcal{B}^{\xi_2}(X) \cap \Sigma^{<\omega}(Y_1))^* \subseteq \Phi$. □

Theorem 2.5. *Let X be a Banach space with a basis, $\Phi \subseteq \Sigma^{<\omega}(X)$ be a tree, Y a block subspace of X, and $\boldsymbol{\delta} = (\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$. The following two cases occur:*

[Case 1] *There exists a block subspace Y_0 of Y such that $\Sigma^{<\omega}(Y_0) \subseteq \Phi$.*

[Case 2] *There exists a countable ordinal*

$\xi_Y^{\Phi} = \min\{1 \leq \xi < \omega_1 : \mathcal{B}^{\xi}(X) \cap \Sigma^{<\omega}(Z) \cap \Phi^c \neq \emptyset$ *for every* $Z < Y\}$ $(< \omega_1)$, *such that:*

2(i) *for every $\xi < \xi_Y^{\Phi}$ there exists a block subspace Y_0 of Y with $\mathcal{B}^{\xi}(X) \cap \Sigma^{<\omega}(Y_0) \subseteq \Phi$;*

2(ii) *for every $\xi \geq \xi_Y^{\Phi}$ and every block subspace Z of Y, there exists a block subspace Y_0 of Z so that the player P has a winning strategy for the game $\big(\mathcal{B}^{\xi}(X) \cap (\Phi^c)_{\boldsymbol{\delta}}\big)[Y_0]$.*

Proof. Firstly, we will prove the theorem for a partition Φ which is a $\mathfrak{T}_d$-closed tree.

[Case 1] Assume that for every $\xi < \omega_1$, there exists a block subspace Y_ξ of Y such that $\mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Y_\xi) \subseteq \Phi$.

According to Proposition 2.4, we can assume that for every $\xi < \omega_1$, there exists a block subspace Y_ξ of Y such that $(\mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Y_\xi))_* \subseteq \Phi_h \subseteq \bar{\Phi}_h$ (Definition 2.1). Since Φ is a $\mathfrak{T}_d$-closed tree, the family $\bar{\Phi}_h \cap \Sigma^{<\omega}(Y)$ is also a $\mathfrak{T}_d$-closed tree in $\Sigma^{<\omega}(X)$. We claim that the tree $\bar{\Phi}_h \cap \Sigma^{<\omega}(Y)$ is not well founded. Indeed, if $\bar{\Phi}_h \cap \Sigma^{<\omega}(Y)$ was well founded, then $o(\bar{\Phi}_h \cap \Sigma^{<\omega}(Y)) = \xi_0 < \omega_1$ and consequently for $\xi_1 > \xi_0$ we would have, according to Proposition 2.3, that $\xi_1 = o((\mathcal{B}^{\xi_1}(X) \cap \Sigma^{<\omega}(Y_{\xi_1}))_*) \leq o(\bar{\Phi}_h \cap \Sigma^{<\omega}(Y)) = \xi_0$, a contradiction. So, $\bar{\Phi}_h \cap \Sigma^{<\omega}(Y)$ is not well founded and consequently there exists a infinite block basis $(s_n)_{n\in\mathbb{N}}$ in Y such that $(s_1, \ldots, s_n) \in \bar{\Phi}_h \cap \Sigma^{<\omega}(Y)$ for every $n \in \mathbb{N}$. The family $\bar{\Phi}_h$ is hereditary, so, if Y_0 is the block subspace of Y with basis $(s_n)_{n\in\mathbb{N}}$, then $\Sigma^{<\omega}(Y_0) \subseteq \bar{\Phi}_h \subseteq \bar{\Phi} = \Phi$.

[Case 2] Assume that there exists $\xi < \omega_1$ such that $\mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Z) \cap \Phi^c \neq \emptyset$ for every block subspace Z of Y. Then ξ_Y^Φ is a countable ordinal.

(2i) Let $\xi < \xi_Y^\Phi$. According to the definition of ξ_Y^Φ, there exists a block subspace Y_0 of Y such that $\mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Y_0) \subseteq \Phi$.

(2ii) Let $\xi \geq \xi_Y^\Phi$ and Z a block subspace of Y. According to Theorem 1.4, there exists a block subspace Y_0 of Z such that either $\mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Y_0) \subseteq \Phi$; or the player P was a winning strategy for the game $\big(\mathcal{B}^\xi(X) \cap (\Phi^c)_{\boldsymbol{\delta}}\big)[Y_0]$. The first alternative is impossible. Indeed, if there exists a block subspace Y_0 of Y such that $\mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Y_0) \subseteq \Phi$, then there exists a block subspace Y_1 of Y_0 such that $\mathcal{B}^{\xi_Y^\Phi}(X) \cap \Sigma^{<\omega}(Y_1) \subseteq \Phi$, according to Proposition 2.4. This is a contradiction to the definition of ξ_Y^Φ.

The proof for the case of a $\mathfrak{T}_d$-closed tree Φ is complete.

Now, let Φ be an arbitrary tree. We use the above arguments for the $\mathfrak{T}_d$-closed tree $\Psi = \overline{\Phi_{-\boldsymbol{\delta}^1}}$ and $\boldsymbol{\delta}^1 = (\delta_n/2)_{n\in\mathbb{N}}$ instead of $\boldsymbol{\delta}$. Then the proof follows from the facts that $\bar{\Psi} \subseteq (\Phi_{-\boldsymbol{\delta}^1})_{\boldsymbol{\delta}^1} \subseteq \Phi$ and that $(\bar{\Psi}^c)_{\boldsymbol{\delta}^1} \subseteq ((\Phi^c)_{\boldsymbol{\delta}^1})_{\boldsymbol{\delta}^1} \subseteq (\Phi^c)_{\boldsymbol{\delta}}$. □

Corollary 2.6. *Let X be a Banach space with a basis, Φ be an arbitrary subset of $\Sigma^{<\omega}(X)$, Y a block subspace of X and $\boldsymbol{\delta} = (\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$. The following cases occur:*

[Case 1] *There exists a block subspace Y_0 of Y such that $\Sigma^{<\omega}(Y_0) \subseteq \Phi$.*

[Case 2] *There exists a countable ordinal ζ_Y^Φ such that:*

for every $\xi < \zeta_Y^\Phi$ there exists a block subspace Y_0 of Y with $\mathcal{B}^\xi(X) \cap \Sigma^{<\omega}(Y_0) \subseteq \Phi$; and

for every $\xi \geq \zeta_Y^\Phi$ and every block subspace Z of Y, there exists a block subspace Y_0 of Z so that the player P has a winning strategy for the game $\big(\mathcal{B}^\xi(X)^* \cap (\Phi^c)_{\boldsymbol{\delta}}\big)[Y_0]$.

Proof. We use Theorem 2.5 for Φ_t, Y and $\boldsymbol{\delta}$. The proof follows using the following fact : If $(s_1, \ldots, s_k) \in \mathcal{B}^\xi(X) \cap ((\Phi_t)^c)_{\boldsymbol{\delta}}$, then there exists $\ell \in \mathbb{N}$ with $\ell \leq k$ such that $(s_1, \ldots, s_\ell) \in \mathcal{B}^\xi(X)^* \cap (\Phi^c)_{\boldsymbol{\delta}}$. Indeed, if $(s_1, \ldots, s_k) \in ((\Phi_t)^c)_{\boldsymbol{\delta}}$, then either $(s_1, \ldots, s_k) \in (\Phi^c)_{\boldsymbol{\delta}}$ or there exists $\ell < k$ such that $(s_1, \ldots, s_\ell) \in (\Phi^c)_{\boldsymbol{\delta}}$. □

Theorem 2.5 is a strengthened form of Gowers's partition theorem:

Theorem (Gowers, [G2]). *Let X be a Banach space with a basis, Φ be an arbitrary subset of $\Sigma^{<\omega}(X)$ and $\boldsymbol{\delta} = (\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$. Then*

either *there exists a block subspace Y_0 of X such that $\Sigma^{<\omega}(Y_0) \subseteq \Phi$;*

or *there exists a block subspace Y_0 of X such that player P has a winning strategy for the game $(\Phi^c)_{\boldsymbol{\delta}}[Y_0]$.*

References

[Bou] J. Bourgain, *On convergent sequences of continuous functions*, Bull Soc. Math. de Belgique **32** (1980), 235–249.

[F1] V. Farmaki, *On Baire-1/4 functions and spreading models*, Mathematika **341**, (1994), 251–265.

[F2] V. Farmaki, *Classifications of Baire-1 functions and c_0-spreading models*, Trans. Amer. Math. Soc. **345** (2), (1994), 819–831.

[F3] V. Farmaki, *Systems of Ramsey families*, Atti Sem. Mat. Fis. Univ. Montena, L, (2002), 363–379.

[G1] W. T. Gowers, *Lipschitz functions on classical spaces*, Europ. J. Combin. **13** (1992), 141–151.

[G2] W. T. Gowers, *An infinite Ramsey theorem and some Banach space dichotomies*, Annals of Mathematics **156** (2002), 797–833.

[M] V. D. Milman, *Spectrum of bounded continuous functions which are given on the unit sphere of a Banach space*, Funct. Anal. Appl. **3** (1969), 67–79.

[NW] C. St. J. A. Nash-Williams, *On well quasiordering transfinite sequences*, Proc. Camb. Phil. Soc. **61** (1965), 33–39.

[OS] E. Odell and T. Schlumprecht, *The distortion problem*, Acta Math. **173** (1994), 259–281.

[R] F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. **30** (2) (1929), 264–286.

[S] J. Schreier, *Ein gegenbeispiel zur theorie der schwachen knovergenz*, Studia Math. **2** (1930), 58–62.

[W] R. Wagner, *Finite high-order games and an inductive approach towards Gowers's dichotomy*, Annals of Pure and Applied Logic **111** (2001), 39–60.

Department of Mathematics, University of Athens, Panepistemiopolis, Athens 157 84, Greece

e-mail address of the author: vfarmaki@math.uoa.gr