A DICHOTOMY PRINCIPLE FOR UNIVERSAL SERIES

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Abstract. Applying results of the infinitary Ramsey theory, namely the dichotomy principle of Galvin-Prikry, we show that for every sequence of scalars $(\alpha_j)_{j=1}^{\infty}$, there exists a subsequence $(\alpha_{k_j})_{j=1}^{\infty}$, such that either every subsequence of $(\alpha_{k_j})_{j=1}^{\infty}$ defines a universal series, or every subsequence of $(\alpha_{k_j})_{j=1}^{\infty}$ does not define a universal series. In particular examples we decide which of the two cases holds.

Introduction

The theory of universal series was initiated by Fekete (1914) (cf. [P]), followed by Menchoff (1945)[M] (on universal trigonometric series) and Seleznev (1951)[Se]. Later, in the results by Luh (1970)[L] and Chui-Parnes (1971)[CP], the approximation by the partial sums of a universal power series holds outside the closure of the domain of definition. Nestoridis (1996)[N] strengthened these results, obtaining approximation on the boundary, as well. There are further results on universal Faber series, Jacobi, Dirichlet and Laurent series, and on harmonic expansions. We refer the reader to the two survey papers by Grosse-Erdmann (1999)[G-E] and Kahane (2000)[K].

An abstract theory of universality is presented in [NP] and [BGNP], according to which the existence of universal series is equivalent to a condition of simultaneous double approximation by a finite linear combination in a family of simple functions forming a vector space. The abstract theory covers most of the previously known cases and leads to simplification of known proofs, since the condition of double approximation follows from various classical approximation theorems (of Mergelyan, Runge, Weierstrass, Walsh etc.). At the same time the abstract approach produces several new cases of universality, such as those defined by means of the normal distribution, or in PDE's.

The theory of Ramsey infinitary combinatorics contains the dichotomy results by Nash-Williams (1965)[N-W] for open partitions, Galvin-Prikry (1973)[GP] for Borel partitions, Silver(1970) [Si] for partitions determined by analytic sets, and the Ellentuck result (1974)[E]. Extensions involving Schreier sets have been given by Farmaki (2004)[F], and

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by Farmaki-Negrepontis (2006)[FN1], (2008)[FN2]. These results, or others of the same type, have found important applications in various branches of mathematics, notably in Banach space theory; we refer the reader to the survey paper by Gowers (2003)[G].

In the present note we apply the Ramsey infinitary combinatorics to the theory of universal series. The meeting point is the fact, shown by an appeal to Baire's category theorem, that under very general conditions, the set of universal series is empty, or equal to a (dense) G_δ set in the suitable space ([G-E], [NP], [BGNP]). It is precisely this fact that makes it possible to employ the Galvin-Prikry dichotomy theorem (stated in Theorem 1.7 below) and prove that every scalar sequence possesses a subsequence, all of whose subsequences are in the universal class $\mathcal{U}$, defined in Section 1 below, or all are in the complement of the class $\mathcal{U}$ in the space of all scalar sequences.

In Section 1 we give definitions of the class $\mathcal{U}$, and of some classes of universal sequences more complicated than $\mathcal{U}$. In Section 2 we prove our main dichotomy result (Theorem 2.1) for these special classes of universal sequences. The fact that these classes are exactly (dense) G_δ subsets of specific Polish spaces, and not only residual, allows for the use of the dichotomy principle of Galvin-Prikry for suitable partitions. Crucial to the proof is the highly non-trivial result, attributed to Lusin and Souslin, mentioned in Theorem 1.6 below, according to which the 1-1 continuous image of a Borel set is Borel. In Section 3 we examine some particular concrete examples of universal series, for which we verify the general dichotomy principle in a direct, elementary way, without recourse to Ramsey theory (and to Theorem 2.1), deciding in addition which horn of the partition actually holds.

There is a number of interesting questions on which we have no answer. It would be desirable to have an effective criterion that allows us to decide which horn of the dichotomy holds in every specific instance, in particular, when hereditary universality actually holds.

1. Preliminaries and notation

We denote by $\mathbb{N} = \{0, 1, 2, ...\}$ the set of natural numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers.

If M is an infinite subset of $\mathbb{N}$, we denote by $[M]$ the set of all infinite subsets of M, considering them as strictly increasing sequences, and if s is a non-empty finite subset of $\mathbb{N}$, we set

$$[s, M] = \{s \cup L \in [\mathbb{N}] : L \in [M] \text{ and } \max s < \min L\} \text{ if } s \neq \emptyset \text{ and } [\emptyset, M] = [M].$$

Universal series. Fix a sequence $(X_k, \rho_k)_{k\geq 1}$ of separable,metrizable topological vector spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, equipped with translation-invariant metrics ρ_k and fix the sequences $(x_n^k)_{n=0}^{\infty} \subseteq X_k$ for every $k \geq 1$.

Definition 1.1. A sequence $(\alpha_n)_{n=0}^{\infty} \in \mathbb{K}^{\mathbb{N}}$ belongs to the class $\mathcal{U}$ if for every $k \geq 1$ the set $\{\sum_{j=0}^{n} \alpha_j x_j^k : n \in \mathbb{N}\}$ is dense in X_k.
If $\mathcal{U} \neq \emptyset$, each $(\alpha_n)_{n=0}^{\infty} \in \mathcal{U}$ is said to generate an unrestricted universal series.

Of special interest is the case where some elements of the class $\mathcal{U}$ satisfy certain restrictions. The restricted universal series are defined as follows:

We fix a vector subspace A of $\mathbb{K}^{\mathbb{N}}$, and assume that A is equipped with a complete metrizable vector space topology, induced by a translation-invariant metric d, such that

(i) the projections $p_m : A \to \mathbb{K}$, $(\alpha_n)_{n=0}^{\infty} \to a_m$ are continuous for every $m \in \mathbb{N}$, and

(ii) the set $c_{00} = \{(\alpha_n)_{n=0}^{\infty} \in \mathbb{K}^{\mathbb{N}} : \{n \in \mathbb{N} : \alpha_n \neq 0\}$ is finite $\}$ is a dense subset of A.

Definition 1.2. A sequence $\alpha = (\alpha_n)_{n=0}^{\infty} \in A$ belongs to the class $\mathcal{U}_A$ if, for every $k \geq 1$ and every $x \in X_k$, there exists a sequence $(\lambda_n)_{n=1}^{\infty} \in \mathbb{N}$ such that

$\rho_k(\sum_{j=0}^{\lambda_n} \alpha_j x_j^k, x) \to 0$ as $n \to \infty$, and

$d(\sum_{j=0}^{\lambda_n} \alpha_j e_j, \alpha) \to 0$ as $n \to \infty$,

where $e_0 = (1, 0, 0, ...)$, $e_1 = (0, 1, 0, ...)$, $e_2 = (0, 0, 1, 0, ...)$,... .

Remark 1.3. In Definition 1.2, we can assume, without loss of generality, that $\lambda_{n+1} < \lambda_n$ for all $n \in \mathbb{N}$.

Definition 1.4. For every $k \geq 1$, let $T_n^k : A \to X_k$, $n \in \mathbb{N}$ be a sequence of continuous functions.

(1) A sequence $\alpha = (\alpha_n)_{n=0}^{\infty} \in A$ belongs to the set $\mathcal{F}_1$ if, for every $k \geq 1$ and every $x \in X_k$, there exists a sequence $(\lambda_n)_{n=1}^{\infty} \in \mathbb{N}$ such that
  $\rho_k(T_{\lambda_n}^k(\alpha), x) \to 0$ as $n \to \infty$.

(2) A sequence $\alpha = (\alpha_n)_{n=0}^{\infty} \in A$ belongs to the set $\mathcal{F}_2$ if, for every $k \geq 1$ and every $x \in X_k$, there exists a sequence $(\lambda_n)_{n=1}^{\infty} \in \mathbb{N}$ such that
  $\rho_k(T_{\lambda_n}^k(\alpha), x) \to 0$ as $n \to \infty$, and
  $d(\sum_{j=0}^{\lambda_n} \alpha_j e_j, \alpha) \to 0$ as $n \to \infty$.

(3) A sequence $\alpha = (\alpha_n)_{n=0}^{\infty} \in A$ belongs to the set $\mathcal{F}_3$ if, for every $k \geq 1$ and every $x \in X_k$, there exists a sequence $(\lambda_n)_{n=1}^{\infty} \in \mathbb{N}$ such that
  $\rho_k(T_{\lambda_n}^k(\alpha), x) \to 0$ as $n \to \infty$,
  $d(\sum_{j=0}^{\lambda_n} \alpha_j e_j, \alpha) \to 0$ as $n \to \infty$, and

$\rho_k(\sum_{j=0}^{\lambda_n} \alpha_j x_j^k, x) \to 0$ as $n \to \infty$.

We refer the reader to [G-E], [NP] and [BGNP] for definitions, results, and interesting examples on the classes $\mathcal{U}$, $\mathcal{U}_A$, and also for the proof of the following Proposition 1.5. A sketch of the proof is included for completeness.

Proposition 1.5. *Under the previous assumptions, the classes $\mathcal{U} \cap A$, $\mathcal{U}_A$, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are G_δ subsets of the space A. In particular for $A = \mathbb{K}^{\mathbb{N}}$, the class $\mathcal{U}$ is a G_δ subset of the space $\mathbb{K}^{\mathbb{N}}$ with the product topology.*

Sketch of the proof. Let $(y_n^k)_{n=0}^{\infty}$ be a dense sequence in X_k for every $k \geq 1$. For $n, l, r, k \in \mathbb{N}$ with $r, k \geq 1$, consider the sets

$E(n, l, r, k) = \{\alpha = (\alpha_n)_{n=0}^{\infty} \in A : \rho_k(\sum_{j=0}^{n} \alpha_j x_j^k, y_l^k) < 1/r\}$,

$D(n, r) = \{\alpha = (\alpha_n)_{n=0}^{\infty} \in A : d(\sum_{j=0}^{n} \alpha_j e_j, \alpha) < 1/r\}$, and

$C(n, l, r, k) = \{\alpha = (\alpha_n)_{n=0}^{\infty} \in A : \rho_k(T_n^k(\alpha), y_l^k) < 1/r\}$.

Under the assumptions for the space A, $E(n, l, r, k), D(n, r), C(n, l, r, k)$ are open subsets of A, and

$\mathcal{U} \cap A = \bigcap_{l,r,k} \bigcup_{n=0}^{\infty} E(n, l, r, k)$,

$\mathcal{U}_A = \bigcap_{l,r,k} \bigcup_{n=0}^{\infty} \big(E(n, l, r, k) \cap D(n, r)\big)$,

$\mathcal{F}_1 = \bigcap_{l,r,k} \bigcup_{n=0}^{\infty} C(n, l, r, k)$,

$\mathcal{F}_2 = \bigcap_{l,r,k} \bigcup_{n=0}^{\infty} \big(C(n, l, r, k) \cap D(n, r)\big)$,

$\mathcal{F}_3 = \bigcap_{l,r,k} \bigcup_{n=0}^{\infty} \big(C(n, l, r, k) \cap D(n, r) \cap E(n, l, r, k)\big)$. □

Borel- Analytic sets. Let X, Y be Polish spaces (i.e. topological Hausdorff spaces, each homeomorphic to a complete, metric, separable space), $f : X \to Y$ a continuous function, and B a Borel subset of X. The image $C = f(B)$ is not always a Borel subset of Y, in fact all such subsets C of Y constitute the class of analytic subsets of Y. Moreover the analytic subsets of Y are characterized as the results of Suslin operation on the class of closed subsets of Y (see [Ke]).

We will use the following highly non-trivial result about Borel sets. A proof can be found in Theorem 15.1 of the Kechris text [Ke].

Theorem 1.6 (Lusin, Souslin)**.** *Let X, Y be Polish spaces and $f : X \to Y$ a continuous function. If B is a Borel subset of X, and f restricted to B is one to one, then $f(B)$ is a Borel subset of Y.*

Infinitary combinatorics. Galvin and Prikry in [GP] proved the following fundamental combinatorial result for infinite sequences of natural numbers.

Theorem 1.7. *Let $\mathcal{R}$ be a family of infinite subsets of the space $[\mathbb{N}]$, endowed with the relative topology in the space $\mathbb{N}^{\mathbb{N}}$ with the product topology. Assume that $\mathcal{R}$ is a Borel subset of $[\mathbb{N}]$, and let s be a finite subset of $\mathbb{N}$ and M an infinite subset of $\mathbb{N}$. Then there exists $L \in [M]$ such that*

either $[s, L] \subseteq \mathcal{R}$,

or $[s, L] \subseteq [\mathbb{N}] \setminus \mathcal{R}$.

Silver in [Si] proved an analogous result in the more general case, where $\mathcal{R}$ is an analytic subset of the space $[\mathbb{N}]$, and Ellentuck in [E] formulated a still more general result.

2. The dichotomy principle

Combining the Galvin and Prikry combinatorial result of Theorem 1.7 with the result of Lusin and Souslin about Borel sets (Theorem 1.6), we can prove a general dichotomy for classes of universal series.

Theorem 2.1. *Let A be a vector subspace of $\mathbb{K}^{\mathbb{N}}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) which is equipped with a complete metrizable vector space topology, induced by a translation-invariant metric d and satisfies properties (i) and (ii) (as given in Section 1) and let $\mathcal{G}$ be a Borel subset of the space A . Then for every sequence $(\alpha_n)_{n=0}^{\infty}$ in $\mathbb{K}$, every finite subset s of $\mathbb{N}$ and every infinite subset M of $\mathbb{N}$ there exists an infinite subset L of M such that*

either all the subsequences $(\alpha_{i_n})_{n=0}^{\infty}$ of $(\alpha_n)_{n=0}^{\infty}$ with $(i_n)_{n=1}^{\infty} \in [s, L]$ belong to $\mathcal{G}$,

or all the subsequences $(\alpha_{i_n})_{n=0}^{\infty}$ of $(\alpha_n)_{n=0}^{\infty}$ with $(i_n)_{n=0}^{\infty} \in [s, L]$ belong to $\mathbb{K}^{\mathbb{N}} \setminus \mathcal{G}$.

In particular, the conclusion holds if we replace the class $\mathcal{G}$ by each of the classes $\mathcal{U} \cap A$, $\mathcal{U}_A$, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ (defined in Section 1). For $A = \mathbb{K}^{\mathbb{N}}$ the conclusion holds for $\mathcal{G} = \mathcal{U} = \mathcal{U} \cap A$ as well.

Proof. We set $\mathcal{R} = \{I = (i_n)_{n=0}^{\infty} \in [\mathbb{N}] : (\alpha_{i_n})_{n=0}^{\infty} \in \mathcal{G}\}$, and we claim that the family $\mathcal{R}$ is a Borel subset of the space $[\mathbb{N}]$, endowed with the relative product topology of $\mathbb{N}^{\mathbb{N}}$.

Indeed, the function $f : [\mathbb{N}] \to \mathbb{K}^{\mathbb{N}}$ with $f((i_n)_{n=0}^{\infty}) = (\alpha_{i_n})_{n=0}^{\infty}$ is continuous if $[\mathbb{N}]$ is endowed with the relative product topology of $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{K}^{\mathbb{N}}$ with the product topology. Notice that $\mathcal{R} = f^{-1}(\mathcal{G})$.

The set $\mathcal{G}$ is a Borel subset of $\mathbb{K}^{\mathbb{N}}$ with the product topology. Indeed, the identity function $g : A \to \mathbb{K}^{\mathbb{N}}$ with $g(\alpha) = \alpha$ is continuous if $\mathbb{K}^{\mathbb{N}}$ is endowed with the product topology, since the space A satisfies property (i), according to which the projections $p_m : A \to \mathbb{K}$, $(\alpha_n)_{n=0}^{\infty} \to a_m$ are continuous for every $m \in \mathbb{N}$. Certainly $\mathbb{K}^{\mathbb{N}}$ with the product topology is a Polish space, and also A is a Polish space, since it is a complete metrizable space and satisfies property (ii). Hence the set $\mathcal{G} = g(\mathcal{G})$ is a Borel subset of $\mathbb{K}^{\mathbb{N}}$, according to the Lusin-Souslin Theorem 1.6.

Since the function $f : [\mathbb{N}] \to \mathbb{K}^{\mathbb{N}}$ is continuous, the set $\mathcal{R} = f^{-1}(\mathcal{G})$ is a Borel subset of $[\mathbb{N}]$ with the relative product topology of $\mathbb{N}^{\mathbb{N}}$.

Now, we can apply the Galvin-Prikry Theorem 1.7 for the family $\mathcal{R}$. It follows that, there exists $L \in [M]$ such that

either $[s, L] \subseteq \mathcal{R}$,

or $[s, L] \subseteq [\mathbb{N}] \setminus \mathcal{R}$.

Hence, all the subsequences $(\alpha_{i_n})_{n=0}^{\infty}$ of $(\alpha_n)_{n=0}^{\infty}$ with $(i_n)_{n=0}^{\infty} \in [s, L]$ either belong to the class $\mathcal{G}$, in case $[s, L] \subseteq \mathcal{R}$ or belong to the class $\mathbb{K}^{\mathbb{N}} \setminus \mathcal{G}$, in case $[s, L] \subseteq [\mathbb{N}] \setminus \mathcal{R}$.

In particular, using, instead of the subclass $\mathcal{G}$ of A, each of the classes $\mathcal{U} \cap A$, $\mathcal{U}_A$, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$, since all these classes, according to Proposition 1.5 are G_δ subsets of the space A, we have the conclusion for each of these classes. In particular, applying the arguments for $\mathcal{G} = \mathcal{U}$ and $A = \mathbb{K}^{\mathbb{N}}$ with the product topology, we have the conclusion for the class $\mathcal{U}$. □

Corollary 2.2. *Let $(X_k)_{k\geq 1}$ be a sequence of separable,metrizable topological vector spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, equipped with translation-invariant metrics ρ_k, for every $k \geq 1$, let A be a vector subspace of $\mathbb{K}^{\mathbb{N}}$, equipped with a complete metrizable vector space topology, induced by a translation-invariant metric d, and satisfying the properties (i) and (ii) (as given in Section 1), and let sequences $(x_n^k)_{n=0}^{\infty}$ be in X_k for every $k \geq 1$. Then, for every sequence $(\alpha_n)_{n=0}^{\infty}$ in $\mathbb{K}$ and $\mathcal{G}$ equal to one of the classes $\mathcal{U} \cap A$, $\mathcal{U}_A$, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$, there exists a subsequence $(\alpha_{l_n})_{n=0}^{\infty}$ of $(\alpha_n)_{n=0}^{\infty}$, such that*

either all the subsequences of $(\alpha_{l_n})_{n=0}^{\infty}$ belong to $\mathcal{G}$,

or all the subsequences of $(\alpha_{l_n})_{n=0}^{\infty}$ belong to $\mathbb{K}^{\mathbb{N}} \setminus \mathcal{G}$.

Proof. We apply Theorem 2.1 in case $s = \emptyset$. □

Theorem 2.1 can be stated in a more general form, assuming the partition family to be analytic instead of Borel.

Theorem 2.3. *Let A be a vector subspace of $\mathbb{K}^{\mathbb{N}}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), equipped with a complete metrizable vector space topology, induced by a translation-invariant metric d and satisfying the properties (i) and (ii) (as given in Section 1), and let $\mathcal{D}$ be an analytic subset of the space A . Then for every sequence $(\alpha_n)_{n=0}^{\infty}$ in $\mathbb{K}$, every finite subset s of $\mathbb{N}$, and every infinite subset M of $\mathbb{N}$, there exists an infinite subset L of M such that*

either all the subsequences $(\alpha_{i_n})_{n=0}^{\infty}$ of $(\alpha_n)_{n=0}^{\infty}$ with $(i_n)_{n=0}^{\infty} \in [s, L]$ belong to $\mathcal{D}$, *or all the subsequences* $(\alpha_{i_n})_{n=0}^{\infty}$ *of* $(\alpha_n)_{n=0}^{\infty}$ *with* $(i_n)_{n=0}^{\infty} \in [s, L]$ *belong to* $\mathbb{K}^{\mathbb{N}} \setminus \mathcal{D}$.

Proof. We note that the family $\mathcal{R} = \{I = (i_n)_{n=0}^{\infty} \in [\mathbb{N}] : (\alpha_{i_n})_{n=0}^{\infty} \in \mathcal{D}\}$ is an analytic subset of the space $[\mathbb{N}]$ endowed with the relative product topology of $\mathbb{N}^{\mathbb{N}}$.

Indeed, the set $\mathcal{D} = g(\mathcal{D})$ is an analytic subset of $\mathbb{K}^{\mathbb{N}}$ with the product topology, as the identity function $g : A \to \mathbb{K}^{\mathbb{N}}$ is continuous and $\mathcal{D}$ is an analytic subset of the Polish space A. Since the function $f : [\mathbb{N}] \to \mathbb{K}^{\mathbb{N}}$ with $f((i_n)_{n=0}^{\infty}) = (\alpha_{i_n})_{n=0}^{\infty}$ is continuous if $[\mathbb{N}]$ is endowed with the relative product topology of $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{K}^{\mathbb{N}}$ with the product topology, we have that $\mathcal{R} = f^{-1}(\mathcal{D})$ is an analytic subset of the space $[\mathbb{N}]$. This last conclusion can be proved easily by using the characterization of analytic subsets of a Polish space X, as those that result from the Souslin operation on the class of closed subsets of X.

Now, we apply the result by Silver in [Si], or by Ellentuck in [E] (in place of the Galvin-Prikry Theorem 1.7, replacing the Borel partition by a partition determined by an analytic set). So, there exists $L \in [M]$ such that

either $[s, L] \subseteq \mathcal{D}$,

or $[s, L] \subseteq [\mathbb{N}] \setminus \mathcal{D}$.

This finishes the proof. □

Remark 2.4. At present, we have no use for the more general results in Theorem 2.1, on partitions determined by an analytic set, but it appears that the full force of the Galvin-Prikry theorem, for Borel partitions is employed, in the proof of our main result 2.1. If this is indeed the case, then Theorem 2.1 is the only "natural theorem" that uses the full strength of the Galvin-Prikry partition theorem. (Cf. the relevant remark after Theorem 5.7 in the Gowers survey paper [G]).

Remark 2.5. For a fixed strictly increacing sequence $(\mu = (\mu_n)_{n=1}^{\infty}$ of $\mathbb{N}$, we can define the classes $\mathcal{U}^{\mu}$, $\mathcal{U}^{\mu} \cap A$, $\mathcal{U}^{\mu}_A$, $\mathcal{F}^{\mu}_1$, $\mathcal{F}^{\mu}_2$, $\mathcal{F}^{\mu}_3$ analogously to the classes $\mathcal{U}$, $\mathcal{U} \cap A$, $\mathcal{U}_A$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ defined in Section 1, with the only difference that the sequence $(\lambda_n)_{n=1}^{\infty} \in \mathbb{N}$ in Definition 1.2 and Definition 1.4 to be a subsequence of μ. All the results which we proved for the classes $\mathcal{U}$, $\mathcal{U} \cap A$, $\mathcal{U}_A$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ also hold for the classes $\mathcal{U}^{\mu}$, $\mathcal{U}^{\mu} \cap A$, $\mathcal{U}^{\mu}_A$, $\mathcal{F}^{\mu}_1$, $\mathcal{F}^{\mu}_2$ and $\mathcal{F}^{\mu}_3$.

3. Particular cases and examples

In this Section we examine some particulal cases of universality in relation with Corollary 2.2, in addition concrete examples are used to show that the classes $\mathcal{U}_A$ and $\mathbb{K}^{\mathbb{N}} \setminus \mathcal{U}_A$, as well as $\mathcal{U} \cap A$ and $\mathbb{K}^{\mathbb{N}} \setminus (\mathcal{U} \cap A)$, are not always hereditary (where a class $\mathcal{F}$ in $\mathbb{K}^{\mathbb{N}}$ is hereditary if every subsequence of any sequence in $\mathcal{F}$ belongs to $\mathcal{F}$).

Fix a separable Banach space X and a sequence $(x_n)_{n=0}^{\infty}$ in X. We consider the particular case of Definition 1.1, where $X_k = X$ and $x_n^k = x_n$ for all $k = 1, 2, \ldots$. In this particular case we prove the following.

Proposition 3.1. *Let X be a Banach space, and $(x_j)_{j=0}^{\infty} \subseteq X$, with $x_j \neq 0$ for every $j \in \mathbb{N}$. Set $A = c_0$ or $A = l^p$, with $0 < p < \infty$, and let $(\alpha_n)_{n=0}^{\infty} \in A$. Then there exists a subsequence $(\alpha_{l_n})_{n=0}^{\infty}$ of $(\alpha_n)_{n=0}^{\infty}$, all of whose subsequences do not belong to the class $\mathcal{U} \cap A = \mathcal{U}_A$.*

Proof. Let $(\alpha_{l_n})_{n=0}^{\infty}$ be a subsequence of $(\alpha_n)_{n=0}^{\infty}$ with $|\alpha_{l_n}| \leq \frac{1}{(n+1)^2.\rho(x_j,0)}$ for every $n \in \mathbb{N}$ and $j = 0, ..., n$. For every subsequence $(\alpha_{k_{l_n}})_{n=0}^{\infty}$ of $(\alpha_{l_n})_{n=0}^{\infty}$ we have that $k_{l_n} \geq n$ for every $n \in \mathbb{N}$, and consequently that $\|\alpha_{k_{l_n}} x_n\| \leq \frac{1}{(l_n+1)^2}$, which implies that the series $\sum_{n=0}^{+\infty} \alpha_{k_{l_n}} x_n$ converges in X. Therefore $(\alpha_{k_{l_n}})_{n=0}^{\infty}$ does not belong to $\mathcal{U} \cap A = \mathcal{U}_A$. □

Proposition 3.2. *Let $1 < R < +\infty$, set $D(0, R) = \{z \in \mathbb{C} : |z| < R\}$, $A = \{(\alpha_n)_{n=0}^{\infty} : \sum_{n=0}^{+\infty} \alpha_n.z^n$ converges in $D(0, R)\}$, and endow A with the metric d which is the image, under the bijective map $g : A \to H(D(0, R))$ with $g((\alpha_n)_{n=0}^{\infty}) = \sum_{n=0}^{+\infty} \alpha_n.z^n$, of the standard metric $\tilde{d}$ on $H(D(0, R))$, a metric compatible with the topology of the uniform convergence on compact subsets of $D(0, R)$. We also assume (cf.* [N], [NP] *and* [BGNP]*) that there exists a sequence $(X_k, \rho_k)_{k\geq 1}$ of separable, metrizable topological vector spaces over $\mathbb{C}$, sequences $(x_n^k)_{n=0}^{\infty} \subseteq X_k$ for every $k \geq 1$, such that the class $\mathcal{U} \cap A = \mathcal{U}_A$ is the class of universal Taylor series , namely the class of all sequences $(\alpha_n)_{n=0}^{\infty} \in A$, for which for every compact set $K \subseteq \mathbb{C}$, $K \cap D(0, R) = \emptyset$, with $\mathbb{C} \setminus K$ connected, and for every function $h : K \to \mathbb{C}$, continuous on K and holomorphic in the interior of K, there exists a sequence $(\lambda_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ with $\sum_{j=0}^{\lambda_n} \alpha_j z^j \xrightarrow{n} h(z)$ uniformly on K. Then, for every sequence $(\alpha_n)_{n=0}^{\infty} \in A$, there exists a subsequence $(\alpha_{l_n})_{n=0}^{\infty}$ of $(\alpha_n)_{n=0}^{\infty}$, all of whose subsequences do not belong to the class $\mathcal{U} \cap A = \mathcal{U}_A$.*

Proof. Set $K = \{R\}$. We can have $X_1 = \mathbb{C} = \{f : K \to \mathbb{C}\}$, ρ_1 the usual metric on $\mathbb{C}$ and $x_j^1 = z^j\big|_K$ for every $j \in \mathbb{N}$. Since $(\alpha_n)_{n=0}^{\infty} \in A$ and $1 < R$, it is easily seen that $\alpha_n \to 0$, hence we can choose $(l_n)_{n=0}^{\infty} \in \mathbb{N}$ with $|\alpha_{l_n}| \leq \frac{1}{(n+1)^2.R^j} = \frac{1}{(n+1)^2.\rho(x_j^1,0)}$ for every $n \in \mathbb{N}$ and $j = 0, ..., n$. The rest of the proof is similar to that of Proposition 3.1. □

Proposition 3.3. *Let X be a separable Banach space and $(x_j)_{j=0}^{\infty} \subseteq X$, with $x_j \neq 0$ for every $j \in \mathbb{N}$. Then for every sequence $(\alpha_n)_{n=0}^{\infty} \in \mathbb{K}^{\mathbb{N}}$, we have the following cases:*

(a) *if* $(\alpha_n)_{n=0}^{\infty}$ *has zero or infinity as accumulation points, then it has a subsequence, all of whose subsequences do not belong to* $\mathcal{U}$*;*

(b) *if* $(\alpha_n)_{n=0}^{\infty}$ *has a subsequence converging to some* $c \in \mathbb{K}$ *with* $c \neq 0$*, and* $(1,1,...) \in \mathcal{U}$*, then it has a subsequence, all of whose subsequences belong to* $\mathcal{U}$*; and,*

(c) *if* $(\alpha_n)_{n=0}^{\infty}$ *has a subsequence converging to some* $c \in \mathbb{K}$ *with* $c \neq 0$*, and* $(1,1,...) \notin \mathcal{U}$*, then it has a subsequence, all of whose subsequences do not belong to* $\mathcal{U}$*.*

Proof. If the sequence $(\alpha_n)_{n=0}^{\infty}$ has a subsequence converging to zero, then, as in the proof of Proposition 3.1, we can construct a subsequence $(\alpha_{l_n})_{n=0}^{\infty}$ of $(\alpha_n)_{n=1}^{\infty}$ whose all subsequences do not belong to the class $\mathcal{U}$.

Let $(\alpha_n)_{n=0}^{\infty}$ have a subsequence converging to some $c \in \mathbb{K}$ with $c \neq 0$. Then we can find a subsequence $(\alpha_{l_n})_{n=0}^{\infty}$ of $(\alpha_n)_{n=0}^{\infty}$ with $|\alpha_{l_n} - c| \leq \frac{1}{(n+1)^2 \cdot \rho(x_j,0)}$ for every $n \in \mathbb{N}$ and $j = 0, ..., n$. Hence, for every subsequence $(\alpha_{k_{l_n}})_{n=0}^{\infty}$ of $(\alpha_{l_n})_{n=0}^{\infty}$ the series $\sum_{n=0}^{+\infty}(\alpha_{k_{l_n}} - c)x_n$ converges in X. Therefore, in case $(c,c,...) \in \mathcal{U}$ all the subsequences of $(\alpha_{l_n})_{n=0}^{\infty}$ belong to $\mathcal{U}$ and in case $(c,c,...) \notin \mathcal{U}$ all the subsequences of $(\alpha_{l_n})_{n=0}^{\infty}$ do not belong to $\mathcal{U}$. But $(c,c,...) \in \mathcal{U}$ if and only if $(1,1,...) \in \mathcal{U}$.

Finally, it remains to examine the case where $(\alpha_n)_{n=0}^{\infty}$ has a subsequence converging to infinity. Then we can find a subsequence $(\alpha_{l_n})_{n=0}^{\infty}$ of $(\alpha_n)_{n=0}^{\infty}$ such that for all $n \in \{1,2,...\}$ and for all $\{F \subseteq \mathbb{N} : F \subseteq \{0,...,n-1\}\}$ to have $\|(\sum_{j\in F} \alpha_{l_j} x_j) + \alpha_{l_n} x_n\| \geq 1$, by choosing $l_n \in \mathbb{N}$ with $\|\alpha_{l_n} x_n\| \geq 1 + \|\sum_{j\in F} \alpha_{l_j} x_j\|$ for every $\{F \subseteq \mathbb{N} : F \subseteq \{0,...,n-1\}\}$. Hence, for every subsequence $(\alpha_{k_{l_n}})_{n=0}^{\infty}$ of $(\alpha_{l_n})_{n=0}^{\infty}$ the set $\{\sum_{n=0}^{N} \alpha_{k_{l_n}} x_n : N \in \mathbb{N}\}$ avoids the open set $\{x \in X : \|x\| < 1\}$, so it is not dense in X. Thus $(\alpha_{k_{l_n}})_{n=0}^{\infty} \notin \mathcal{U}$.

This completes the proof. □

The following theorem extends Corollary 2.2 for the class $\mathcal{U} \cap A$ in the particular case where A is a hereditary family, thus providing a criterion on whether all the subsequences are in $\mathcal{U} \cap A$ or in its complement.

Theorem 3.4. *Let* X *be a separable Banach space,* $(x_j)_{j=0}^{\infty} \subseteq X$*, with* $x_j \neq 0$ *for every* $j \in \mathbb{N}$*, and* A *a hereditary vector subspace of* $\mathbb{K}^{\mathbb{N}}$ *(with* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*), equipped with a complete metrizable vector space topology, induced by a translation-invariant metric* d *and satisfying properties (i) and (ii) (as given in Section 1). Then for every sequence* $(\alpha_n)_{n=0}^{\infty}$ *in* $\mathbb{K}$ *we have the following cases:*

(a) *if* $(1,1,...) \in \mathcal{U}$*, and* $(\alpha_n)_{n=0}^{\infty}$ *has a subsequence, included in* A*, and converging to some* $c \in \mathbb{K}$*, with* $c \neq 0$*, then* $(\alpha_n)_{n=0}^{\infty}$ *has a subsequence, all of whose subsequences belong to the class* $\mathcal{U} \cap A$*; and*

(b) *if either* $(1,1,...) \notin \mathcal{U}$*, or* ($(1,1,...) \in \mathcal{U}$ *and*) $(\alpha_n)_{n=0}^{\infty}$ *has no subsequence in* A *converging to some* $c \in \mathbb{K}$*, with* $c \neq 0$*, then* $(\alpha_n)_{n=0}^{\infty}$ *has a subsequence, all of whose subsequences do not belong to the class* $\mathcal{U} \cap A$.

Moreover, if $(\alpha_n)_{n=0}^{\infty}$ *has a subsequence converging to zero or to infinity, then* $(\alpha_n)_{n=0}^{\infty}$ *has a subsequence, all of whose subsequences do not belong to the class* $\mathcal{U} \cap A$.

Proof. If $(1,1,...) \notin \mathcal{U}$, then, according to Proposition 3.3, $(\alpha_n)_{n=0}^{\infty}$ has a subsequence, all of whose subsequences do not belong to the class $\mathcal{U}$ and therefore do not belong to $\mathcal{U} \cap A$.

Assume $(1,1,...) \in \mathcal{U}$. If $(\alpha_n)_{n=0}^{\infty}$ has a subsequence converging to zero or to infinity, then, according to Proposition 3.3, $(\alpha_n)_{n=0}^{\infty}$ has a subsequence whose all subsequences do not belong to the class $\mathcal{U}$ and therefore to $\mathcal{U} \cap A$.

If $(1,1,...) \in \mathcal{U}$ and $(\alpha_n)_{n=0}^{\infty}$ has all its accumulation points in $\mathbb{K} \setminus \{0\}$, then, according to Proposition 3.3, it has a subsequence $(\alpha_{l_n})_{n=0}^{\infty}$ whose all subsequences belongs to the class $\mathcal{U}$. If one subsequence $(\alpha_{k_{l_n}})_{n=0}^{\infty}$ belong to the class A, since A is hereditary, all its subsequences belong to $\mathcal{U} \cap A$. The remaining case is when all the subsequences of $(\alpha_{l_n})_{n=0}^{\infty}$ do not belong to A. Then all the subsequences of $(\alpha_{l_n})_{n=0}^{\infty}$ do not belong to $\mathcal{U} \cap A$.

This completes the proof. □

Remark 3.5. (1) In order to give an example of a sequence belonging to the class $\mathcal{U} \cap A$ with a subsequence not belonging to $\mathcal{U}$, we apply Proposition 3.1 for $A = c_0$ and we notice that $\mathcal{U} \cap c_0 \neq \emptyset$ in the case of universal trigonometric series in the sense of Menchoff ([KN], [BGNP]), as well as in the case of universal Taylor series in the open unit disk in the sense of Luh ([L]) and Chui-Parnes ([CP]),where the universal approximation is not required on the boundary ([KKN], [MN], [BGNP]).

For another example, start with a sequence $(x_n)_{n=0}^{\infty}$ in a Banach space X such that the set $\{\sum_{n=0}^{N} x_{2n}) : N \in \mathbb{N}\}$ is dense in X. Then the sequence $(1,0,1,0,...)$ belongs to $\mathcal{U}$ and obviously its subsequence $(0,0,0,0,...)$ does not belong to $\mathcal{U}$.

(2) In order to give an example of a sequence not belonging to the class $\mathcal{U}$ with a subsequence belonging to $\mathcal{U}$, we start with a sequence $(x_n)_{n=0}^{\infty}$ dense in a Banach space X and we consider the sequence $(y_n)_{n=0}^{\infty} \in X$ where $y_0 = x_0$ and $y_{2n-1} = y_{2n} = x_n$ for every $n \geq 1$, which is also dense in X. Fix $(z_n)_{n=0}^{\infty} \in X$, where $z_0 = y_0 = x_0$ and $z_n = y_n - y_{n-1}$ for every $n \geq 1$. Then $z_{2n} = 0$ and $z_{2n-1} = y_n - y_{n-1}$ for every $n \geq 1$.

The sequence $(\alpha_n)_{n=0}^{\infty} \subseteq \mathbb{R}$ with $\alpha_{2n+1} = 0$ and $\alpha_{2n} = 1$ for every $n \in \mathbb{N}$ does not belong to $\mathcal{U}$. Indeed, for every $N \in \mathbb{N}$ we have that $\sum_{n=0}^{N} \alpha_n z_n = y_0$. On the other hand, the subsequence $(\alpha_{2n})_{n=0}^{\infty}$ of $(\alpha_n)_{n=0}^{\infty}$ belongs to $\mathcal{U}$, since $\sum_{n=0}^{2N} \alpha_{2n} z_n = \sum_{n=0}^{2N} z_n = y_N$ for every $N \in \mathbb{N}$ and $(y_N)_{N=0}^{\infty} \subseteq X$ is dense in X.

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