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The Uniform Convergence Ordinal Index and the l^1-Behavior of a Sequence of Functions

VASSILIKI FARMAKI
Department of Mathematics, University of Athens, Panepistimiopolis, 6-15784 Athens, Greece.
E-mail: vfarmaki@math.uoa.gr

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Abstract. In this paper we introduce and study two indices of a uniformly bounded sequence (f_n) of real valued functions defined on a set Γ and converging pointwise to a function f. The first index $\xi^{(f_n)}$ measures uniform convergence of (f_n), while the second index $\xi_+^{(f_n-f)}$ measures the relation of the sequence (f_n-f) to the positive face of the usual basis of ℓ^1. There is a close connection between these two indices, indicated by:

(a) $\xi^{(f_n)}<\omega_1 \Leftrightarrow \xi_+^{(f_n-f)}<\omega_1$; and
(b) if $\xi^{(f_n)}<\omega_1$ then $\xi_+^{(f_n-f)}=\omega^\zeta$ where ζ is the least ordinal with $\xi^{(f_n)}\leqslant\omega^\zeta$.

Using this connection the following dichotomies hold:

either [**Case** $\xi^{(f_n)}=\omega_1$](f_n-f) has an l^1_+-subsequence;
or [**Case** $\xi^{(f_n)}<\omega_1$](f_n) converges weakly to f in the Banach space $l^\infty(\Gamma)$. Fixing the least countable ordinal ζ with $\xi^{(f_n)}\leqslant\omega^\zeta$ we obtain for every countable ordinal α the further dichotomy:

either [**Case** $\alpha<\zeta$] there exist a subsequence of (f_n-f) with l^1-spreading model of order α;
or [**Case** $\alpha\geqslant\zeta$] the sequence (f_n) converges ω^α-uniformly to f; equivalently every subsequence of (f_n) has an $\mathcal{A}_{\omega^\alpha}$-convex block subsequence converging uniformly to f.

$((\mathcal{A}_\xi)_{1\leqslant\xi<\omega_1}$ is the complete thin Schreier system introduced previously by the author).
There are applications of these results to Banach space theory.

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Introduction

In this paper we study uniformly bounded sequences (f_n) of real valued functions defined on a set Γ (with no topological structure) and converging pointwise to a function f. The classical modes of convergence (uniform or pointwise) and of the corresponding Banach-space behavior (such as l^1-embedding) of such sequences with the aid of suitable ordinal indices, take on, in this paper, their position as extreme states (the ordinal 1 for uniform, the first uncountable ordinal ω_1 for pointwise) between which there is a whole spectrum of intermediate states, precisely quantified and characterized by the intermediate countable ordinals.

The main tools in our proofs, combinatorial in nature, consist of the Ramsey-type principle for every countable ordinal and the Pták-type theorem for every countable ordinal proved in [7].

First (in Section 2 below) we study an ordinal index, denoted $\xi^{(f_n)}$, measuring the degree of uniform convergence of (f_n). It may assume as its value, every countable ordinal and also ω_1. However, if we consider the sequence (f_n) as a sequence in the Banach space $l^\infty(\Gamma)$ (equivalently, as a sequence of functions defined on the closed unit ball of the dual space $(l^\infty(\Gamma))^*$), then the index can only be equal to ω_1 or to a (limit) countable ordinal of the form ω^ζ for some $\zeta < \omega_1$ (Theorem 2.10). This is a set-theoretic differentiation between Banach space theory and topology. The nearer the sequence index is to the value 1, the more its convergence resembles uniform convergence, while the nearer it is to ω_1 the more its convergence resembles pointwise convergence. For the definition of this index we employ the complete, thin Schreier system $(\mathcal{A}_\xi)_{1\leqslant\xi<\omega_1}$, introduced previously by the author in [7].

Next (in Section 3) we study the ℓ^1_+-index $\xi_+^{(f_n-f)}$ of the sequence (f_n-f) (and $\xi_+^{(\chi_n)}$ for a bounded sequence (χ_n) in a Banach space). The index may assume as its value, only ordinals of the form ω^ζ for some countable ordinal ζ and ω_1 (Proposition 3.6) and measures the relation of the sequence (say (χ_n) in a Banach space) with the positive face of the usual basis of l^1. It is countable if and only if the sequence (f_n-f) is weakly null in $l^\infty(\Gamma)$.

The main results can be summarized as follows:

THEOREM 1. *Let (f_n) be a uniformly bounded sequence of real valued functions on a set Γ which converges pointwise to a function f. Then,*

(a) $\xi^{(f_n)} < \omega_1$ if and only if $\xi_+^{(f_n-f)} < \omega_1$.
(b) If $\xi^{(f_n)} < \omega_1$, then $\xi_+^{(f_n-f)} = \omega^\zeta$, where ζ is the least ordinal with $\xi^{(f_n)} \leqslant \omega^\zeta$ (Theorem 3.11).

Using the close connection of these two indices we state (as Theorem 3.14) the following result, which can be considered as a natural extension of Rosenthal's classical l^1-Theorem from the ω_1-ordinal to the countable ordinals.

THEOREM. *Let (f_n) be a uniformly bounded sequence of real valued functions on a set Γ which converges pointwise to a function f. Then, either*
(1) [**Case** $\xi^{(f_n)} = \omega_1$] *(f_n-f) has an l^1_+ subsequence (Theorem 2.5);*
or
(2) [**Case** $\xi^{(f_n)} < \omega_1$] *(f_n) converges weakly to f (in the Banach space $l^\infty(\Gamma)$).*
In case (2), let $\xi_+^{(f_n-f)} = \omega^\zeta$. For each countable ordinal α we obtain the further dichotomy:

either (2i) [**Case** $\alpha < \zeta$] *there exists a subsequence of* $(f_n - f)$ *with* l^1*-spreading model of order* α *(Proposition 3.10);*
or (2ii) [**Case** $\zeta \leqslant \alpha$] *the sequence* (f_n) *converges* ω^α*-uniformly to* f*, or equivalently every subsequence of* (f_n) *has a concrete* $\mathcal{A}_{\omega^\alpha}$*-convex block subsequence converging uniformly to* f.

Applying the previous results to the general Banach space theory we state (in Theorem 3.15) a corresponding result for the case of a weak-Cauchy sequence in a Banach space.

This second theorem provides dichotomies for every countable order α as well as a limiting dichotomy for the ω_1-case. The existence of analogous dichotomies have been proved, using different approaches, for the case $\alpha = 1$ in [17] and [13] and for the case of a successor (non limit) countable ordinal in [3]. Additionally the previous theorem yields an effective criterion for deciding which of the two legs of the dichotomy actually occurs, using the l^1_+-index. A method of calculating the l^1_+-index of sequences in certain Banach spaces is given in [9] and [12].

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NOTATION. We denote by $\mathbb{N}$ the set of all natural numbers and by $\mathbb{R}$ the set of all real numbers. For an infinite subset M of $\mathbb{N}$ we denote by $[M]^{<\omega}$ the set of all finite subsets of M and by [M] the set of all infinite subsets of M (considering them as strictly increasing sequences).

If H, F are finite subsets of $\mathbb{N}$ then we write $H \leqslant F$ if $\max H \leqslant \min F$, while $H < F$ if $\max H < \min F$. By $|H|$ we denote the cardinality of H.

Identifying every subset of $\mathbb{N}$ with its characteristic function, we topologize the set of all subsets of $\mathbb{N}$ by the topology of pointwise convergence. For a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ and $M \in [\mathbb{N}]$ we write:

(i) $\mathcal{F}(M) = \{(m_{n_1}, \dots, m_{n_k}) \in [M]^{<\omega} : (n_1, \dots, n_k) \in \mathcal{F}\}$.

(ii) $\mathcal{F}_* = \{H \in [\mathbb{N}]^{<\omega} : H \subseteq F$ for some $F \in \mathcal{F}\}$. $\mathcal{F}$ is **hereditary** if $\mathcal{F}_* = \mathcal{F}$.

(iii) $\mathcal{F}^* = \{H \in [\mathbb{N}]^{<\omega} : H$ is an initial segment of some $F \in \mathcal{F}\}$. $\mathcal{F}$ is **thin** if it contains no proper initial segment of any of its elements.

1. The basic combinatorial tools

The main tools in our proofs, combinatorial in nature, consist of the Ramsey-type principle for every countable order and the Prák-type theorem for every countable order. These results have been proved in [7]. We recall them here for completeness.

DEFINITION 1.1 ([7]). (**The complete thin Schreier system** $(\mathcal{A}_\xi)_{\xi<\omega_1}$). For every non zero, limit ordinal λ we choose and fix a strictly increasing sequence (λ_n) of successor ordinals smaller than λ with $\sup_n \lambda_n = \lambda$.

We define the system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ recursively as follows:

(1) [**Case** $\xi=0$]

$$\mathcal{A}_0=\{\emptyset\};$$

(2) [**Case** $\xi=\zeta+1$]

$$\mathcal{A}_\xi=\mathcal{A}_{\zeta+1}=\{s\subset\mathbb{N}: s=\{n\}\cup s_1,\ \text{where } n\in\mathbb{N}, \{n\}<s_1 \text{and} s_1\in\mathcal{A}_\zeta\};$$

(3) [**Case** $\xi=\omega^{\beta+1}$, β countable ordinal]

$$\mathcal{A}_\xi=\mathcal{A}_{\omega^{\beta+1}}=\left\{s\subset\mathbb{N}: s=\bigcup_{i=1}^{n} s_i \text{ with } n=\min s_1, s_1<\cdots<s_n,\right.$$

$$\left.\text{and } s_1,\dots,s_n\in\mathcal{A}_{\omega^\beta}\right\};$$

(4) [**Case** $\xi=\omega^\lambda$, λ non-zero, countable limit ordinal]

$$\mathcal{A}_\xi=\mathcal{A}_{\omega^\lambda}=\{s\subseteq\mathbb{N}: s\in\mathcal{A}_{\omega^{\lambda_n}} \text{ with } n=\min s\},$$

(where (λ_n) is the sequence of ordinals, converging to λ, fixed above); and

(5) [**Case** ξ limit, $\omega^\alpha<\xi<\omega^{\alpha+1}$ for some $0<\alpha<\omega_1$]

Let $\xi=p\omega^\alpha+\sum_{i=1}^m p_i\omega^{\alpha_i}$ be the canonical representation of ξ, where $m\geqslant 0$, $p,p_1,\dots,p_m\geqslant 1$ are natural numbers so that either $p>1$ or $p=1$ and $m\geqslant 1$ and $\alpha>\alpha_1>\cdots>\alpha_m>0$ are countable ordinals. Then

$$\mathcal{A}_\xi=\left\{s\subseteq\mathbb{N}: s=s_0\cup\left(\bigcup_{i=1}^{m} s_i\right) \text{ with } s_m<\cdots<s_1<s_0,\right.$$

$$s_0=s_1^0\cup\cdots\cup s_p^0 \text{ with } s_1^0<\cdots<s_p^0, s_j^0\in\mathcal{A}_{\omega^\alpha}, 1\leqslant j\leqslant p,$$

$$s_i=s_1^i\cup\cdots\cup s_{p_i}^i, \text{ with } s_1^i<\cdots<s_{p_i}^i,$$

$$\left.s_j^i\in\mathcal{A}_{\omega^{\alpha_i}}, 1\leqslant i\leqslant m, 1\leqslant j\leqslant p_i\right\}.$$

We set $\mathcal{B}_\alpha=\mathcal{A}_{\omega^\alpha}$ for each $1\leqslant\alpha<\omega_1$.

REMARK 1.2. (i) Each family $\mathcal{A}_\xi$ for $1\leqslant\xi<\omega_1$ is **thin** (does not contain proper initial segments of its elements).

(ii) ([7]) Each finite subset F of $\mathbb{N}$ has a **canonical representation** with respect to the family $\mathcal{A}_\xi$. This means that for every $1\leqslant\xi<\omega_1$ there exist unique $n\in\mathbb{N}$, sets $s_1,\dots,s_n\in\mathcal{A}_\xi$ and s_{n+1}, a proper initial segment of some element of $\mathcal{A}_\xi$, with $s_1<\cdots<s_n<s_{n+1}$, such that $F=\cup_{i=1}^{n+1}s_i$. The number n is called the **type** $t_\xi(F)$ of F with respect to $\mathcal{A}_\xi$.

THEOREM 1.3 ([7]). ξ-**Ramsey type theorem**. *Let $\mathcal{F}$ be an arbitrary family of finite subsets of $\mathbb{N}$, M an infinite subset of $\mathbb{N}$ and ξ a countable ordinal number. Then, there exists an infinite subset L of M such that*

$$\textit{either}\, \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}.$$

Using the strong Cantor-Bendixson index we have developed in [7] a refined form of the above theorem (Theorem 1.6).

DEFINITION 1.4. ([6], [4], [3]) Let $\mathcal{F}$ be a hereditary and pointwise closed family of finite subsets on $\mathbb{N}$. For $M \in [\mathbb{N}]$ we define the **strong Cantor-Bendixson derivatives** $(\mathcal{F})^\xi_M$ of $\mathcal{F}$ on M for every $\xi < \omega_1$ as follows:

$$(\mathcal{F})^1_M = \{F \in \mathcal{F}[M] : F \text{ is a cluster point of } \mathcal{F}[F \cup L] \text{ for each } L \in [M]\};$$

(where, $\mathcal{F}[M] = \mathcal{F} \cap [M]^{<\omega}$).

$$(\mathcal{F})^{\xi+1}_M = ((\mathcal{F})^\xi_M)^1_M.$$

If ξ is a limit ordinal,

$$(\mathcal{F})^\xi_M = \bigcap_{\beta<\xi} (\mathcal{F})^\beta_M.$$

The **strong Cantor-Bendixson index of $\mathcal{F}$ on M** is defined to be the smallest countable ordinal ξ such that $(\mathcal{F})^\xi_M = \emptyset$. We denote this index by $s_M(\mathcal{F})$.

REMARK 1.5.
(i) The strong Cantor-Bendixson index $s_M(\mathcal{F})$ is a successor countable ordinal.
(ii) If $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $s_M(\mathcal{F}_1) \leqslant s_M(\mathcal{F}_2)$ for every $M \in [\mathbb{N}]$.
(iii) $s_M(\mathcal{F}) = s_M(\mathcal{F} \cap [M]^{<\omega})$ for every $M \in [\mathbb{N}]$.
(iv) For every $M \in [\mathbb{N}]$ and $F \in [M]^{<\omega}$, according to a remark in [8], we have:

$$F \in (\mathcal{F})^1_M \text{ if and only if the set } \{m \in M : F \cup \{m\} \notin \mathcal{F}\} \text{ is finite.}$$

(v) If L is almost contained in M, then $s_L(\mathcal{F}) \geqslant s_M(\mathcal{F})$.
(vi) ([7]) $s_M((\mathcal{A}_\xi)_*) = \xi + 1$ for every $1 \leqslant \xi < \omega_1$ and $M \in [\mathbb{N}]$.

THEOREM 1.6 ([7]). **Refined ξ-Ramsey type theorem.** *Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}$ and M an infinite subset of $\mathbb{N}$. We have the following cases*:
Case 1 *If the family $\mathcal{F} \cap [M]^{<\omega}$ is not pointwise closed, then there exists $L \in [M]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$.*
Case 2 *If the family $\mathcal{F} \cap [M]^{<\omega}$ is pointwise closed, then there exists $L \in [M]$ such that $[L]^{<\omega} \subseteq ([\mathbb{N}]^{<\omega} \setminus \mathcal{F})_*$. Moreover setting*

$$\xi^{\mathcal{F}}_M = \sup\{s_L(\mathcal{F}) : L \in [M]\};$$

the following obtain:

2(i) *For every countable ordinal* ξ *with* $\xi+1<\xi_M^{\mathcal{F}}$ *there exists* $L\in[M]$ *such that*

$$(\mathcal{A}_\xi)_*\cap[L]^{<\omega}\subseteq\mathcal{F}$$

2(ii) *For every countable ordinal* ξ *with* $\xi_M^{\mathcal{F}}<\xi+1$ *there exists* $L\in[M]$ *such that*

$$\mathcal{F}\cap[L]^{<\omega}\subseteq(\mathcal{A}_\xi)^*\setminus\mathcal{A}_\xi;$$

and equivalently,

$$\mathcal{A}_\xi\cap[L]^{<\omega}\subseteq[\mathbb{N}]^{<\omega}\setminus\mathcal{F}.$$

2(iii) *If* $\xi_M^{\mathcal{F}}=\xi+1$, *then both alternatives may materialize.*

Now we recall the ξ-Pták type theorem for some $1\leqslant\xi<\omega_1$, which has been proved in [7], using the notion of the weight of a finite subset F of $\mathbb{N}$ with respect of a set of the family $\mathcal{A}_\xi$. The classical Pták's theorem is the limiting ω_1-case.

DEFINITION 1.7. For every finite subset F of $\mathbb{N}$, every countable ordinal ξ, and every $s\in\mathcal{A}_\xi$ we define recursively the ξ-**weight** $w_\xi(F;s)$ **of** F **with respect to** s, to be a real (in fact, a rational) number in the real interval [0,1], as follows:

(1) [**Case** $\xi=1$] Since $\mathcal{A}_1=\{\{n\}:n\in\mathbb{N}\}$, we set for every $n\in\mathbb{N}$

$$w_1(F;\{n\})=1 \text{ if } n\in F \text{ and } w_1(F;\{n\})=0 \text{ otherwise.}$$

(2) [**Case** $\xi=\zeta+1$] Let $s\in\mathcal{A}_{\zeta+1}$. Then, $s=\{n\}\cup s_1$, where $n\in\mathbb{N}$, $\{n\}<s_1$ and $s_1\in\mathcal{A}_\zeta$. We set

$$w_{\zeta+1}(F;s)=w_\zeta(F;s_1)\cdot w_1(F;\{n\}).$$

(3) [**Case** $\xi=\omega^{\beta+1}$ for $0\leqslant\beta<\omega_1$] Let $s\in\mathcal{A}_{\omega^{\beta+1}}$. Then $s=s_1\cup\cdots\cup s_n$, with $n=\min s_1, s_1<\cdots<s_n$ and $s_1,\ldots,s_n\in\mathcal{A}_{\omega^\beta}$. We set

$$w_{\omega^{\beta+1}}(F;s)=\frac{1}{n}\sum_{i=1}^{n}w_{\omega^\beta}(F;s_i).$$

(4) [**Case** $\xi=\omega^\lambda$ for λ non-zero, countable limit ordinal]. Let $s\in\mathcal{A}_{\omega^\lambda}$. Then $s\in\mathcal{A}_{\omega^{\lambda_n}}$ with $n=\min s$, where (λ_n) is the fixed sequence of ordinals "converging" to λ, (Definition 1.1). So, $w_{\omega^\lambda}(F;s)=w_{\omega^{\lambda_n}}(F;s)$, $n=\min s$.

(5) [**Case** ξ limit, $\omega^{\alpha_0}<\xi<\omega^{\alpha_0+1}$ for some $0<\alpha_0<\omega_1$]. In this case, ξ has a unique representation of ordinals as follows: $\xi=p_0\omega^{\alpha_0}+\sum_{i=1}^{m}p_i\omega^{\alpha_i}$, where $m\in\mathbb{N}$, $\alpha_0>\alpha_1>\cdots>\alpha_m>0$ are ordinal numbers and $p_0,p_1,\ldots,p_m\geqslant1$ are natural numbers, so that either $p_0>1$ or $p_0=1$ and $m\geqslant1$.

Let $s\in\mathcal{A}_\xi$. Then $s=s_0\cup s_1\cup\cdots\cup s_m$ with $s_m<\cdots<s_1<s_0$, where $s_i=s_1^i\cup\cdots\cup s_{p_i}^i$ with $s_1^i<\cdots<s_{p_i}^i$ and $s_j^i\in\mathcal{A}_{\omega^{\alpha_i}}$ for every $0\leqslant i\leqslant m$ and $1\leqslant j\leqslant p_i$. We set

$$w_\xi(F;s)=\prod_{i=0}^{m}\prod_{j=1}^{p_i}w_{\omega^{\alpha_i}}(F;s_j^i).$$

REMARK 1.8 ([2], [7]). For every countable ordinal α and $s \in \mathcal{A}_{\omega^\alpha} = \mathcal{B}_\alpha$ we define recursively the functions $\varphi_\alpha^s : \mathbb{N} \to [0, +\infty)$ as follows:

$$\varphi^0_{\{k\}}(n) = 1 \text{ if } n = k; \text{ and } \varphi^0_{\{k\}}(n) = 0 \text{ otherwise, for every } \{k\} \in \mathcal{B}_0.$$

$$\varphi_s^{\alpha+1} = \frac{1}{k}\sum_{i=1}^{k} \varphi_{s_i}^\alpha, \text{ for every } s = s_1 \cup \cdots \cup s_k \in \mathcal{B}_{\alpha+1}.$$

$$\varphi_s^\lambda = \varphi_s^{\lambda_k}, k = \min s, \text{for every } s \in \mathcal{B}_\lambda, \text{ where } \lambda \text{ is a non-zero, countable limit ordinal.}$$

It is easy to see that $\sum_{n\in\mathbb{N}} \varphi_s^\alpha(n) = 1$ and that $s = \{n \in \mathbb{N} : \varphi_\alpha^s(n) \neq 0\}$. Moreover $w_{\omega^\alpha}(F, s) = \sum_{n\in F} \varphi_\alpha^s(n)$ for every $F \in [\mathbb{N}]^{<\omega}$.

THEOREM 1.9 ([7]). (**ξ-Pták theorem**). *Let $\mathcal{F}$ be a hereditary and pointwise closed family of finite subsets of $\mathbb{N}$, $M \in [\mathbb{N}]$, ξ a non-zero, countable ordinal and $0 < \varepsilon < 1$. If for every $s \in \mathcal{A}_\xi \cap [M]^{<\omega}$ there exists $F \in \mathcal{F}$ such that $w_\xi(F; s) > \varepsilon$, then*:

(i) *there exists $L \in [M]$ such that $s_L(\mathcal{F}) \geqslant \xi + 1$;*

(ii) *$\xi_M^{\mathcal{F}} \leqslant \xi + 1$, and*

(iii) *for every ordinal ζ with $\zeta < \xi$ there exists $L \in [M]$ such that*

$$\mathcal{A}_\zeta \cap [L]^{<\omega} \subset \mathcal{F}.$$

THEOREM 1.10 ([15]). **(Pták's theorem).** *Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}$ and $0 < \varepsilon < 1$. If for every non-negative, real valued function φ on $\mathbb{N}$ with finite support and such that $\sum_{n\in\mathbb{N}} \varphi(n) = 1$ there exists $F \in \mathcal{F}$ such that $\sum_{n\in F} \varphi(n) > \varepsilon$, then there exists $L \in [\mathbb{N}]$ such that $[L]^{<\omega} \subset \mathcal{F}$.*

Finally, we give the definition of the generalized Schreier families.

DEFINITION 1.11 ([1], [2], [20]). **(Generalized Schreier Families).**

$$\mathcal{F}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\};$$

$$\mathcal{F}_{\alpha+1} = \left\{ F \subseteq \mathbb{N} : F = \bigcup_{i=1}^{k} F_i, \{k\} \leqslant F_1 < \cdots < F_k, \text{ and } F_i \in \mathcal{F}_\alpha \right\} \cup \{\emptyset\};$$

If α is a limit ordinal choose and fix $(\alpha_n)_{n\in\mathbb{N}}$ strictly increasing to α and set

$$\mathcal{F}_\alpha = \{F \subseteq \mathbb{N} : F \in \mathcal{F}_{\alpha_k} \text{ with } k \leqslant \min F\} \cup \{\emptyset\}.$$

REMARK 1.12.

(i) The families $\mathcal{F}_\alpha$ for every $1 \leqslant \alpha < \omega_1$ are **hereditary**.

(ii) If $(n_1, \ldots, n_k) \in \mathcal{F}_\alpha$ and $m_i \geqslant n_i$ for $i = 1, \ldots, k$, then $(m_1, \ldots, m_k) \in \mathcal{F}_\alpha$.

(iii) ([7]) For every $0 \leqslant \alpha < \omega_1$ and $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that
$$\mathcal{F}_\alpha(L) \subseteq (\mathcal{B}_\alpha)_* \subseteq \mathcal{F}_\alpha.$$
(iv) ([3]) $s_M(\mathcal{F}_\alpha) = \omega^\alpha + 1$, for every $1 \leqslant \alpha < \omega_1$ and $M \in [\mathbb{N}]$.

(v) ([3], [8], [7]). If $\mathcal{F}$ is a hereditary and pointwise closed family of finite subsets of $\mathbb{N}$ and $M \in [\mathbb{N}]$ such that $s_M(\mathcal{F}) \geqslant \omega^\alpha$, then there exists $L \in [M]$ such that $\mathcal{F}_\alpha(L) \subseteq \mathcal{F}$.

2. The ordinal index of uniform convergence

In this section we introduce an ordinal index, denoted, $\xi^{(f_n)}$, measuring the degree of uniform convergence of a sequence (f_n) of real valued functions defined on a set Γ (with no topological structure) and converging pointwise to a function f. In general the index can take any countable ordinal or ω_1 as its value; however, if we consider, the sequence (f_n) as a sequence in the Banach space $l^\infty(\Gamma)$ (equivalently, as a sequence of functions defined on the closed unit ball of the dual space $(l^\infty(\Gamma))^*$) then the index can only be equal to ω_1 or to a (limit) countable ordinal of the form ω^ζ for some $\zeta < \omega_1$. This is an interesting fact, differentiating Banach space theory and topology.

We characterize (in Theorem 2.5 below) the sequences of functions with countable index as the weakly convergent sequences of $l^\infty(\Gamma)$, equivalently as the sequences without l^1_+-sequences and finally as those sequences whose every subsequence has a uniform converging convex block subsequence.

In case of sequences of continuous functions defined on a compact metric space other related indices have been defined previously in [10].

DEFINITION 2.1. Let (f_n) be a sequence of real valued functions defined on a set Γ and converging pointwise to a function f. For every $\varepsilon > 0$ we set
$$\mathcal{U}_\varepsilon^{(f_n)} = \{F \in [\mathbb{N}]^{<\omega} : \text{ there exists } \gamma \in \Gamma \text{ such that } |f_i(\gamma) - f(\gamma)| \geqslant \varepsilon \text{ for every } i \in F\}.$$
All the families $\mathcal{U}_\varepsilon^{(f_n)}$ for $\varepsilon > 0$ are hereditary.

We then define the **index of uniform convergence** $\xi^{(f_n)}$ **of** (f_n) **on** Γ as follows:

(i) If there exists $\varepsilon > 0$ such that the family $\mathcal{U}_\varepsilon^{(f_n)}$ is not pointwise closed, we set
$$\xi^{(f_n)} = \omega_1.$$
(ii) If all the families $\mathcal{U}_\varepsilon^{(f_n)}$ for $\varepsilon > 0$ are pointwise closed, we set
$$\xi^{(f_n)} = \sup\{s_M(\mathcal{U}_\varepsilon^{(f_n)}) : M \in [\mathbb{N}] \text{ and } \varepsilon > 0\},$$
where $\xi^{(f_n)}$ is a countable ordinal.

We say that the sequence (f_n) **converges ξ-uniformly to f on Γ**, for some countable ordinal ξ, if $\xi^{(f_n)} \leqslant \xi$.

REMARK 2.2.
(i) A sequence (f_n) of real valued functions converges uniformly to a function f if and only if $\xi^{(f_n)} = 1$. Indeed, (f_n) converges uniformly to f if and only if the families $\mathcal{U}_\varepsilon^{(f_n)}$ for every $\varepsilon > 0$ are finite sets and equivalently if $s_M(\mathcal{U}_\varepsilon^{(f_n)}) = 1$ for every $\varepsilon > 0$ and $M \in [\mathbb{N}]$.
(ii) Every sequence of continuous functions defined on a compact, Hausdorff space which converges pointwise to a continuous function has countable index of uniform convergence. This happens since $\mathcal{U}_\varepsilon^{(f_n)}$ is not pointwise closed if and only if there exists $M \in [\mathbb{N}]$ with $[M]^{<\omega} \subseteq \mathcal{U}_\varepsilon^{(f_n)}$ (Theorem 1.6).
(iii) If (g_n) is a subsequence of a pointwise converging sequence (f_n), then $\xi^{(g_n)} \leqslant \xi^{(f_n)}$. So, if (f_n) converges ξ-uniformly to a function f for some $1 \leqslant \xi < \omega_1$, then (g_n) also converges ξ-uniformly to f.

In the following examples we indicate that there exist pointwise converging sequences of real valued functions (as well as continuous functions on compact metric spaces) either with uncountable or with countable index of uniform convergence. Moreover this index can be either a limit or a successor ordinal.

EXAMPLES 2.3. (1) For $n \in \mathbb{N}$ we define the function $f_n : [\mathbb{N}]^{<\omega} \to \mathbb{N}$ with $f_n(F) = 1$ if $n \in F$ and $f_n(F) = 0$ otherwise. The sequence (f_n) converges pointwise to zero and has index of uniform convergence equal to ω_1.

(2) Let ξ be any countable ordinal and $(\mathcal{A}_\xi)_*$ be the hereditary family of finite subsets of $\mathbb{N}$ corresponding to the thin Schreier family $\mathcal{A}_\xi$ (Definition 1.1). The family $(\mathcal{A}_\xi)_*$, considered as a subspace of the Cantor space $\{0,1\}^{\mathbb{N}}$, is a compact and metric space. For every $n \in \mathbb{N}$ set $f_n : (\mathcal{A}_\xi)_* \to \mathbb{R}$ with $f_n(F) = 1$ if $n \in \mathcal{F}$ and $f_n(F) = 0$ otherwise. Of course (f_n) is a sequence of continuous functions on $(\mathcal{A}_\xi)_*$, which converges pointwise to zero. The index of uniform convergence of (f_n) is equal to $\xi + 1$, as $\mathcal{U}_\varepsilon^{(f_n)} = (\mathcal{A}_\xi)_*$ for every $0 < \varepsilon \leqslant 1$, $\mathcal{U}_\varepsilon^{(f_n)} = \emptyset$ for $1 < \varepsilon$ and $s_M((\mathcal{A}_\xi)_*) = \xi + 1$ for every $M \in [\mathbb{N}]$ (Remark 1.5).

(3) Let

$$B = \left\{ (\alpha_i)_{i \in \mathbb{N}} : \sum_{i \in \mathbb{N}} |\alpha_i| \leqslant 1 \right\}.$$

This is a compact metric space endowed with the pointwise convergence topology. We set $f_n : B \to \mathbb{R}$ with $f_n((\alpha_i)_{i \in \mathbb{N}} = \alpha_n$ for every $n \in \mathbb{N}$. The sequence (f_n) of continuous functions on B, converges pointwise to zero and has index of uniform convergence equal to ω, since $\mathcal{U}_{1/n}^{(f_n)} = ([\mathbb{N}]^n)_*$ for every $n \in \mathbb{N}$ and consequently $s_M(\mathcal{U}_{1/n}^{(f_n)}) = n$ for every $n \in \mathbb{N}$.

Now, we will characterize the class of uniformly bounded, pointwise converging sequences of functions which have countable index of uniform convergence. In this characterization we make use of the notion of an ℓ^1_+-sequence in a Banach space, which we define below. We note that the linear space of all bounded functions defined on a set Γ endowed with the supremum norm is denoted by $l^\infty(\Gamma)$ and it is a Banach space.

DEFINITION 2.4. A bounded sequence (χ_n) in a Banach space X is an **l^1_+-sequence** in X if there exists $\varepsilon > 0$ such that

$$\varepsilon \sum_{i=1}^{n} \lambda_i \leqslant \left\| \sum_{i=1}^{n} \lambda_i \chi_i \right\| \text{ for every } n \in \mathbb{N} \text{ and } \lambda_1, \ldots, \lambda_n \geqslant 0.$$

THEOREM 2.5. *Let (f_n) be a uniformly bounded sequence of real valued functions defined on a set Γ which converges pointwise to a function f. The following are equivalent*:

(i) $\xi^{(f_n)} < \omega_1$;

(ii) *the sequence $(f_n - f)$ in $l^\infty(\Gamma)$ does not have any l^1_+-subsequence*;

(iii) *every subsequence of (f_n) has a convex block subsequence converging uniformly to f; and*

(iv) *the sequence (f_n) converges weakly to f in the Banach space $l^\infty(\Gamma)$.*

Proof. (i) $\Rightarrow$ (ii) Let $\xi^{(f_n)} < \omega_1$. We assume that there exists a subsequence (g_n) of the (f_n) and $\varepsilon > 0$ such that

$$\varepsilon \sum_{i=1}^{n} \lambda_i \leqslant \left\| \sum_{i=1}^{n} \lambda_i (g_i - f) \right\| \text{ for every } n \in \mathbb{N} \text{ and } \lambda_1, \ldots, \lambda_n \geqslant 0.$$

Let $\varphi : \mathbb{N} \to [0, +\infty)$ be a function such that $\sum_{n\in\mathbb{N}} \varphi(n) = 1$ and the set $\{n \in \mathbb{N} : \varphi(n) : \varphi(n) \neq 0\}$ is finite. Then there exists $\gamma \in \Gamma$ such that

$$\frac{\varepsilon}{2} < \left| \sum_{n\in\mathbb{N}} \varphi(n)(g_n - f)(\gamma) \right|.$$

Set $F = \{n \in \mathbb{N} : |(g_n - f)(\gamma)| > \frac{\varepsilon}{4}\}$. Obviously $F \in \mathcal{U}^{(g_n)}_{\frac{\varepsilon}{4}}$. Since

$$\frac{\varepsilon}{2} < \left| \sum_{n\in\mathbb{N}} \varphi(n)(g_n - f)(\gamma) \right| \leqslant \sum_{n\in F} \varphi(n) + \frac{\varepsilon}{4},$$

where $C = \sup_n \|f_n - f\| < \infty$, we have $\sum_{n\in F} \varphi(n) > \frac{\varepsilon}{8C}$.

From Pták's classical theorem (Theorem 1.10) it follows that there exists $M \in [\mathbb{N}]$ such that $[M]^{<\omega} \subseteq \mathcal{U}^{(g_n)}_{\delta}$, with $\delta = \varepsilon/4$. If $g_n = f_{k_n}$ for every $n \in \mathbb{N}$ and $M = (m_n)_{n\in\mathbb{N}}$, then setting $L = \{k_{m_n} : n \in \mathbb{N}\}$ we have that $[L]^{<\omega} \subseteq \mathcal{U}^{(f_n)}_{\delta}$; and consequently that $\mathcal{U}^{(f_n)}_{\delta}$ is not pointwise closed family (Theorem 1.6). This gives that $\xi^{(f_n)} = \omega_1$.

A contradiction; hence we have proved that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i) Let $(f_n - f)$ has not any l^1_+-subsequence. We assume that $\xi^{(f_n)} = \omega_1$. Then there exists $\varepsilon > 0$ such that the family $\mathcal{U}^{(f_n)}_\varepsilon$ is not pointwise closed. Hence, there exists $M = (m_n)_{n \subseteq \mathbb{N}} \in \mathbb{N}$ such that $[M]^{<\omega} \subseteq \mathcal{U}^{(f_n)}_\varepsilon$. Set $h_n = f_{m_n}$ for every $n \in \mathbb{N}$ and

$$\mathcal{U}^{(h_n)}_{\varepsilon,1} = \{F \subseteq \mathbb{N} \ : \ \text{there exists } \gamma \in \Gamma \text{ such that } f(\gamma) - h_i(\gamma) \geqslant \varepsilon \text{ for every } i \in F\},$$

$$\mathcal{U}^{(h_n)}_{\varepsilon,2} = \{F \subseteq \mathbb{N} \ : \ \text{there exists } \gamma \in \Gamma \text{ such that } h_i(\gamma) - f(\gamma) \geqslant \varepsilon \text{ for every } i \in F\}.$$

Let $\varphi: \mathbb{N} \to [0, +\infty)$ be a function such that $\sum_{n\in\mathbb{N}} \varphi(n) = 1$ and the set $F = \{n \in \mathbb{N}: \varphi(n) \neq 0\}$ is finite. Since $(m_n)_{n\in F} \in \mathcal{U}^{(f_n)}_\varepsilon$, there exist $F_1 \in \mathcal{U}^{(h_n)}_{\varepsilon,1}$ and $F_2 \in \mathcal{U}^{(h_n)}_{\varepsilon,2}$ such that $F = F_1 \cup F_2$ and $F_1 \cap F_2 = \emptyset$. Hence,

$$\text{either} \sum_{n\in F_1} \varphi(n) > \frac{1}{3} \text{ or } \sum_{n\in F_2} \varphi(n) > \frac{1}{3}.$$

From Pták's theorem, there exists $L \in [\mathbb{N}]$ such that $[L]^{<\omega} \subseteq \mathcal{U}^{(h_n)}_{\varepsilon,1} \cup \mathcal{U}^{(h_n)}_{\varepsilon,2}$. Let $L = (l_n)_{n\in\mathbb{N}}$ and $g_n = h_{l_n}$ for every $n \in \mathbb{N}$. Then $(g_n - f)$ is an l^1_+-sequence. Indeed, let $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \geqslant 0$. Since $(l_1, \ldots, l_n) \in \mathcal{U}^{(h_n)}_{\varepsilon,1} \cup \mathcal{U}^{(h_n)}_{\varepsilon,2}$, there exist $\gamma \in \Gamma$ and $\alpha \in \{-1, 1\}$ such that $\alpha(g_i(\gamma) - f(\gamma)) \geqslant \varepsilon$ for every $i = 1, \ldots, n$. Hence, $\varepsilon \sum_{i=1}^n \lambda_i \leqslant \|\sum_{i=1}^n \lambda_i (g_i - f)\|$. A contradiction; which proves that $\xi^{(f_n)} < \omega_1$.

(ii) $\Rightarrow$ (iii) Let $(f_n - f)$ has not any l^1_+-subsequence and let (g_n) be a subsequence of (f_n). Then we can construct inductively a convex block subsequence (h_n) of (g_n) such that$\|h_n - f\| < \frac{1}{n}$ for every $n \in \mathbb{N}$.

(iii) $\Rightarrow$ (iv) We assume that $(f_n - f)$ does not converge weakly to zero in $l^\infty(\Gamma)$. Then there exists $\chi^* \in (l^\infty(\Gamma))^*$ with $\|\chi^*\| \leqslant 1$ and $\varepsilon > 0$ such that the set $M = \{n \in \mathbb{N}: |\chi^*(f_n - f)| \geqslant \varepsilon\}$ is infinite. Replacing χ^* by $-\chi^*$, if necessary, we can find $M_1 \in [M]$ such that $\chi^*(f_n - f) \geqslant \varepsilon$ for every $n \in M_1$. Set $M_1 = (m_n)_{n\in\mathbb{N}}$ and $g_n = f_{m_n}$ for every $n \in \mathbb{N}$. Then the subsequence (g_n) of (f_n) has no convex block subsequence converging uniformly to f, since $\|\sum_{i=1}^n \lambda_i g_i - f\| \geqslant \varepsilon$ for every $\lambda_1, \ldots, \lambda_n \geqslant 0$ with $\sum_{i=1}^n \lambda_i = 1$.

(iv) $\Rightarrow$ (ii) We assume that $(g_n - f)$ is an l^1_+-subsequence of $(f_n - f)$. Then there exists $\varepsilon > 0$ such that $\varepsilon \sum_{i=1}^n \lambda_i \leqslant \|\sum_{i=1}^n \lambda_i (g_i - f)\|$ for every $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \geqslant 0$.

Let $\varphi: \mathbb{N} \to [0, +\infty)$ be a function such that $\sum_{n=1}^\infty \varphi(n) = 1$ and the set $\{n \in \mathbb{N}: \varphi(n) \neq 0\}$ is finite. Then there exists $\chi^* \in Y^*$, where Y is the closed linear subspace of $l^\infty(\Gamma)$ generated by $(g_n - f)$, with $\|\chi^*\| \leqslant 1$ and $\frac{\varepsilon}{2} < |\sum_{n=1}^\infty \varphi(n) \ \chi^*(g_n - f)|$. Set $F = \{n \in \mathbb{N}: \ |\chi^*(g_n - f)| \ \geqslant \ \frac{\varepsilon}{4}\}$. Then $\frac{\varepsilon}{2} < |\sum_{n=1}^\infty \varphi(n) \ \chi^*(g_n - f)| \leqslant C \sum_{n\in F} \varphi(n) + \frac{\varepsilon}{4}$, where $C = \sup_n \|f_n - f\| < \infty$. Hence, $\sum_{n\in F} \varphi(n) > \frac{\varepsilon}{4C} = \delta$. Set $\mathcal{F} = \{F \in [\mathbb{N}]^{<\omega}: \text{there exist } \chi^* \in Y^* \text{ with } \|\chi^*\| \leqslant 1 \text{ and } |\chi^*(g_n - f)| \geqslant \frac{\varepsilon}{4} \text{ for every } n \in F\}$.

From Pták's classical theorem there exists $M \in [\mathbb{N}]$ such that $[M]^{<\omega} \subset \mathcal{F}$. Since the closed unit ball of Y^* is weak*-compact and metrizable, there exists $\chi^* \in Y^*$ with $\|\chi^*\| \leqslant 1$ such that $\chi^*(g_n - f)| \geqslant \frac{\varepsilon}{4}$ for every $n \in M$. This gives that $(g_n - f)$ does not converge weakly to zero. That finishes the proof of the theorem.

REMARKS. (i) ℓ^1_+-sequences which are also basic sequences are thoroughly studied, under the term wide-(s) sequences, by H. Rosenthal in [18] and [19]. These sequences were originally introduced by I. Singer in [21], under the term P^*-sequences.

(ii) As pointed out by the referee, the following dichotomy can be easily proved using a classical result by Kadec-Pelczyński ([11]): every bounded sequence in a Banach space has a basic subsequence which is either l^1_+, or weakly null. This result can be further refined using Rosenthal's l^1-dichotomy theorem to obtain a subsequence which is either a weak-Cauchy, l^1_+-sequence or an l^1-sequence (i.e. equivalent to the l^1-basis); c.f. Proposition 1 in [19].

Using the previous theorem we can now characterize those uniformly bounded and pointwise convergent sequences of continuous functions defined on a compact Hausdorff space which have countable index of uniform convergence precisely as those with continuous limit function.

COROLLARY 2.6. *Let (f_n) be a uniformly bounded sequence of* **continuous**, *real valued functions defined on a compact Hausdorff space K which converges pointwise to a function f. Then $\xi^{(f_n)} < \omega_1$ if and only if f is a continuous function.*

Proof. If $\xi^{(f_n)} < \omega_1$, then f is continuous, according to implication (i)$\Rightarrow$(iii) of Theorem 2.5. On the other hand if f is a continuous function then $\xi^{(f_n)} < \omega_1$, according to Remark 2.2(ii).

For a Banach space X let K be the closed unit ball of its dual space X^*. If we endow K with the weak* topology (a sequence (χ^*_n) in K converges weak* to some $\chi^* \in K$ if and only if $\chi^*_n(\chi) \to \chi^*(\chi)$ for every $\chi \in X$), then K is a compact Hausdorff space. Via the natural embedding $e: X \to C(K)$ with $e(\chi)(\chi^*) = \chi^*(\chi)$ for every $\chi \in X$ and $\chi^* \in K$, we have that X is isometric to a subspace of the Banach space $C(K)$ of all the continuous functions on K with the supremum norm. So, a sequence (χ_n) in X can be considered as a sequence in $C(K)$.

The sequences (χ_n) in X which converge pointwise on K are called **weak-Cauchy**. So, we can define the index of uniform convergence $\xi^{(\chi_n)}$ on K for every weak-Cauchy sequence (χ_n) in a Banach space X. As we will prove below, this index is countable if and only if (χ_n) converges weakly in X.

COROLLARY 2.7. *Let (χ_n) be a bounded, weak Cauchy sequence in a Banach space X. Then $\xi^{(\chi_n)} < \omega_1$ if and only if the sequence (χ_n) converges weakly in X to some element χ of X.*

Proof. Since (χ_n) is a weak-Cauchy sequence in X, there exists $\chi^{**} \in X^{**}$ such that $\chi^*(\chi_n) \to \chi^{**}(\chi^*)$ for every $\chi^* \in X^*$. As (χ_n) is considering as a sequence of continuous functions on the closed unit ball K of X^* converging pointwise to χ^{**}/K we have, according to the previous corollary, that $\xi^{(\chi_n)} < \omega_1$ if and only if $\chi^{**}/K \in C(K)$. This equivalence gives that $\xi^{(\chi_n)} < \omega_1$ if and only if (χ_n) converges weakly to an element χ of X.

We have already seen (see Remark 2.2) that there exist pointwise convergent sequences of continuous functions on compact metric spaces with indices of uniform convergence non-limit countable ordinals. It is, then, a remarkable fact that every weakly convergent sequence in a Banach space has always index of uniform convergence (equal to a limit ordinal and moreover) of the form ω^ζ for some countable ordinal ζ, (Theorem 2.9 below). This is a set-theoretic differentiation between Banach space theory and topology.

LEMMA 2.8. *Let (f_n) be a uniformly bounded sequence of real valued functions defined on a set Γ which converges pointwise to a function f and ζ a countable ordinal. If*

(i) $\omega^\zeta < s_I(\mathcal{U}_\varepsilon^{(f_n)})$ *for some $I \in [\mathbb{N}]$ and $\varepsilon > 0$, then*

(ii) *there exist a subsequence (g_n) of $(f_n)_{n \in I}$ and $\theta \in \{-1, 1\}$ such that, for every $F \in \mathcal{F}_\zeta$, there exists $\gamma \in \Gamma$so that $\theta \cdot (g_i(\gamma) - f(\gamma)) \geqslant \varepsilon$ for every $i \in F$.*

Proof. Let $\omega^\zeta < s_I(\mathcal{U}_\varepsilon^{(f_n)})$. According to Remark 1.11 (vii) there exists $L \in [I]$ such that $\mathcal{F}_\zeta(L) \subseteq \mathcal{U}_\varepsilon^{(f_n)}$. If $L = (l_n)$, we set $h_n = f_{l_n} - f$ for every $n \in \mathbb{N}$ and

$$\mathcal{U}_{\varepsilon,1}^{(h_n)} = \{F \subseteq \mathbb{N} \colon \text{ there exists } \gamma \in \Gamma \text{ such that } -h_i(\gamma) \geqslant \varepsilon \text{ for every } i \in F\},$$
$$\mathcal{U}_{\varepsilon,1}^{(h_n)} = \{F \subseteq \mathbb{N} \colon \text{ there exists } \gamma \in \Gamma \text{ such that} h_i(\gamma) \geqslant \varepsilon \text{ for every } i \in F\}.$$

Since $\mathcal{B}_\zeta \subseteq \mathcal{F}_\zeta \subseteq \mathcal{U}_\varepsilon^{(h_n)}$, for every $s \in \mathcal{B}_\zeta$ there exist $F_1 \in \mathcal{U}_{\varepsilon,1}^{(h_n)}$ and $F_2 \in \mathcal{U}_{\varepsilon,2}^{(h_n)}$ such that $s = F_1 \cup F_2$ and $F_1 \cap F_2 = \emptyset$. Hence, for every $s \in \mathcal{B}_\zeta$ there exists $F \in \mathcal{U}_{\varepsilon,1}^{(h_n)} \cup \mathcal{U}_{\varepsilon,2}^{(h_n)}$ such that $\frac{1}{3} < w_{\omega^\zeta}(F; s)$.

Set

$$\mathcal{L}_1 = \left\{ s \in \mathcal{B}_\zeta \colon \text{ there exists } F \in \mathcal{U}_{\varepsilon,1}^{(h_n)} \text{ such that } \frac{1}{3} < w_{\omega^\zeta}(F; s) \right\}$$

and

$$\mathcal{L}_2 = \left\{ s \in \mathcal{B}_\zeta \colon \text{ there exists } F \in \mathcal{U}_{\varepsilon,2}^{(h_n)} \text{ such that} \frac{1}{3} < w_{\omega^\zeta}(F; s) \right\}.$$

Then $\mathcal{B}_\zeta = \mathcal{L}_1 \cup \mathcal{L}_2$ and, according to the ω^ζ-Ramsey theorem (Theorem 1.3), there exist $p \in \{1, 2\}$ and $N \in [\mathbb{N}]$ such that $\mathcal{B}_\zeta \cap [N]^{<\omega} \subseteq \mathcal{L}_p$. Applying the ω^ζ-Pták type theorem (Theorem 1.9), we can find $M \in [N]$ such that $\omega^\zeta < s_M(\mathcal{U}_{\varepsilon,p}^{(h_n)})$. Then there exists $N_1 \in [M]$ such that $\mathcal{F}_\zeta(N_1) \subseteq \mathcal{U}_{\varepsilon,p}^{(h_n)})$. If $N_1 = (n_k)_{k \in \mathbb{N}}$, then we set $g_k = h_{n_k}$ for every $k \in \mathbb{N}$. For every $F \in \mathcal{F}_\zeta$ we have $(n_i)_{i \in F} \in \mathcal{U}_{\varepsilon,p}^{(h_n)})$, hence there exists $\gamma \in \Gamma$ such that $(-1)^p(g_i(\gamma) - f(\gamma) \geqslant \varepsilon$ for every $i \in F$.

THEOREM 2.9. *Let (χ_n) be a sequence in a Banach space X and $\chi \in X$. If (χ_n) converges weakly to χ, then $\xi^{(\chi_n)} = \omega^\zeta$ for some countable ordinal ζ.*

Proof. Let (χ_n) be a sequence in the Banach space X which converges weakly to $\chi \in X$. We assume that $\chi = 0$, otherwise we replace (χ_n) by $(\chi_n - \chi)$. We have $\xi^{(\chi_n)} < \omega_1$ (Corollary 2.7), hence, there exists a unique countable ordinal ζ such that $\omega^\zeta \leqslant \xi^{(\chi_n)} < \omega^{\zeta+1}$. Arguing by contradiction suppose that $\omega^\zeta < \xi^{(\chi_n)}$. Then there exist $M \in [\mathbb{N}]$ and $\varepsilon > 0$ such that $\omega^\zeta < s_M(\mathcal{U}_\varepsilon^{(\chi_n)})$. Since $1 < s_M(\mathcal{U}_\varepsilon^{(\chi_n)})$, there exists $I \in [M]$ such that $\inf\{\|\chi_n\| : n \in I\} \geqslant \varepsilon$. Of course $\omega^\zeta < s_I(\mathcal{U}_\varepsilon^{(\chi_n)})$.

Using the previous lemma, we can find a subsequence (y_n) of $(\chi_n)_{n \in I}$ such that for every $F \in \mathcal{F}_\zeta$ there exists $\chi^* \in X^*$ with $\|\chi^*\| \leqslant 1$ and $\chi^*(y_n) \geqslant \varepsilon$ for every $n \in \mathcal{F}_\zeta$. Since (y_n) is weakly null we can assume that (y_n) is a basic sequence ([5]).

According to Lemma 1.2 in [3] for every $0 < \delta < 1$ there exists a subsequence (z_n) of (y_n) such that for every $n \in \mathbb{N}$, $F \subseteq \{1, \ldots, n\}$ and $\chi^* \in X^*$ with $\|\chi^*\| \leqslant 1$ and $\chi^*(z_i) \geqslant \varepsilon$ for every $i \in F$ there exists $y^* \in X^*$ with $\|y^*\| \leqslant 1$ satisfying the following two conditions:

(i) $y^*(z_i) > (1 - \delta)\varepsilon$ for every $i \in F$; and

(ii) $|y^*(z_i)| < \varepsilon \cdot \delta$ for every $i \in \{1, \ldots, n\} \setminus F$.

Let $k \in \mathbb{N}$ and $s_1, \ldots, s_k \in \mathcal{B}_\zeta$ with $s_1 < \cdots < s_k$. Since $\mathcal{B}_\zeta \subseteq \mathcal{F}_\zeta$, for every $\lambda \in \{1, \ldots, k\}$ there exists $\chi_\lambda^* \in X^*$ with $\|\chi_\lambda^*\| \leqslant 1$ and $\chi_\lambda^*(z_i) \geqslant \varepsilon$ for every $i \in s_\lambda$. Hence, for every $\lambda \in \{1, \ldots, k\}$ there exists $y_\lambda^* \in X^*$ with $\|y_\lambda^*\| \leqslant 1$ satisfying:

(i) $y_\lambda^*(z_i) > (1 - \frac{1}{2k})\varepsilon$ for every $i \in s_\lambda$; and

(ii) $|y_\lambda^*(z_i)| < \frac{\varepsilon}{2k}$ for every $i \in \bigcup_{n=1}^k s_n \setminus s_\lambda$.

Set $y^* = \frac{y_1^* + \cdots + y_k^*}{k}$. Of course $\|y^*\| \leqslant 1$ and for every $\lambda \in \{1, \ldots, k\}$ and $i \in s_\lambda$ we have:

$$y^*(z_i) = \frac{1}{k}\left(y_\lambda^*(z_i) + \sum_{\substack{n=1\\n\neq\lambda}}^{k} y_n^*(z_i)\right) \geqslant \frac{\varepsilon}{k}\left(1 - \frac{1}{2k}\right) - \frac{1}{k}\sum_{\substack{n=1\\n\neq\lambda}}^{k} |y_n^*(z_i)|$$

$$\geqslant \frac{\varepsilon}{k}\left(1 - \frac{1}{2k}\right) - \frac{\varepsilon}{2k}\frac{(k-1)}{k} = \frac{\varepsilon}{2k}.$$

This means that $s_1 \cup \cdots \cup s_k \in \mathcal{U}_{\frac{\varepsilon}{2k}}^{(z_n)}$. Hence, $\mathcal{A}_{k\omega^\zeta} \subseteq \mathcal{U}_{\frac{\varepsilon}{2k}}^{(z_n)}$ for every $k \in \mathbb{N}$. From Remark 1.5 we have that $s_\mathbb{N}(\mathcal{U}_{\frac{\varepsilon}{2k}}^{(z_n)}) \geqslant k\omega^\zeta$ for every $k \in \mathbb{N}$ and consequently that $\xi^{(\chi_n)} \geqslant \xi^{(z_n)} \geqslant \omega^{\zeta+1}$.

A contradiction to our hypothesis that $\xi^{(\chi_n)} < \omega^{\zeta+1}$; hence $\xi^{(\chi_n)} = \omega^\zeta$.

We can further clarify the difference in behavior of the index of uniform convergence between Banach space and topology by the following consideration. If (f_n) is a uniformly bounded sequence of real valued functions defined on a set Γ which converges pointwise to a function f, then its index of uniform convergence $\xi^{(f_n)}$ on Γ can be equal to any countable ordinal ξ (Examples 2.3). On the other hand if we consider (f_n) as a bounded sequence in the Banach space $l^\infty(\Gamma)$ or equivalently as a sequence of continuous functions defined on the closed unit ball K of $(l^\infty(\Gamma))^*$, then (f_n) has index of uniform convergence equal to ω_1 if $\xi^{(f_n)}=\omega_1$ and of the form ω^ζ, where ζ is the least countable ordinal with $\xi^{(f_n)}<\omega^\zeta$ in case $\xi^{(f_n)}<\omega_1$.

THEOREM 2.10. *Let (f_n) be a uniformly bounded sequence of real valued function defined on a set Γ which converges pointwise to a function f. If $\xi^{(f_n)}$ is the index of uniform convergence of (f_n) on Γ and $\xi_B^{(f_n)}$ is the index of uniform convergence of (f_n) as a sequence in $l^\infty(\Gamma)$, then*:
(i) $\xi^{(f_n)}=\omega_1$ *if and only if* $\xi_B^{(f_n)}=\omega_1$; *and*
(ii) *If* $\xi^{(f_n)}<\omega_1$, *then* $\xi_B^{(f_n)}=\omega^\zeta$, *where ζ is the least countable ordinal with* $\xi^{(f_n)}\leqslant\omega^\zeta$.

Proof. (i) If follows from Theorem 2.5 and Corollary 2.7.

(ii) Let $\xi^{(f_n)}<\omega_1$. According to Theorem 2.9 and (i) there exists a countable ordinal ζ such that $\xi_B^{(f_n)}=\omega^\zeta$. Obviously $\xi^{(f_n)}\leqslant\xi_B^{(f_n)}$.

Let $\alpha<\omega_1$ such that $\omega^\alpha<\xi_B^{(f_n)}$. According to Lemma 2.8, there exist $\varepsilon>0$, $\theta\in\{-1,1\}$ and a subsequence (g_n) of (f_n), such that for every $s\in\mathcal{B}_\alpha$ there exists $\chi^*\in K$ (K is the closed unit ball of $(l^\infty(\Gamma))^*$) such that $\theta(\chi^*(g_n)-\chi^*(f))\geqslant\varepsilon$ for every $n\in s$. Then,

$$\left\|\sum_{n\in s}\varphi_s^\alpha(n)g_n-f\right\|\geqslant\varepsilon \text{ for every } s\in\mathcal{B}_\alpha,$$

where $\varphi_s^\alpha:\mathbb{N}\to[0,1]$ is the convex combination with respect to s, defined in Remark 1.8. This gives that for every $s\in\mathcal{B}_\alpha$ there exists $\gamma\in\Gamma$ with

$$\frac{\varepsilon}{2}<\left|\sum_{n\in s}\varphi_s^\alpha(n)(g_n(\gamma)-f(\gamma))\right|\leqslant\left\|\sum_{n\in s}\varphi_s^\alpha(n)g_n-f\right\|.$$

Set $F=\{n\in s:|(g_n-f)(\gamma)|>\frac{\varepsilon}{4}$. Then $F\in\mathcal{U}_\varepsilon^{(g_n)}$ and

$$\frac{\varepsilon}{2}<\left|\sum_{n\in s}\varphi_s^\alpha(n)g_n(\gamma)-f(\gamma)\right|\leqslant\left|\sum_{n\in F}\varphi_s^\alpha(n)g_n(\gamma)-f(\gamma)\right|$$
$$+\left|\sum_{n\in s\setminus F}\varphi_s^\alpha(n)g_n(\gamma)-f(\gamma)\right|\leqslant Cw_{\omega^a}(F;s)+\frac{\varepsilon}{4};$$

where $C=\sup_n\|g_n-f\|$. Hence, $w_{\omega^\alpha}(F,s)>\frac{\varepsilon}{4C}$. According to the ω^α-Pták type theorem (Theorem 1.9) there exists $L\in[\mathbb{N}]$ such that $s_L(\mathcal{U}_{\frac{\varepsilon}{4}}^{(g_n)})>\omega^\alpha$. Hence $s_M(\mathcal{U}_{\frac{\varepsilon}{4}}^{(f_n)})>\omega^\alpha$ for $M=(i_n)_{n\in L}$. So, $\omega^\alpha<\xi^{(f_n)}$. This gives that ζ is the least countable ordinal with $\xi^{(f_n)}\leqslant\omega^\zeta$.

3. Uniform convergence for every countable ordinal

In this section, firstly we study and obtain a characterization of the ξ-uniform convergence of sequences of functions (for all limit ordinals $\xi<\omega_1$) defined (in the general topological setting) on an arbitrary set Γ, in terms of the complete thin Schreier system $(\mathcal{A}_\xi, \xi<\omega_1)$ (in Theorem 3.1 and Proposition 3.2 below). We believe that this characterization is so natural and basic that can take the place of a definition.

Secondly, restricting ourselves to the Banach space setting we obtain (in Theorem 3.13) a stronger characterization of the ω^α-uniformly convergent sequences as those sequences, for which every subsequence has a uniformly converging (not simply convex but specifically) $\mathcal{B}_\alpha$-convex block subsequence, refining Theorem 2.5.

Combining all these results we obtain a dichotomy theorem which can be considered as the natural extension of Rosenthal's l^1-theorem.

THEOREM 3.1. *Let (f_n) be a sequence of real valued functions defined on a set Γ which converges pointwise to a function f. Also, let ξ be a limit countable ordinal and (ξ_n) be a sequence of ordinals strictly increasing to ξ. The following are equivalent*:

(i) *(f_n) converges ξ-uniformly to f (i.e. or $\xi^{(f_n)}\leqslant\xi$); and*

(ii) *for every $M\in[\mathbb{N}]$ there exists a strictly increasing sequence $\varphi:\mathbb{N}\to M$ such that for every $M\in[\mathbb{N}]$ there exists a strictly increasing sequence $\varphi:\mathbb{N}\to M$ such that for every $\varepsilon>0$ there exists $n_0=n_0(\varepsilon)\in\mathbb{N}$ so that*

$$\{\varphi(n):n\geqslant n_0 \text{ and } |f_{\varphi(n)}(\gamma)-f(\gamma)|\geqslant\varepsilon\}\in(\mathcal{A}_{\xi_{n_0}})^*\setminus\mathcal{A}_{\xi_{n_0}} \text{ for every } \gamma\in\Gamma.$$

Proof. (i) $\Rightarrow$ (ii) Let $\xi^{(f_n)}\leqslant\xi$ and $M\in[\mathbb{N}]$. We claim that for every $L\in[\mathbb{N}]$ and $\varepsilon>0$ there exists $I\in[L]$ such that

$$\xi^I_{\mathcal{U}^{(f_n)}_\varepsilon}=\sup\{s_N(\mathcal{U}^{(f_n)}_\varepsilon):N\in[I]\}<\xi.$$

Indeed, suppose that it does not hold. Then there exists $L\in[\mathbb{N}]$ and $\varepsilon>0$ such that for every $I\in[L]$ we have $\sup\{s_N(\mathcal{U}^{(f_n)}_\varepsilon):N\in[I]\}=\xi$. Hence, there exists a decreasing sequence $(I_n)_{n\in\mathbb{N}}$ in $[L]$ such that $s_{I_n}(\mathcal{U}^{(f_n)}_\varepsilon)>\xi_n$ for every $n\in\mathbb{N}$. Setting $I=(i^n_n)_{n\in\mathbb{N}}$ if $I_n=(i^n_k)_{k\in\mathbb{N}}$ for every $n\in\mathbb{N}$ we have that $s_I(\mathcal{U}^{(f_n)}_\varepsilon)\geqslant s_{I_n}(\mathcal{U}^{(f_n)}_\varepsilon)>\xi_n$ for every $n\in\mathbb{N}$. So, $s_I(\mathcal{U}^{(f_n)}_\varepsilon)\geqslant\xi$. Since $s_I(\mathcal{U}^{(f_n)}_\varepsilon)$ is a successor ordinal we have $s_I(\mathcal{U}^{(f_n)}_\varepsilon)\geqslant\xi+1$. A contradiction, which proves our claim.

Using the refined form of the ξ-Ramsey type theorem, (Theorem 1.6), we can find inductively an increasing sequence (k_n) in $\mathbb{N}$ and a decreasing sequence (I_n) in $[M]$ such that

$$\mathcal{U}^{(f_n)}_{\frac{1}{n}}\cap[I_n]^{<\omega}\subseteq(\mathcal{A}_{\xi_{k_n}})^*\setminus\mathcal{A}_{\xi_{k_n}} \text{ for every } n\in\mathbb{N}.$$

Indeed, for $\varepsilon=1$, according to the previous claim, there exists $M_1\in[M]$ such that $\xi^{M}_{\mathcal{U}_1^{(f_n)}}<\xi_{k_1}+1<\xi$. According to the refined ξ_{k_1}-Ramsey type theorem there exists $I_1\in[M_1]$ such that $\mathcal{U}_1^{(f_n)}\cap[I_1]^{<\omega}\subset(\mathcal{A}_{\xi_{k_1}})^*\setminus\mathcal{A}_{\xi_{k_1}}$. For the same reasons there exist $M_2\in[I_1]$ and $k_2\in\mathbb{N}$ such that $\xi^{M_2}_{\mathcal{U}_{\frac{1}{2}}^{(f_n)}}<\xi_{k_2}+1<\xi$ and consequently $I_2\in[I_1]$ such that $\mathcal{U}_{\frac{1}{2}}^{(f_n)}\cap[I_2]^{<\omega}\subset(\mathcal{A}_{\xi_{k_2}})^*\setminus\mathcal{A}_{\xi_{k_2}}$. We continue in analogous way.

Set $\varphi(n)=i_n^n$ if $I_n=(i_m^n)_{m\in\mathbb{N}}$ for every $n\in\mathbb{N}$. For every $\varepsilon>0$ there exists $\lambda\in\mathbb{N}$ with $\frac{1}{\lambda}<\varepsilon$. Set $n_0(\varepsilon)=n_0=k_\lambda$. For each $\gamma\in\Gamma$ we have

$$\{\varphi(n):n\geqslant n_0 \text{ and } |f_{\varphi(n)}(\gamma)-f(\varphi)|\geqslant\varepsilon\}\in(\mathcal{A}_{\xi_{n_0}})^*\setminus\mathcal{A}_{\xi_{n_0}}.$$

(ii) $\Rightarrow$ (i) Let the condition (ii) holds. Suppose that $\xi^{(f_n)}>\xi$. Then there exist $\varepsilon>0$ and $M\in[\mathbb{N}]$ such that $s_M(\mathcal{U}_\varepsilon^{(f_n)})>\xi>\xi_n+1$ for every $n\in\mathbb{N}$. According to the condition (ii) there exist $L\in[M]$ and $n_0\in\mathbb{N}$ such that

$$\mathcal{U}_\varepsilon^{(f_n)}\cap[L]^{<\omega}\subseteq(\mathcal{A}_{\xi_{n_0}})^*\setminus\mathcal{A}_{\xi_{n_0}}.$$

We have $s_L(\mathcal{U}_\varepsilon^{(f_n)})\geqslant s_M(\mathcal{U}_\varepsilon^{(f_n)})\xi_{n_0}+1$, so applying the refined ξ_{n_0}-Ramsey type theorem there exists $I\in[L]$ such that $\mathcal{A}_{\xi_{n_0}}\cap[I]^{<\omega}\subseteq\mathcal{U}_\varepsilon^{(f_n)}\cap[L]^{<\omega}$. This is a contradiction; hence $\xi^{(f_n)}\leqslant\xi$.

For an arbitrary countable ordinal ξ, not necessarily limit, the ξ-uniform convergent sequences have an analogous property, which however does not constitute a characterization. It constitutes a characterization in case we know that $\xi^{(f_n)}$ is a limit ordinal, as in the case of weakly convergent sequences in Banach spaces.

PROPOSITION 3.2. *Let (f_n) be a sequence of real valued functions on a set Γ converging pointwise to a function f and ξ a countable ordinal. If*

(i) *(f_n) converges ξ-uniformly to f, then*

(ii) *for every $M\in[\mathbb{N}]$ there exists a strictly increasing function $\varphi:\mathbb{N}\to M$ such that for every $\varepsilon>0$ there exists $n_0=n_0(\varepsilon)\in\mathbb{N}$ so that*

$$\{\varphi(n):n\geqslant n_0 \text{ and } |f_{\varphi(n)}(\gamma)-f(\gamma)|\geqslant\varepsilon\}\in(\mathcal{A}_\xi)^*\setminus\mathcal{A}_\xi \text{ for every } \gamma\in\Gamma.$$

Statements (i) and (ii) are not equivalent in general. If (ii) holds then either $\xi^{(f_n)}\leqslant\xi$ or $\xi^{(f_n)}=\xi+1$.

Proof. Let ξ be a countable ordinal number with $\xi^{(f_n)}\leqslant\xi$. Then

$$\xi^{M}_{\mathcal{U}_{\frac{1}{n}}^{(f_n)}}=\sup\{s_L(\mathcal{U}_{\frac{1}{n}}^{(f_n)}):L\in[M]\}<\xi+1$$

for every $n\in\mathbb{N}$ and $M\in[\mathbb{N}]$.

According to the refined ξ-Ramsey type theorem, (Theorem 1.6), for every $n\in\mathbb{N}$ and $M\in[\mathbb{N}]$ there exists $L\in[M]$ such that $\mathcal{U}_{\frac{1}{n}}^{(f_n)}\cap[L]^{<\omega}\subseteq(\mathcal{A}_\xi)^*\setminus\mathcal{A}_\xi$.

Inductively, we can find a decreasing sequence $(L_n)_{n\in\mathbb{N}}$ in $[\mathbb{N}]$ such that

$$\mathcal{U}^{(f_n)}_{\frac{1}{n}} \cap [L_n]^{<\omega} \subseteq (\mathcal{A}_\xi)^* \backslash \mathcal{A}_\xi \text{ for every } n\in\mathbb{N}.$$

Set $L = (l^n_n)_{n\in\mathbb{N}}$ if $L_n = (l^n_i)_{i\in\mathbb{N}}$ for every $n \in \mathbb{N}$, and $\varphi(n)=l^n_n$ for every $n\in\mathbb{N}$. For every $\varepsilon>0$ there exists $n_0=n_0(\varepsilon)$ such that $\frac{1}{n_0}<\varepsilon$. Hence, for each $\gamma\in\Gamma$ we have that

$$\{\varphi(n):n\geqslant n_0 \text{ and } |f_{\varphi(n)}(\gamma)-f(\gamma)|\geqslant\varepsilon\}\in\mathcal{U}^{(f_n)}_{\frac{1}{n_0}}\cap[L_{n_0}]^{<\omega}\subseteq(\mathcal{A}_\xi)^*\backslash\mathcal{A}_\xi.$$

Now suppose that (ii) holds and hat $\xi^{(f_n)}\neq\xi+1$. If $\xi^{(f_n)}>\xi+1$, then there exist $\varepsilon>0$ and $M\in[\mathbb{N}]$ such that $s_M(\mathcal{U}^{(f_n)}_\varepsilon)>\xi+1$. According to the refined ξ-Ramsey type theorem there exists $L\in[M]$ such that $\mathcal{A}_\xi[L]^{<\omega}\cap[L]^{<\omega}\subseteq\mathcal{U}^{(f_n)}_\varepsilon$. But according to (ii) there exists $I\in[L]$ such that $\mathcal{U}^{(f_n)}_\varepsilon\cap[I]^{<\omega}\subseteq(\mathcal{A}_\xi)^*\backslash\mathcal{A}_\xi$. This is a contradiction; hence $\xi^{(f_n)}\leqslant\xi$.

With the following example we will prove that the statements (i) and (ii) are not equivalent in general.

EXAMPLE. Let $\Gamma=\{F\in[\mathbb{N}]^{<\omega}:2\leqslant F \text{ and } |F|=\min F-1\}$($|F|$ denotes the cardinality of F). For every $I\in[\mathbb{N}]$ we have $s_I(\Gamma_*)=\omega+1$. Indeed, the family Γ is ω-uniform on $M=\mathbb{N}\backslash\{1\}$ (for the definition see [14] and [7]), so $s_L(\Gamma_*)=\omega+1$ for every $L\in[M]$ according to [7]. This gives that $s_I(\Gamma_*)=\omega+1$ for every $I\in[\mathbb{N}]$ (see Remark 1.5 (v)).

For every $n\in\mathbb{N}$ set $f_n:\Gamma\to\mathbb{R}$ with $f_n(F)=1$ if $n\in F$ and $f_n(F)=0$ otherwise. Of course, the sequence (f_n) converges pointwise to zero and $\mathcal{U}^{(f_n)}_\varepsilon=\Gamma_*$ for every $0<\varepsilon\leqslant1$. Hence, $\xi^{(f_n)}=\omega+1$. But for every $\varepsilon>0$ and $F\in\Gamma$ we have

$$\{n\in\mathbb{N}:\ |f_n(F)|\geqslant\varepsilon\}\in(\mathcal{A}_\omega)^*\backslash\mathcal{A}_\omega.$$

This finishes the proof of the proposition.

As a corollary of Theorem 3.1 we have the following characterization of the ω-uniformly convergent sequences in terms of the uniform pointwise convergence, in the sense given by Mercourakis in [13].

COROLLARY 3.3. *Let (f_n) be a sequence of real valued functions on a set Γ converging pointwise to a function f. The following are equivalent*:

(i) (f_n) *converges ω-uniformly to f (equivalently $\xi^{(f_n)}\leqslant\omega$); and*

(ii) *every subsequence (g_n) of (f_n) has a subsequence (h_n) with the following property: for every $\varepsilon>0$ there exists $n_0(\varepsilon)\in\mathbb{N}$ such that*

$$|\{n\in\mathbb{N}:|h_n(\gamma)-f(\gamma)|\geqslant\varepsilon\}|<n_0(\varepsilon) \textit{ for all } \gamma\in\Gamma.$$

Proof. (i) $\Rightarrow$ (ii) If $\xi^{(f_n)} \leqslant \omega$, then according to Theorem 3.1 every subsequence (g_n) of (f_n) has a further subsequence (h_n) such that for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ so that $\{n \in \mathbb{N} : n \geqslant n_0$ and $|h_n(\gamma) - f(\gamma)| \geqslant \varepsilon\} \in (\mathcal{A}_{n_0})^* \setminus \mathcal{A}_{n_0}$ for every $\gamma \in \Gamma$. This gives that $|\{n \in \mathbb{N} : |h_n(\gamma) - f(\gamma)| \geqslant \varepsilon\}| \leqslant 2n_0$ for every $\gamma \in \Gamma$.

(ii) $\Rightarrow$ (i) This is a consequence of Theorem 3.1, setting $\xi_n = n$ for every $n \in \mathbb{N}$.

REMARK 3.4. (i) For every limit ordinal ξ, by choosing appropriate sequences strictly increasing to ξ (as in Theorem 3.1), we can obtain interesting descriptions of the ξ-uniform convergence. For example in case $\xi = \omega^{\alpha+1}$, choosing the sequence $(n\omega^\alpha)_{n\in\mathbb{N}}$ we obtain that: a sequence (f_n) converges $\omega^{\alpha+1}$-uniformly to f if and only if for every $M \in [\mathbb{N}]$ there exists a strictly increasing sequence $\varphi : \mathbb{N} \to M$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that the type (see Remark 1.2 (ii)) of the set

$$\{\varphi(n) : n \geqslant n_0 \text{ and } |f_{\varphi(n)}(\gamma) - f(\gamma)| \geqslant \varepsilon\}$$

with respect to $\mathcal{B}_\alpha$ is at most n_0.

(ii) Papanastassiou and Kiriakouli in [14], using the generalized Schreier families $(\mathcal{F}_\alpha)_{\alpha<\omega_1}$ instead of $(\mathcal{A}_{\omega^\alpha})_{\alpha<\omega_1}$, defined in a similar way the ω^α-uniform pointwise convergence of a sequence of functions, extending the uniform pointwise convergence of Mercourakis (case $\xi = \omega$).

In order to obtain our principal result (in Theorem 3.13), according to which the ω^α-uniformly convergent sequences are identified with those sequences, for which every subsequence has a uniformly converging (not simply convex, but specifically) $\mathcal{B}_\alpha$-convex block subsequence, we must analyze the close connection that exists between the index of uniform convergence, and the l^1_+-(and l^1-) index of sequences. We define these indices below.

DEFINITION 3.5. Let (χ_n) be a bounded sequence in a Banach space X. For every $\varepsilon > 0$ we set

$$\mathcal{W}_\varepsilon^{(\chi_n)} = \left\{ F \in [\mathbb{N}]^{<\omega} : \varepsilon \sum_{i\in F} \lambda_i \leqslant \left\| \sum_{i\in F} \lambda_i \chi_i \right\| \text{ for every } (\lambda_i)_{i\in F} \subseteq [0, +\infty) \right\}.$$

$$\mathcal{C}_\varepsilon^{(\chi_n)} = \left\{ F \in [\mathbb{N}]^{<\omega} : \varepsilon \sum_{i\in F} |\lambda_i| \leqslant \left\| \sum_{i\in F} \lambda_i \chi_i \right\| \text{ for every } (\lambda_i)_{i\in F} \subseteq \mathbb{R} \right\}.$$

We define the $\mathbf{l^1_+}$**-index** $\xi_+^{(n)}$ of (χ_n) (resp. the $\mathbf{l^1}$**-index** $\xi_1^{(\chi_n)}$ of (χ_n)) as follows:

(1) If there exists $\varepsilon > 0$ such that the family $\mathcal{W}_\varepsilon^{\chi_n)}$ (resp. the family $\mathcal{C}_\varepsilon^{(\chi_n)}$) is not pointwise closed, then we set

$$\xi_+^{(\chi_n)} = \omega_1 \text{ (resp. } \xi_1^{(\chi_n)} = \omega_1); \text{ and}$$

(2) if all the families $\mathcal{W}_\varepsilon^{(\chi_n)}$ (resp. the families $\mathcal{C}_\varepsilon^{(\chi_n)}$) for $\varepsilon>0$ are pointwise closed, then we set

$$\xi_+^{(\chi_n)}=\sup\{s_M(\mathcal{W}_\varepsilon^{(\chi_n)}):M\in[\mathbb{N}] \text{ and } \varepsilon>0\}; \text{ and respectively}$$

$$\xi_1^{(\chi_n)}=\sup\{s_M(\mathcal{C}_\varepsilon^{(\chi_n)}):M\in[\mathbb{N}] \text{ and } \varepsilon>0\}.$$

which are countable ordinals.

PROPOSITION 3.6. *Let* (χ_n) *be a bounded sequence in a Banach space* X. *Then*

(i) $\xi_1^{(\chi_n)}\leqslant\xi_+^{(\chi_n)}$. *These indices are not equal in general.*

(ii) $\xi_1^{(\chi_n)}<\omega_1$ *if and only if* (χ_n) *has not any subsequence equivalent to the unit basis of* l^1 *and equivalently if* (χ_n) *is weak Cauchy.*

(iii) $\xi_+^{(\chi_n)}<\omega_1$ *if and only if* (χ_n) *has not any* l_+^1*-subsequence and equivalently if* (χ_n) *converges weakly to zero.*

(iv) *If* $\xi_+^{(\chi_n)}<\omega_1$, *then* $\xi_+^{(\chi_n)}=\xi_1^{(\chi_n)}=\omega^\zeta$ *for some* $0\leqslant\zeta<\omega_1$.

Proof. (i) $\mathcal{C}_\varepsilon^{(\chi_n)}\subseteq\mathcal{W}_\varepsilon^{(\chi_n)}$ for every $\varepsilon>0$, hence $\xi_1^{(\chi_n)}\leqslant\xi_+^{(\chi_n)}$. The two indices are not equal in general. Indeed, let (s_n) be the summing basis of c_0. Of course $\|s_n\|=1$ for every $n\in\mathbb{N}$. It is easily proved that $\xi_1^{(\chi_n)}=\omega$, but $\xi_+^{(\chi_n)}=\omega_1$.

(ii) $\xi_1^{(\chi_n)}=\omega_1$ if and only if (χ_n) has an l^1-subsequence (Theorem 1.6). So, $\xi_1^{(\chi_n)}<\omega_1$ if and only if (χ_n) is weak Cauchy according to Rosenthal's l^1-theorem [16].

(iii) $\xi_+^{(\chi_n)}=\omega_1$ if and only if (χ_n) has an l_+^1-subsequence, (Theorem 1.6). Hence $\xi_+^{(\chi_n)}<\omega_1$ if and only if (χ_n) converges weakly to zero, according to Theorem 2.5.

(iv) Let $\xi_+^{(\chi_n)}<\omega_1$. Firstly, we will prove that $\xi_+^{(\chi_n)}=\omega^\zeta$ for some $0\leqslant\zeta<\omega_1$. Since $\xi_+^{(\chi_n)}<\omega_1$ there exists a unique countable ordinal ζ with $\omega^\zeta\leqslant\xi_+^{(f_n)}<\omega^{\zeta+1}$. Arguing by contradiction, we suppose that $\omega^\zeta<\xi_+^{(\chi_n)}$. Then there exist $M\in[\mathbb{N}]$ and $\varepsilon>0$ so that $\omega^\zeta<s_M(\mathcal{W}_\varepsilon^{(\chi_n)})$. Since $1<s_M(\mathcal{W}_\varepsilon^{(\chi_n)})$, there exists $I\in[M]$ with $\inf\{\|\chi_n\|:n\in I\}\geqslant\varepsilon$. Of course $\omega^\zeta<s_M(\mathcal{W}_\varepsilon^{(\chi_n)})\leqslant s_I(\mathcal{W}_\varepsilon^{(\chi_n)})$. So, according to Remark 1.12, there exists $L\in[I]$ such that $\mathcal{F}_\zeta(L)\subseteq\mathcal{W}_\varepsilon^{(\chi_n)}$. If $L=(l_n)_{n\in\mathbb{N}}$, we set $y_n=\chi_{l_n}$ for every $n\in\mathbb{N}$. Since $\xi_+^{(y_n)}<\omega_1$, the sequence (y_n) is weakly null and seminormalized, so it has a basic subsequence (z_n) with basic constant $C\geqslant 1$.

Let $k\in\mathbb{N}$ and $s_1,\dots,s_k\in\mathcal{B}_\zeta$ with $s_1<\cdots<s_k$ and $s=\bigcup_{n=1}^k s_n$. Since (z_n) is basic, for every $(\lambda_i)_{i\in s}\subseteq[0,+\infty)$ we have $\|\sum_{i\in s}\lambda_i z_i\|\geqslant\frac{1}{4C}\|\sum_{i\in s_n}\lambda_i z_i\|$ for every $n=1,\dots,k$, and consequently $\|\sum_{i\in s}\lambda_i z_i\|\geqslant\frac{1}{4kC}\sum_{n=1}^k\|\sum_{i\in s_n}\lambda_i z_i\|\geqslant\frac{\varepsilon}{4kC}\sum_{i\in s}\lambda_i$, since $\mathcal{B}_\zeta\subseteq\mathcal{F}_\zeta\subseteq\mathcal{W}_\varepsilon^{(z_n)}$. This gives that $\mathcal{A}_{k\omega^\zeta}\subseteq\mathcal{W}_{\delta_k}^{(z_n)}$ for every $k\in\mathbb{N}$, where $\delta_k=\frac{\varepsilon}{4kC}$. So, $s_\mathbb{N}(\mathcal{W}_{\delta_k}^{(z_n)})\geqslant k\omega^\zeta$ for every $k\in\mathbb{N}$. If $z_n=\chi_{m_n}$ for every $n\in\mathbb{N}$, then setting $N=\{m_n:n\in\mathbb{N}\}$, we can verify that $s_N(\mathcal{W}_{\delta_k}^{(\chi_n)})\geqslant k\omega^\zeta$ for every $k\in\mathbb{N}$. Hence, $\xi_+^{(\chi_n)}\geqslant k\omega^\zeta$ for every $k\in\mathbb{N}$. But it is impossible, since we assumed that $\xi_+^{(\chi_n)}<\omega^{\zeta+1}$. Thus $\xi_+^{(\chi_n)}=\omega^\zeta$.

Now, we will prove that $\xi_1^{(\chi_n)}=\xi_+^{(\chi_n)}$. If $\zeta=0$, then $\|\chi_n\|\to 0$ and obviously $\xi_1^{(\chi_n)}=1=\xi_+^{(\chi_n)}$. We assume that $1\leqslant\zeta$. Let ξ be an ordinal with $1<\xi<\omega^\zeta$. Then there exists $M\in[\mathbb{N}]$ and $\varepsilon>0$ such that $\xi<s_M(\mathcal{W}_\varepsilon^{(\chi_n)})$ and moreover $\inf\{\|\chi_n\|:n\in M\}\geqslant\varepsilon$. The sequence $(\chi_n)_{n\in M}$ is weakly null and seminormalized. According to the "l^1-unconditional" theorem of Argyros-Mercourakis-Tsarpalias in [3], there exist $L\in[M]$ and $C(\varepsilon)>0$ so that $\mathcal{W}_\varepsilon^{(\chi_n)}\cap[L]^{<\omega}\subseteq\mathcal{C}_{C(\varepsilon)}^{(\chi_n)}$. Since $\xi<s_M(\mathcal{W}_\varepsilon^{(\chi_n)})\leqslant s_L(\mathcal{W}_\varepsilon^{(\chi_n)})\leqslant s_L(\mathcal{C}_{C(\varepsilon)}^{(\chi_n)})$, we have that $\xi_1^{(\chi_n)}>\xi$, and consequently that $\xi_1^{(\chi_n)}\geqslant\omega^\zeta=\xi_+^{(\chi_n)}$. Hence $\xi_1^{(\chi_n)}=\xi_+^{(\chi_n)}=\omega^\zeta$.

This finishes the proof.

Now we will prove the close connection that exists between the index of uniform convergence and the l_+^1-index of a sequence of functions. Already, we have proved in Theorem 2.5 that the index of uniform convergence of a sequence (f_n) (defined on a set Γ and converging pointwise to a function f) is countable if and only if the $\ell_+^1=$-index of (f_n-f) is countable. This relation which corresponds to the limiting ω_1-case, is extended to any ordinal, by proving (in Theorem 3.11) that if the index of uniform convergence is a countable ordinal ξ, then the l_+^1-index (as well as the l^1-index) is equal to ω^ζ, where ζ is the least ordinal with $\xi<\omega^\zeta$, (turns out to be equal to $\xi_B^{(f_n)}$, c.f. Theorem 2.10).

In order to prove this relation, firstly we will characterize all the ordinals α with $\omega^\alpha<\xi^{(f_n)}$ in Proposition 3.10 below. In this characterization we use the notion of the l^1-spreading model of order α, for some $1\leqslant\alpha<\omega_1$, (Definition 3.7), and also the notion of the convex combination of a sequence with respect to an element of the family $\mathcal{B}_\alpha$ (Definition 3.9).

DEFINITION 3.7. Let (χ_n) be a bounded, sequence in a Banach space X and α be a countable ordinal with $1\leqslant\alpha<\omega_1$. We say that (χ_n) has **l^1-spreading model** of order α if there exists $\varepsilon>0$ such that

$$\varepsilon\sum_{i\in F}|\lambda_i|\leqslant\left\|\sum_{i\in F}\lambda_i\chi_i\right\|\quad\text{for every } F\in\mathcal{F}_\alpha \text{ and } (\lambda_i)_{i\in F}\subseteq\mathbb{R}.$$

REMARK 3.8. (i) A sequence (χ_n) has a subsequence with l^1-spreading model of order α if and only if $\omega^\alpha<\xi_1^{(\chi_n)}$. Indeed, if $\omega^\alpha<\xi_1^{(\chi_n)}$, then there exist $\varepsilon>0$ and $M\in[\mathbb{N}]$ such that $\omega^\alpha<s_M(\mathcal{C}_\varepsilon^{(\chi_n)})$ and consequently $L\in[M]$ with $\mathcal{F}_\alpha(L)\subseteq\mathcal{C}_\varepsilon^{(\chi_n)}$ (Remark 1.12 (v)). On the other hand, if $(y_n)=(\chi_n)_{n\in M}$ has l^1-spreading model of order α, then there exists $\varepsilon>0$ so that $\mathcal{B}_\alpha\subseteq\mathcal{C}_\varepsilon^{(y_n)}$. Hence $\xi_1^{(\chi_n)}\geqslant s_M(\mathcal{C}_\varepsilon^{(\chi_n)})\geqslant s_{\mathbb{N}}(\mathcal{C}_\varepsilon^{(y_n)})\geqslant s_{\mathbb{N}}(\mathcal{B}_\alpha)=\omega^\alpha+1>\omega^\alpha$.

(ii) A bounded sequence (χ_n) has a subsequence with l^1-spreading model of the greatest possible order if and only if either $\xi_1^{(\chi_n)}=\omega_1$, or $\xi_1^{(\chi_n)}=\omega^{\zeta+1}$ for some countable ordinal ζ.

(iii) If a bounded sequence (χ_n) has for every countable ordinal α a subsequence with l^1-spreading model of order α, then it has a subsequence equivalent to the usual basis of l^1.

DEFINITION 3.9. *Let (χ_n) be a sequence in a Banach space X and α be a countable ordinal. Then*

(1) The **convex combination** $c_s^\alpha(\chi_n)$ *of the sequence* (χ_n) **with respect to an element** s **of** $\mathcal{B}_\alpha$ *is defined as follows*:

$$c_s^\alpha(\chi_n)=\sum_{n\in s}\varphi_s^\alpha(n)\chi_n;$$

where $\varphi_s^\alpha:\mathbb{N}\to[0,\infty)$ is the finite supported function defined in Remark 1.8.

(2) A sequence (z_k) in X is a $\mathcal{B}_\alpha$-**convex block subsequence of** (χ_n), *if and only if $z_k=c_{s_k}^\alpha(\chi_n)$ for every $k\in\mathbb{N}$, where $(s_k)\subseteq\mathcal{B}_\alpha$ with $s_1<s_2<\cdots$. The set $\bigcup_{k\in\mathbb{N}}s_k$ is called the* **support** *of (z_k).*

Using the canonical representation with respect to the family $\mathcal{B}_\alpha$ (Remark 1.2), it can be proved that for every $M\in[\mathbb{N}]$ there exists a **unique** $\mathcal{B}_\alpha$-convex block subsequence of (χ_n) with support M.

PROPOSITION 3.10. *Let (f_n) be a uniformly bounded sequence of real valued functions defined on a set Γ which converges pointwise to a function f and α be a countable ordinal. If $\xi^{(f_n)}<\omega_1$ the following are equivalent*:

(i) $\omega^\alpha<\xi^{(f_n)}$;

(ii) *there exists a subsequence (g_n) of (f_n), $\varepsilon>0$ and $\theta\in\{-1,1\}$ such that, for every $F\in\mathcal{F}_\alpha$, there exists $\gamma\in\Gamma$ so that $\theta\cdot(g_i(\gamma)-f(\gamma)\geqslant\varepsilon$ every $i\in F$;*

(iii) $\omega^\alpha<\xi_+^{(f_n-f)}=\xi_1^{(f_n-f)}$;

(iv) *there exists a subsequence of (f_n-f) with l^1-spreading model or order α*;

(v) *there exist a subsequence (g_n) of (f_n), $\varepsilon>0$ and $I\in[\mathbb{N}]$ such that*

$$\varepsilon\leqslant\|c_s^\alpha(g_n)-f\| \text{ for every } s\in\mathcal{B}_\alpha\cap[I]^{<\omega}.$$

Proof. (i) $\Rightarrow$ (ii) This is already proved in Lemma 2.8.

(ii) $\Rightarrow$ (iii) According to condition (ii) there exist a subsequence $(g_n)=(f_n)_{n\in M}$ of (f_n) and $\varepsilon>0$ such that $\mathcal{B}_\alpha\subseteq\mathcal{F}_\alpha\subseteq\mathcal{W}_\varepsilon^{(f_n)}$. Then

$$\omega^\alpha<\omega^\alpha+1=s_{\mathbb{N}}(\mathcal{W}_\varepsilon^{(g_n-f)})\leqslant s_M(\mathcal{W}_\varepsilon^{(f_n-f)})\leqslant\xi_+^{(f_n-f)}.$$

Proposition 3.6 finishes the proof.

(iii) $\Rightarrow$ (iv) It is proved in Remark 3.8 (i) .

(iv) $\Rightarrow$ (v) It is obvious.

(v) $\Rightarrow$ (i) Let (g_n) be a subsequence of $(f_n),\varepsilon>0$ and $I\in[\mathbb{N}]$ such that $\varepsilon\leqslant\|c_s^\alpha(g_n)-f\|$ for every $s\in\mathcal{B}_\alpha\cap[I]^{<\omega}$, and let $s\in\mathcal{B}_\alpha\cap[I]^{<\omega}$. Then there exists $\gamma\in\Gamma$ such that $\frac{\varepsilon}{2}<|\sum_{n\in s}\varphi_s^\alpha(n)(g_n-f)(\gamma)|\leqslant\|c_s^\alpha(g_n)-f\|$.

Set $F=\{n\in s:|(g_n-f)(\gamma)|>\frac{\varepsilon}{4}\}$. Then $F\in\mathcal{U}^{(g_n)}_{\frac{\varepsilon}{4}}$ and

$$\frac{\varepsilon}{2}<\left|\sum_{n\in F}\varphi^{\alpha}_s(n)(g_n-f)(\gamma)\right|+\left|\sum_{n\in s\setminus F}\varphi^{\alpha}_s(n)(g_n-f)(\gamma)\right|\leqslant w_{\omega^\alpha}(F;s)\delta+\frac{\varepsilon}{4},$$

where $\delta=\sup\|g_n-f\|$. Hence, $w_{\omega^\alpha}(F;s)>\frac{\varepsilon}{4\delta}$.

According to the ω^α-Pták type theorem (Theorem 1.9) there exists $L\in[I]$ such that $s_L(\mathcal{U}^{(g_n)}_{\frac{\varepsilon}{4}})>\omega^\alpha$. So we have $s_M(\mathcal{U}^{(f_n)}_{\frac{\varepsilon}{4}})>\omega^\alpha$ for $M=\{m_n:n\in L\}$, and consequently $\omega^\alpha<\xi^{(f_n)}$, as required.

This finishes the proof.

THEOREM 3.11. *Let (f_n) be a uniformly bounded sequence of real valued functions defined on set Γ which converges pointwise to a function f. Then*

(i) $\xi^{(f_n)}=\omega_1$ *if and only if* $\xi^{(f_n-f)}_+=\omega_1$.

(ii) *If* $\xi^{(f_n)}<\omega_1$, *then* $\xi^{(f_n-f)}_1=\xi^{(f_n-f)}_+=\omega^\zeta$, *where ζ is the least countable ordinal with* $\xi^{(f_n)}\leqslant\omega^\zeta$.

Proof. (i) It follows from Theorem 2.5 and Proposition 3.6.

(ii) Let $\xi^{(f_n)}<\omega_1$. Then $\xi^{(f_n-f)}_+<\omega_1$ (according to (i)) and moreover $\xi^{(f_n-f)}_1=\xi^{(f_n-f)}_+=\omega^\zeta$ for some countable ordinal ζ (Proposition 3.6). According to Proposition 3.10 we have that

$$\{0\leqslant\alpha<\omega_1:\omega^\alpha<\xi^{(f_n)}\}=\{0\leqslant\alpha<\omega_1:\alpha<\zeta\}.$$

This equality gives that ζ is the least ordinal with $\xi^{(f_n)}\leqslant\omega^\zeta$

COROLLARY 3.12. *Let (χ_n) be a sequence in a Banach space X and $\chi\in X$. If (χ_n) converges weakly to χ, then* $\xi^{(\chi_n)}=\xi^{(\chi_n-\chi)}_+=\xi^{(\chi_n-\chi)}_1=\omega^\zeta$ *for some countable ordinal ζ.*

Proof. It follows from Theorem 3.11 and Theorem 2.9.

Now, we can state our principal result, according to which the ω^α-uniformly convergent sequences of functions are identified with the sequences whose every subsequence has a uniformly convergent $\mathcal{B}_\alpha$-convex block subsequence (thus completing Theorem 3.1) and also with the sequences which do not have subsequences with l^1-spreading model of order α.

THEOREM 3.13. *Let (f_n) be a uniformly bounded sequence of real valued functions defined on a set Γ which converges pointwise to a function f. For a countable ordinal α the following are equivalent*:

(i) *The sequence (f_n) converges ω^α-uniformly to f (or equivalently $\xi^{(f_n)}\leqslant\omega^\alpha$)*;

(ii) $\xi^{(f_n-f)}_+=\xi^{(f_n-f)}_1\leqslant\omega^\alpha$;

(iii) *the sequence (f_n-f) in $l^\infty(\Gamma)$ does not have any subsequence with l^1-spreading model of order α*;

(iv) *for every subsequence* (g_n) *of* (f_n) *and* $M\in[\mathbb{N}]$ *there exists* $L\in[M]$ *such that*

$$\|c_s^{\alpha}(g_n)-f\|<\frac{1}{\min s}\ \textit{for every } s\in\mathcal{B}_\alpha\cap[L]^{<\omega};\ \textit{and}$$

(v) *every subsequence* (g_n) *of* (f_n) *has a* $\mathcal{B}_\alpha$*-convex block subsequence converging uniformly to* f.

Proof. (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii) They follow from Proposition 3.10.

(i) $\Rightarrow$ (iv) Let $\xi^{(f_n)}\leqslant\omega^\alpha$. According to Proposition 3.10 ((i) $\Leftrightarrow$ (v)) for every subsequence (g_n) of (f_n), $\varepsilon>0$ and $L\in[\mathbb{N}]$ there exists $s\in\mathcal{B}_\alpha\cap[L]^{<\omega}$ such that $\|c_s^{\alpha}(g_n)-f\|<\varepsilon$. Let (g_n) be a subsequence of (f_n) and $M\in[\mathbb{N}]$. We set $\mathcal{F}_1=\{F\in\mathcal{B}_\alpha:\|c_s^{\alpha}(g_n)-f\|<1\}$. Then $\mathcal{B}_\alpha\cap[L]^{<\omega}\cap\mathcal{F}_1\neq\emptyset$ for every $L\in[M]$. According to ω^α-Ramsey type theorem (Theorem 1.3) there exists $L_1\in[M]$ such that $\|c_s^{\alpha}(g_n)-f\|<1$ for every $s\in\mathcal{B}_\alpha\cap[L_1]^{<\omega}$. Set $\mathcal{F}_2=\{s\in\mathcal{B}_\alpha:\|c_s^{\alpha}(g_n)-f)\|<\frac{1}{2}\}$. Since $\mathcal{B}_\alpha\cap[L]^{<\omega}\cap\mathcal{F}_2\neq\emptyset$ for every $L\in[L_1]$, there exists $L_2\in[L_1]$ such that $\|c_s^{\alpha}(g_n)-f)\|<\frac{1}{2}$ for every $s\in\mathcal{B}_\alpha\cap[L_2]^{\omega}$.

Inductively, we can find a decreasing sequence (L_n) in $[\mathbb{N}]$ such that

$$\|c_s^{\alpha}(g_n)-f\|<\frac{1}{n}\ \text{for every } n\in\mathbb{N} \text{ and } s\in\mathcal{B}_\alpha\cap[L_n]^{<\omega}.$$

Set $L=(l_n^n)_{n\in\mathbb{N}}$ if $L_n=(l_i^n)_{i\in\mathbb{N}}$ for every $n\in\mathbb{N}$.

For every $s=(n_1,\dots,n_k)\in\mathcal{B}_\alpha\cap[L]^{<\omega}$ we have that $s\in\mathcal{B}_\alpha\cap[L_{n_1}]^{<\omega}$, so

$$\|c_s^{\alpha}(g_n)-f\|<\frac{1}{n_1}=\frac{1}{\text{mins}}.$$

(iv) $\Rightarrow$ (v) Let (g_n) be a subsequence of (f_n). Then there exists $L\in[\mathbb{N}]$ such that

$$\|c_s^{\alpha}(g_n)-f\|<\frac{1}{\text{mins}}\ \text{for every } s\in\mathcal{B}_\alpha\cap[L]^{<\omega}.$$

According to [7] (see Definition 3.9) there exists a (unique) sequence (s_k) in $\mathcal{B}_\alpha$ with $s_1<s_2<\cdots$ and $L=\bigcup_{k=1}^{\infty}s_k$. Obviously, the $\mathcal{B}_\alpha$-convex block subsequence $(c_{s_k}^{\alpha}(g_n))$ of (g_n) converges uniformly to f.

(v) $\Rightarrow$ (iii) If a subsequence of (f_n) has a $\mathcal{B}_\alpha$-convex block subsequence converging uniformly to f, then obviously it has not l^1-spreading model of order α.

Now, we are in position to gather previous results and hereby state a dichotomy result that can be considered as the natural extension of Rosenthal's classical l^1-theorem [16] (recalled here for convenience) from the ω_1-ordinal to the countable ordinals.

ROSENTHAL'S THEOREM ([16]). *Let* (f_n) *be a uniformly bounded sequence of real valued functions defined on a set* Γ. *Then*

either (1) (f_n) *has a subsequence equivalent to the unit basis of* l^1;

or (2) (f_n) *has a pointwise convergent subsequence.*

THEOREM 3.14. *Let (f_n) be a uniformly bounded sequence of real valued functions defined on a set Γ and pointwise convergent to a function f. Then*
either [**Case** $\xi^{(f_n)} = \omega_1$]$(f_n - f)$ *has an l^1_+-subsequence*;
or [**Case** $\xi^{(f_n)} < \omega_1$](f_n) *converges weakly to f in the Banach space $l^\infty(\Gamma)$. Fixing the least countable ordinal ζ for which $\xi^{(f_n)} \leqslant \omega^\zeta$, we obtain for every countable ordinal α the further dichotomy*:
either [**Case** $\alpha < \zeta$] *there exists a subsequence of $(f_n - f)$ with l^1-spreading model of order α*;
or [**Case** $\alpha \geqslant \zeta$] *the sequence (f_n) converges ω^α-uniformly to f or equivalently every subsequence of (f_n) has a $\mathcal{B}_\alpha$-convex block subsequence converging uniformly to f.*

Applying the previous result to the general Banach space theory we have the following:

THEOREM 3.15. *Let (χ_n) be a bounded sequence in a Banach space X.*

(1) *either the (χ_n) has a subsequence equivalent to the unit basis of l^1; or (χ_n) has a weak Cauchy subsequence.*
(2) *If (χ_n) is a weak Cauchy sequence converging weak* to an element χ^{**} of x^{**}, either $(\chi_n - x^{**})$ has an l^1_+-subsequence; or (χ_n) converges weakly to some element χ of X.*
(3) *If (χ_n) converges weakly to some element χ of X, then there exists a (unique) ordinal ζ such that $\xi^{(\chi_n)} = \xi_+^{(\chi_n - \chi)} = \xi_1^{(\chi_n - \chi)} = \omega^\zeta$. So, for each countable ordinal α we obtain the following*:
    (3i) *either* [**Case** $\alpha < \zeta$] *there exists a subsequence of $(\chi_n - \chi)$ with l^1-spreading model of order α*;
    (3ii) *or* [**Case** $\alpha \geqslant \zeta$] *the sequence (χ_n) converges ω^α-uniformly to χ, or equivalently every subsequence of (χ_n) has a $\mathcal{B}_\alpha$-convex block subsequence norm converging to χ.*

We conclude with the observation that we classify the class W^X of all the weakly convergent sequences of a Banach space X into an increasing hierarchy $(W^X_\alpha)_{0 \leqslant \alpha < \omega_1}$, where

$$W^X_\alpha = \{(\chi_n) \subseteq X : (\chi_n) \text{ is weakly convergent and } \xi^{(\chi_n)} \leqslant \omega^\alpha\}.$$

With the help of the previous theorem these classes can be characterized as follows.

COROLLARY 3.16. *Let X be a Banach space, (χ_n) a sequence in X converging weakly to some $\chi \in X$. For each countable ordinal number α*
(i) *$(\chi_n) \in W^X_\alpha$ if and only if for every subsequence (y_n) of (χ_n) and $M \in [\mathbb{N}]$ there exists a B_a-convex block subsequence of (y_n) supported on M and norm convergent to χ; and*

(ii) $(\chi_n) \in W^X \setminus W^X_\alpha$ *if and only if there exists a subsequence of* (χ_n) *with* l^1-*spreading model of order* α.

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