Ramsey and Nash-Williams combinatorics via Schreier families

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(Preliminary Version)

Abstract

The main results of this paper (a) extend the finite Ramsey partition theorem, and (b) employ this extension to obtain a stronger form of the infinite Nash-Williams partition theorem, and also a new proof of Ellentuck's, and hence Galvin-Prikry's partition theorem. The proper tool for this unification of the classical partition theorems at a more general and stronger level is the system of Schreier families $(\mathcal{A}_\xi)$ of finite subsets of the set of natural numbers, defined for every countable ordinal ξ.

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Introduction

The main results of this papers (Theorem A) extend the finite Ramsey ([R], 1929) partition theorem, and employ this extension to obtain a stronger form of the infinite Nash-Williams ([N-W], 1965) partition theorem (Theorems B, B′, C), and also a new proof of Ellentuck's ([E], 1974), and hence Galvin-Prikry's ([G-P], 1973) partition theorems.

Rather unexpectedly the ideas that lead to this purely combinatorial result have developed, in parallel, in Banach space theory, and they involve the system of Schreier families ($\mathcal{A}_\xi$) of finite subsets of the set $\mathbb{N}$ of natural numbers defined for every countable ordinal ξ. For the purpose of constructing a famous counterexample in the theory of Banach spaces, Schreier ([S], 1930) devised the classical Schreier family $\mathcal{F}_1$. This family definitely came into the attention of Banach space theory with the also famous Tsirelson counterexample ([T], 1974), a construction that uses the Schreier family. In 1992 Alspach-Argyros ([A-A]) introduced the generalized Schreier families $\mathcal{F}_\alpha$ for every countable ordinal α, and they used this family for the construction of Tsirelson type spaces, described by countable ordinals. In my 1994 paper ([F1]), working on some refinements of Rosenthal's ([R])paper on c_0 , it was first realized that the generalized Schreier families $\mathcal{F}_\alpha$, corresponded to the ordinal ω^α, and that these was room for defining the intermediate generalized Schreier families $\mathcal{A}_\xi$ for every countable ordinal ξ, thus filling the internals between ω^α and $\omega^{\alpha+1}$, and in such a way that the earlier generalized Schreier families $\mathcal{F}_\alpha$ essentially coincide with $\mathcal{A}_{\omega^\alpha}$. Independently, and working on a quite different problem (on distortable Banach spaces), Tomczak-Jaegermann ([TJ], 1996), considered, at the suggestion of B. Maurey as she mentions, a variation of such families.

At this point we might say that the complete system of thin Schreier families ($\mathcal{A}_\xi$) (defined in 1.3 below), although a purely combinatorial object, arose in Banach space theory. There were some Banach space results that in retrospect can be considered as witnesses of a Ramsey type dichotomy nature of these families or of their predecessors (Kiriakouli-Negrepontis ([M-N], 1992), Farmaki ([F1,F2], 1994), Argyros-Mercourakis-Tsarpalias ([A-M-T], 1998), Judd ([J], 1999)).

In 1998 ([F4]) the study of the families ($\mathcal{A}_\xi$) was refined and employed for the proof of a far-reaching extension of the classical Ramsey theorem (Theorem A in this paper), one that holds for every countable ordinal ξ, of which the initial part, concerned with finite ordinals, coincides with the classical Ramsey theorem. Denoting by $[L]^{<\omega}$ the family of all finite subsets of a set L, here is the statement of the theorem:

Theorem A (Ramsey partition theorem extended to countable ordinals). Let $\mathcal{F}$ be an arbitrary family of finite subsets of $\mathbb{N}$, M an infinite subset of $\mathbb{N}$ and ξ a countable ordinal number. Then, there exists an infinite subset L of M such that

$$\text{either } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F} .$$

Since $\mathcal{A}_n = [\mathbb{N}]^n$ for any finite ordinal $n < \omega$, Theorem A (= Theorem 1.5 below) is in fact an extension of the classical Ramsey partition theorem from partitions on the families of n-tuples to (roughly speaking) partitions on the families $\mathcal{A}_\xi$ for any countable ordinal ξ.

The extended Ramsey Theorem A implies, strengthened forms of the Nash-Williams partition Theorem (Theorems B,B′,C). We will employ Theorem A not for arbitrary, but only for hereditary families $\mathcal{F}$ of finite subsets of $\mathbb{N}$. For such families $\mathcal{F}$ the strong Cantor-Bedixson index $s_L(\mathcal{F})$ (Proposition 2.9) together with the canonical decomposition of any subset of $\mathbb{N}$ w.r.t. $(\mathcal{A}_\xi)$ (Proposition 2.4) imply a criterion that allows us to decide (in most cases) which horn of the dichotomy provided by Theorem A will actually hold. Denoting by [L] the family of all infinite subsets of a set L, and by $\mathcal{A}_\xi^\star$ the family of all the initial segments of elements of $\mathcal{A}_\xi$, here is the statement of the theorem:

Theorem B (Stronger form of Nash-Williams partition theorem for hereditary families). Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}$ and M an infinite subset of $\mathbb{N}$. We have the following cases:

[Case 1] If the family $\mathcal{F} \cap [M]^{<\omega}$ is not pointwise closed, then there exists $L \in [M]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$.

[Case 2] If the family $\mathcal{F} \cap [M]^{<\omega}$ is pointwise closed, then setting

$$\xi_M^{\mathcal{F}} = sup\{s_L(\mathcal{F}) : L \in [M]\} ;$$

which is a countable ordinal, the following subcases obtain:

2(i) For every countable ordinal ξ with $\xi + 1 < \xi_M^{\mathcal{F}}$ there exists $L \in [M]$ such that

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} .$$

2(ii) For every countable ordinal ξ with $\xi_M^{\mathcal{F}} < \xi + 1$ there exists $L \in [M]$ such that

$$\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi ;$$

and equivalently,

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F} .$$

2(iii) If $\xi_M^{\mathcal{F}} = \xi + 1$, then both alternatives may materialize.

It is probably not apparent to the reader, why Theorem B (= Theorem 3.7 below) is in fact a result that deserves to be called a (strong) form of the Nash-Williams partition theorem. A convenient way to see this is by considering the reformulation that Gowers([G], 2002) gave of that theorem, and which can be stated as follows:

Nash-Williams partition theorem (in Gowers reformulation). Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$. Then there exists an infinite subset L of $\mathbb{N}$, such that

either (i) $[L]^{<\omega} \subseteq \mathcal{F}$;

or (ii) for every infinite subsets I of L, there exists an initial segment s of I which belongs to $[\mathbb{N}]^{<\omega} \setminus \mathcal{F}$.

Furthermore we remark that it is easy to see that WLOG we may assume in this reformulation that $\mathcal{F}$ be a **tree** of finite subsets of $\mathbb{N}$ (cf. Remark 3.12).

A slightly weaker version of Theorem B, Theorem B$'$, (= Theorem 3.10) concerns trees, and not necessarily hereditary families, of finite subsets of $\mathbb{N}$, bringing our result to a closer relation with the (tree form) Gowers reformulation of the Nash-Williams partition theorem.

Let us consider a further consequence of Theorem B$'$ that brings forth in a clear manner the way in which our approach yields a result substantially stronger than the classical Nash-Williams result. The statement involves the decomposition (mentioned above) of any subset of $\mathbb{N}$ w.r.t. the system $(\mathcal{A}_\xi)_{\xi<\omega_1}$. In fact, by Proposition 2.4 below, every (infinite or finite) subset I of $\mathbb{N}$ has a unique canonical representation w.r.t. each Schreier family $\mathcal{A}_\xi$, in such a way that for for every $\xi < \omega_1$, there is a unique initial segment $s_{\xi,I}$ of I that belongs to the Schreier family $\mathcal{A}_\xi$.

Theorem C (Stronger form of Nash-Williams partition theorem in Gowers reformulation). Let $\mathcal{F}$ be a tree of finite subsets of $\mathbb{N}$. Then there exists an infinite subset L of $\mathbb{N}$, such that

either (i) $[L]^{<\omega} \subseteq \mathcal{F}$;

or (ii) there is a countable ordinal ξ_0, such that for every infinite subsets I of L, there exists an initial segment s of I which belongs to $[\mathbb{N}]^{<\omega} \setminus \mathcal{F}$, and which is that unique initial segment of I that belongs to $\mathcal{A}_{\xi_0}$.

Compare this with the treeform Gowers reformulation of the Nash-Williams theorem stated above. It is seen that our strengthened version provides, in the second horn of the dichotomy, not only the existence of the finite initial segments s_I if I (for all infinite subsets I of L), but their determination by a countable ordinal ξ_0 in a **unique** and **uniform** way: thus the segment s_I of I, does not simply exist as provided by the classical Nash-Williams result, but is that unique finite initial segment of I, which, according to the general decomposition of every subset of $\mathbb{N}$ w.r.t. the system $(\mathcal{A}_\xi)_{\xi<\omega_1}$, is an element of the family $\mathcal{A}_{\xi_0}$.

Ellentuck's theorem (not in a stronger form though), and thus the Galvin-Prikry partition theorem, follows also from our Theorem B (cf. Theorem 4.6, Corollary 4.9, Remark 4.10).

On the basis of these results, it is reasonable to conclude that the Schreier system $(\mathcal{A}_\xi)_{\xi<\omega_1}$, proves to be the correct combinatorial tool for the unification, extension and strengthening of all the finite Ramsey and infinite Nash-Williams partition theorems. These extended and strengthened partition theorems will no doubt find many applications, not only in the theory of Banach spaces, but in all the various areas where the classical combinatorial partition theorems have proved amply fruitful.

Notation. We denote by $\mathbb{N}$ the set of all natural numbers. For an infinite subset M of $\mathbb{N}$ we denote by $[M]^{<\omega}$ the set of all finite subsets of M, for $k \in \mathbb{N}$ we denote by $[M]^k$ the set of all $k-$element subsets of M, and by $[M]$ the set of all infinite subsets of M (considering them as strictly increasing sequences).

If s, t are non empty subsets of $\mathbb{N}$, then $s \preceq t$ means that s is an initial segment of t, while $s \prec t$ means that s is a proper initial segment of t. We write $s \leq t$ if $max\ s \leq min\ t$, while $s < t$ if $max\ s < min\ t$.

Identifying every subset of $\mathbb{N}$ with its characteristic function, we topologize the set of all subsets of $\mathbb{N}$ by the topology of pointwise convergence.

1. The complete thin Schreier system and the Ramsey partition theorem extended to countable ordinals

The main result in this section is a Ramsey type theorem for every countable ordinal ξ (Theorem 1.5 - Theorem A), which can be considered as the countable ordinal analogue of the classical Ramsey theorem. This theorem is stated for the complete thin Schreier system $(\mathcal{A}_\xi)_{\xi<\omega_1}$, defined 1.3.

We recall Ramsey's classical partition theorem.

Theorem 1.1 (**Ramsey** [R]). Let $\mathcal{F}$ be an arbitrary family of finite subsets of $\mathbb{N}$, M an infinite subset of $\mathbb{N}$ and k a natural number. Then there exists an infinite subset L of M such that either $[L]^k \subseteq \mathcal{F}$ or $[L]^k \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$.

This classical Ramsey partition theorem will prove to be, in Theorem A below, the initial segment of a whole family of Ramsey type partition results, one for every countable ordinal ξ.

In order to arrive at the statement of the Ramsey partition theorem for any countable ordinal ξ we need a ξ-ordinal analogue $\mathcal{A}_\xi$ of $[\mathbb{N}]^k = \mathcal{A}_k$. This is accomplished for every $\xi < \omega_1$, by a rather laborious transfinite induction, that depends essentially on a (classical) representation of (limit) ordinals, involving the ordinal analogue of Euclidean algorithm as follows:

Proposition 1.2 (**Representation of ordinals**, [C2] [L]). Let α be a non-zero, countable ordinal. For every limit ordinal ξ, so that $\omega^\alpha < \xi < \omega^{\alpha+1}$, there exist a unique natural number $m \geq 0$, a sequence of ordinals $\alpha > \alpha_1 \ldots > \alpha_m > 0$ and natural numbers $p, p_1, \ldots, p_m \geq 1$ (so that either $p > 1$ or $p = 1$ and $m \geq 1$), such that $\xi = p\omega^\alpha + \sum_{i=1}^m p_i \omega^{\alpha_i}$.

We are now ready to define the families $\mathcal{A}_\xi$, for $\xi < \omega_1$, which for reasons that will be explained later, will be collectively called the complete thin Schreier system.

Definition 1.3 (**The complete thin Schreier system** $(\mathcal{A}_\xi)_{\xi<\omega_1}$).

For every non zero, limit ordinal λ we choose and fix a strictly increasing sequence (λ_n) of successor ordinals smaller than λ with $sup_n\ \lambda_n = \lambda$.

We will define the system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ recursively as follows:

(1) [**Case** $\xi = 0$]

$\mathcal{A}_0 = \{\emptyset\}$;

(2) [**Case** $\xi = \zeta + 1$]

$\mathcal{A}_\xi = \mathcal{A}_{\zeta+1} = \{s \subseteq \mathbb{N} : s = \{n\} \cup s_1 \ ,\ \text{where } n \in \mathbb{N}, \{n\} < s_1 \text{ and } s_1 \in \mathcal{A}_\zeta\}$;

(3) [**Case** $\xi = \omega^{\beta+1},\ \beta$ countable ordinal]

$$\mathcal{A}_\xi = \mathcal{A}_{\omega^{\beta+1}} = \mathcal{B}_{\beta+1} = \{s \subseteq \mathbb{N} : s = \bigcup_{i=1}^n s_i \text{ with } n = min\ s_1\ ,\quad s_1 < \ldots < s_n,$$
$$\text{and } s_1, \ldots, s_n \in \mathcal{A}_{\omega^\beta}\};$$

(4) [**Case** $\xi = \omega^\lambda,\ \lambda$ non-zero, countable limit ordinal]

$\mathcal{A}_\xi = \mathcal{A}_{\omega^\lambda} = \mathcal{B}_\lambda = \{s \subseteq \mathbb{N} : s \in \mathcal{A}_{\omega^{\lambda_n}} \text{ with } \ n = min\ s\}$,

(where (λ_n) is the sequence of ordinals, converging to λ , fixed above); and

(5) [**Case** ξ limit $, \omega^\alpha < \xi < \omega^{\alpha+1}$ for some $0 < \alpha < \omega_1$]

Let $\xi = p\omega^\alpha + \sum_{i=1}^m p_i \omega^{\alpha_i}$ be the above representation (Proposition 1.2).

$$\mathcal{A}_\xi = \{s \subseteq \mathbb{N} : s = s_0 \cup (\bigcup_{i=1}^m s_i) \ \text{ with } s_m < \ldots < s_1 < s_0\ ,$$
$$s_0 = s_1^0 \cup \ldots \cup s_p^0 \text{ with } s_1^0 < \ldots < s_p^0,\ s_j^0 \in \mathcal{A}_{\omega^\alpha}, 1 \leq j \leq p\ ,$$
$$s_i = s_1^i \cup \ldots \cup s_{p_i}^i,\ \text{with } s_1^i < \ldots < s_{p_i}^i,\ s_j^i \in \mathcal{A}_{\omega^{\alpha_i}}, 1 \leq i \leq m,\ 1 \leq j \leq p_i\ \}\ .$$

Remark 1.4 (i) $\mathcal{A}_\xi \subseteq [\mathbb{N}]^{<\omega}$ for every $\xi < \omega_1$ and $\emptyset \notin \mathcal{A}_\xi$ for every $\xi > 0$.

(ii) $\mathcal{A}_k = [\mathbb{N}]^k$ for $k = 1, 2, \ldots$

(iii) $\mathcal{B}_1 = \mathcal{A}_\omega = \{s \in [\mathbb{N}]^{<\omega} : s = (n_1 < \ldots < n_k) \quad \text{with} \quad n_1 = k\}$.

Thus $\mathcal{A}_\omega$ is a modification of the **classical Schreier family** ([S])

$\mathcal{F}_1 = \{s \subseteq \mathbb{N} : s = (n_1 < \ldots < n_k) \text{ with } n_1 \geq k\}$.

In this sense $\mathcal{A}_\omega$ is a thin Schreier family (this notion, used also, by Pudlak - Rödl, will be defined precisely later on in Definition 2.1).

(iv) $\mathcal{B}_\alpha = \mathcal{A}_{\omega^\alpha}$, for $\alpha < \omega_1$, is defined using only $\mathcal{B}_\beta = \mathcal{A}_{\omega^\beta}$ for $\beta < \alpha$, and not using all previously defined families $\mathcal{A}_\xi,\ \xi < \omega^\alpha$. $\mathcal{B}_k$, for $k \in \mathbb{N}$, is a modification of generalized Schreier families defined by Alspach-Odell ([A-O]); and more generally $\mathcal{B}_\alpha$, for $\alpha < \omega_1$ is a modification of the families $\mathcal{F}_\alpha$, defined by Alspach-Argyros ([A-A]).

Now that the definition of the complete thin Schreier system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ is given, we are ready to state the first Ramsey partition theorem, for any countable ordinal ξ, a theorem whose scope can be appreciated by the fact that the classical Ramsey theorem corresponds to a finite ordinal $\xi < \omega$.

Theorem 1.5 (=Theorem A, Ramsey partition theorem extended to countable ordinals). Let $\mathcal{F}$ be an arbitrary family of finite subsets of $\mathbb{N}$, M an infinite subset of $\mathbb{N}$ and ξ a countable ordinal number. Then, there exists an infinite subset L of M such that

$$\text{either } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F} .$$

In order to prove this theorem, we must find a way to relate the complete thin Schreier system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ with Ramsey type partition. This is done in Proposition 1.7. The connecting concept that suits this purpose turns out to be the ξ-uniform families, which were defined by Pudlák and Rödl ([P-R]), an inductive concept incorporated in Proposition 1.7 below.

Definition 1.6 Let $\mathcal{L}$ be a family of finite subsets of $\mathbb{N}$. We set
$\mathcal{L}(n) = \{s \in [\mathbb{N}]^{<\omega} : \{n\} < s \text{ and } \{n\} \cup s \in \mathcal{L}\}$ for every $n \in \mathbb{N}$.

Proposition 1.7 For every countable ordinal ξ there exists a concrete sequence (ξ_n) of countable ordinals such that $\mathcal{A}_\xi(n) = \mathcal{A}_{\xi_n} \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega}$for every $n \in \mathbb{N}$. Moreover, $\xi_n = \zeta$ for every $n \in \mathbb{N}$ if $\xi = \zeta + 1$ is a successor ordinal and (ξ_n) is a strictly increasing sequence with $sup_n \xi_n = \xi$ if ξ is a limit ordinal.

Proof We will prove it by recursion on ξ.

(1) [**Case** $\xi = 1$] For every $n \in \mathbb{N}$ we have
$\mathcal{A}_1(n) = \{\emptyset\} = \mathcal{A}_0 \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega}$.

(2) [**Case** $\xi = \zeta + 1$] For every $n \in \mathbb{N}$ we have
$\mathcal{A}_\xi(n) = \{s \subseteq \mathbb{N} : \{n\} < s \text{ and } s \in \mathcal{A}_\zeta\} = \mathcal{A}_\zeta \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega}$.

(3) [Case $\xi = \omega^{\beta+1}$ for $0 \leq \beta < \omega_1$] For every $n \in \mathbb{N}$ we have

$$\mathcal{A}_\xi(n) = \{s \subseteq \mathbb{N} : \{n\} < s, \{n\} \cup s = \bigcup_{i=1}^{n} s_i \ , s_1 < \ldots < s_n \text{ and } s_1, \ldots, s_n \in \mathcal{A}_{\omega^\beta}\}$$
$$= \{s \subseteq \mathbb{N} : s = s_0 \cup (\bigcup_{i=2}^{n} s_i) \text{ with } s_0 \in \mathcal{A}_{(\omega^\beta)_n} \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega},$$
$$s_0 < s_2 < \ldots < s_n \text{ and } s_2, \ldots, s_n \in \mathcal{A}_{\omega^\beta}\}$$
$$= \mathcal{A}_{(n-1)\omega^\beta + (\omega^\beta)_n} \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega} \ ,$$

according to the induction hypothesis. Hence, $\xi_n = (n-1)\omega^\beta + (\omega^\beta)_n$ for every $n \in \mathbb{N}$ and obviously $sup_n \ \xi_n = \omega^{\beta+1}$. We note that in case $\xi = \omega$ we have $\xi_n = n - 1$, since $\omega^0 = 1$.

(4) [Case $\xi = \omega^\lambda$ for λ non-zero, countable limit ordinal] Let (λ_n) be the sequence of successor ordinals converging to λ fixed in the definition of the system $(\mathcal{A}_\xi)_{\xi<\omega_1}$. For every $n \in \mathbb{N}$ we have

$$\mathcal{A}_\xi(n) = \{s \subseteq \mathbb{N} : \{n\} < s \ \text{ and } \{n\} \cup s \in \mathcal{A}_{\omega^{\lambda_n}}\}$$
$$= \{s \subseteq \mathbb{N} : s \in \mathcal{A}_{\omega^{\lambda_n}}(n)\} = \mathcal{A}_{(\omega^{\lambda_n})_n} \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega} \ ,$$

according to the induction hypothesis. If $\lambda_n = \mu_n + 1$ for every $n \in \mathbb{N}$, then

$$\xi_n = (\omega^{\lambda_n})_n = (\omega^{\mu_n+1})_n = (n-1)\omega^{\mu_n} + (\omega^{\mu_n})_n \ \text{ for every } n \in \mathbb{N} \ .$$

Of course $sup_n \ \xi_n = \omega^\lambda$, since $\omega^{\mu_n} \leq (n-1)\omega^{\mu_n} \leq (n-1)\omega^{\mu_n} + (\omega^{\mu_n})_n$.

(5) [Case ξ limit, $\omega^\alpha < \xi < \omega^{\alpha+1}$ for some $0 < \alpha < \omega_1$] For simplicity we examine firstly the case $\xi = p\omega^\alpha$ for some $p \in \mathbb{N}$ with $p > 1$. For every $n \in \mathbb{N}$ we have

$$\mathcal{A}_\xi(n) = \{s \subseteq \mathbb{N} : \{n\} < s \text{ and } \{n\} \cup s = \bigcup_{i=1}^{p} s_i \text{ with } s_1 < \ldots < s_p$$
$$\text{and } s_i \in \mathcal{A}_{\omega^\alpha} \text{ for } 1 \leq i \leq p\}$$
$$= \{s \subseteq \mathbb{N} : s = s_0 \cup (\bigcup_{i=2}^{p} s_i) \text{ with } \{n\} < s_0 < s_2 < \ldots < s_p,$$
$$s_0 \in \mathcal{A}_{(\omega^\alpha)_n} \text{ and } s_2, \ldots, s_p \in \mathcal{A}_{\omega^\alpha}\}$$
$$= \mathcal{A}_{(p-1)\omega^\alpha + (\omega^\alpha)_n} \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega} \ .$$

Thus $\xi_n = (p\omega^\alpha)_n = (p-1)\omega^\alpha + (\omega^\alpha)_n$ for every $n \in \mathbb{N}$. Of course, $sup_n \ \xi_n = \xi$, since $sup_n \ (\omega^\alpha)_n = \omega^\alpha$.

Now, let $\xi = \beta + p\omega^\alpha$, where $p \in \mathbb{N}$ with $p \geq 1$ and β is an ordinal number with $0 < \beta < \xi$ (see Proposition 1.2). Then for every $n \in \mathbb{N}$ we have

$$\begin{aligned}\mathcal{A}_\xi(n) &= \{s \subseteq \mathbb{N} : \{n\} < s, \{n\} \cup s = s_1 \cup s_2,\ s_1 < s_2,\ s_1 \in \mathcal{A}_{p\omega^\alpha} \text{ and } s_2 \in \mathcal{A}_\beta\} \\ &= \{s \subseteq \mathbb{N} : s = s_0 \cup s_2 \text{ with } \ \{n\} < s_0 < s_2\ , s_0 \in \mathcal{A}_{p\omega^\alpha}(n) \text{ and } s_2 \in \mathcal{A}_\beta\} \\ &= \{s \subseteq \mathbb{N} : s = s_0 \cup s_2 \text{ with } \{n\} < s_0 < s_2,\ s_0 \in \mathcal{A}_{p\omega^\alpha}(n) \text{ and } s_2 \in \mathcal{A}_\beta\} \\ &= \mathcal{A}_{\beta + (p\omega^\alpha)_n} \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega}\ .\end{aligned}$$

Hence, $\xi_n = \beta + (p\omega^\alpha)_n = \beta + (p-1)\omega^\alpha + (\omega^\alpha)_n$ for every $n \in \mathbb{N}$. Of course, $sup_n\ \xi_n = \xi$.
This finishes the proof.

We now, mimicking the standard proof of the classical Ramsey theorem, prove the ξ-Ramsey type theorem for every countable ordinal ξ.

Proof of Theorem 1.5 We will prove it by recursion on ξ. Let $\xi = 1$. Then, $\mathcal{A}_1 = \{\{n\} : n \in \mathbb{N}\}$. Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ and $M \in [\mathbb{N}]$. Set $I = \{m \in M : \{m\} \in \mathcal{F}\}$ and consequently set $L = I$ in case I is infinite and $L = M \setminus I$ otherwise. Of course, either $\mathcal{A}_1 \cap [L]^{<\omega} \subseteq \mathcal{F}$ or $\mathcal{A}_1 \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$.

Let $\xi > 1$. Assume that the theorem is valid for all ordinal $\zeta < \xi$. Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ and $M \in [\mathbb{N}]$. Set $m_1 = minM$, $M_1 = M \setminus \{m_1\}$ and $\mathcal{F}_1 = \{s \subseteq M_1 : \{m_1\} \cup s \in \mathcal{F}\}$. According to the previous proposition, $\mathcal{A}_\xi(m_1) = \mathcal{A}_{\xi_{m_1}} \cap [\mathbb{N} \setminus \{1, \ldots, m_1\}]^{<\omega}$ with $\xi_{m_1} < \xi$. Therefore, from the induction hypothesis, there exists $L_1 \in [M_1]$ such that either $\mathcal{A}_{\xi_{m_1}} \cap [L_1]^{<\omega} \subseteq \mathcal{F}_1$ or $\mathcal{A}_{\xi_{m_1}} \cap [L_1]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}_1$. Hence,

either $\mathcal{A}_\xi(m_1) \cap [L_1]^{<\omega} \subseteq \mathcal{F}_1$ or $\mathcal{A}_\xi(m_1) \cap [L_1]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}_1$.

Set $m_2 = minL_1$, $M_2 = L_1 \setminus \{m_2\}$ and $\mathcal{F}_2 = \{s \subseteq M_2 : \{m_2\} \cup s \in \mathcal{F}\}$. Since $\mathcal{A}_\xi(m_2) = \mathcal{A}_{\xi_{m_2}} \cap [\mathbb{N} \setminus \{1, \ldots, m_2\}]^{<\omega}$ with $\xi_{m_2} < \xi$, and according to the induction hypothesis, there exists $L_2 \in [M_2]$ such that

either $\mathcal{A}_\xi(m_2) \cap [L_2]^{<\omega} \subseteq \mathcal{F}_2$ or $\mathcal{A}_\xi(m_2) \cap [L_2]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}_2$.

Set $m_3 = minL_2$, $M_3 = L_2 \setminus \{m_3\}$ and proceed analogously.
In this way we can construct a strictly increasing sequence $I = (m_n)_{n\in\mathbb{N}}$ in M, two decreasing sequences $(M_n)_{n\in\mathbb{N}}$, $(L_n)_{n\in\mathbb{N}}$ in [M] such that:

(i) $m_\kappa \in L_n$ for every $\kappa > n$;

(ii) $L_n \subseteq M_n$ for every $n \in \mathbb{N}$; and

(iii) If $\mathcal{F}_n = \{s \subseteq M_n : \{m_n\} \cup s \in \mathcal{F}\}$, then

either $\mathcal{A}_\xi(m_n) \cap [L_n]^{<\omega} \subseteq \mathcal{F}_n$ or $\mathcal{A}_\xi(m_n) \cap [L_n]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}_n$.

Set $I_1 = \{m_n \in I : \mathcal{A}_\xi(m_n) \cap [L_n]^{<\omega} \subseteq \mathcal{F}_n\}$ and $I_2 = I \setminus I_1$. If I_1 is infinite, then $\mathcal{A}_\xi \cap [I_1]^{<\omega} \subseteq \mathcal{F}$. Indeed, let $F \in \mathcal{A}_\xi \cap [I_1]^{<\omega}$ and let $m_n = minF$. Then $F = \{m_n\} \cup s$ for some $s \in \mathcal{A}_\xi(m_n)$. Since $m_n < s$ and $s \in [I]^{<\omega}$ we have that

$s \subseteq L_n$ using (i). Hence, $s \in \mathcal{A}_\xi(m_n) \cap [L_n]^{<\omega}$. Since $m_n \in I_1$, we have that $s \in \mathcal{F}_n$, and consequently $F = \{m_n\} \cup s \in \mathcal{F}$.
If I_2 is infinite, then, analogously can be proved that $\mathcal{A}_\xi \cap [I_2]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$.
Setting $L = I_1$ if I_1 is infinite, and $L = I \setminus I_1$ otherwise we have that

either $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$ or $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$.

This finishes the proof.

Corollary 1.8 Let M be an infinite subset of $\mathbb{N}, \{P_1, \ldots, P_n\}$ a finite partition of $[M]^{<\omega}$ and ξ a countable ordinal number. Then there exists $L \in [M]$ and $i \in \{1, \ldots, n\}$ such that $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq P_i$.

Corollary 1.9 Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$, $M \in [\mathbb{N}]$ and ξ a countable ordinal. If $\mathcal{A}_\xi \cap \mathcal{F} \cap [I]^{<\omega} \neq \emptyset$ for every $I \in [M]$, then there exists $L \in [M]$ such that $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$.

2. Three basic properties of the complete thin Schreier system

In this section we will prove three basic properties of the complete thin Schreier systems $(\mathcal{A}_\xi)_{\xi<\omega_1}$, namely

(a) Each family $\mathcal{A}_\xi$ is thin (Proposition 2.2);

(b) every (finite of infinite) subset of $\mathbb{N}$ has (unique) canonical representation with respect to each family $\mathcal{A}_\xi$ (Proposition 2.4); and

(c) the strong Cantor-Bendixson index of $\mathcal{A}_\xi$ (given in Definition 2.6 below) is precisely $\xi + 1$; and this index is stable if $\mathcal{A}_\xi$ is restricted to $\mathcal{A}_\xi \cap [M]^{<\omega}$ for any infinite subset M of $\mathbb{N}$ (Proposition 2.9).

These properties will be necessary for establishing, Theorem B in the next section 3, in case we have not an arbitrary, but only a hereditary family $\mathcal{F}$ of finite subsets, an effective criterion that allows us to decide (is most cases) which horn of the dichotomy provided by Theorem A will actually hold. Although not immediately apparent, it turns out, as we shall see in section 3, that Theorem B is in fact a strengthened Nash-Williams partition theorem.

We start with some definitions.

Definition 2.1 Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$.

(i) $\mathcal{F}$ is **thin** if there are no elements $s, t \in \mathcal{F}$ with s a proper initial segment (in the order of the natural numbers) of t.

(ii) $\mathcal{F}^\star = \{t \in [\mathbb{N}]^{<\omega} : t \text{ is an initial segment of some } s \in \mathcal{F}\} \cup \{\emptyset\}$.

(iii) $\mathcal{F}_\star = \{t \in [\mathbb{N}]^{<\omega} : t \text{ is a subset of some } s \in \mathcal{F}\}$.

(iv) $\mathcal{F}$ is **hereditary** if $\mathcal{F}_\star = \mathcal{F}$.

(v) $\mathcal{F}$ is a **tree** if $\mathcal{F}^\star = \mathcal{F}$.

Proposition 2.2 Every family, $\mathcal{A}_\xi$ for $\xi < \omega_1$, is thin.

Proof We will prove it by induction on ξ. The family $\mathcal{A}_1 = \{\{n\} : n \in \mathbb{N}\}$ is obviously thin. Let $\xi > 1$. Assume that $\mathcal{A}_\zeta$ is thin, for every $\zeta < \xi$. Let $s \in \mathcal{A}_\xi$ and t a proper initial semgent of s. If $t = \emptyset$, then $t \notin \mathcal{A}_\xi$ (Remark 1.4 (i)). Let $t \neq \emptyset$. If $m = mins$, then $t \setminus \{m\}$ is a proper initial segment of $s \setminus \{m\}$. Since $s \setminus \{m\} \in \mathcal{A}_\xi(m) \subseteq \mathcal{A}_{\xi_m}$ for some $\xi_m < \xi$ (Proposition 1.7), and $\mathcal{A}_{\xi_m}$ is thin, we have that $t \setminus \{m\} \notin \mathcal{A}_{\xi_m}$ and consequently $t \notin \mathcal{A}_\xi$. This proves that $\mathcal{A}_\xi$ is a thin family.

In the following we will prove that every subset of $\mathbb{N}$ has canonical representation with respect to each family $\mathcal{A}_\xi$.

Definition 2.3 Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$.

(i) A non-empty, finite subset s of $\mathbb{N}$ has **canonical representation** $R_\mathcal{F}(s) = \{s_1, \ldots, s_n, s_{n+1}\}$ with type $t_\mathcal{F}(s) = n$ with respect to $\mathcal{F}$, if there exist unique $n \in \mathbb{N}$, $s_1, \ldots, s_n \in \mathcal{F}$ and s_{n+1} a proper initial segment of some element of $\mathcal{F}$ with $s_1 < \ldots < s_n < s_{n+1}$ and such that $s = \bigcup_{i=1}^{n+1} s_i$.

(ii) A infinite subset I of $\mathbb{N}$ has **canonical representation** $R_\mathcal{F}(I) = (s_n)_{n\in\mathbb{N}}$ with respect to $\mathcal{F}$ if there exists unique sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ with $I = \bigcup_{n=1}^{\infty} s_n$ and such that $s_1 < s_2 < \ldots$,

Proposition 2.4 (**Canonical representation with respect to** $\mathcal{A}_\xi$) Let ξ be a countable ordinal number. Every non-empty subset of $\mathbb{N}$ has canonical representation with respect to the family $\mathcal{A}_\xi$.

Proof Let I be an infinite subset of $\mathbb{N}$. We will prove, by induction on ξ, that I has canonical representation with respect to each family $\mathcal{A}_\xi$, $\xi < \omega_1$. Of course I has canonical representation with respect to the family $\mathcal{A}_1 = \{\{n\} : n \in \mathbb{N}\}$. Let $\xi > 1$ and let (ξ_n) be the corresponding sequence defined in Proposition 1.7. Assume that the assertion holds for every $\zeta < \xi$. Set $m_1 = minI$ and $I_1 = I \setminus \{m_1\}$. According to the induction hypothesis, I_1 has canonical representation $(s_n^1)_{n\in\mathbb{N}}$ with respect to $\mathcal{A}_{\xi_{m_1}}$, since $1 \leq \xi_{m_1} < \xi$. Of course, $s_1 = \{m_1\} \cup s_1^1 \in \mathcal{A}_\xi$. Set $m_2 = min(I \setminus s_1)$

and $I_2 = I \setminus (s_1 \cup \{m_2\})$. According to the induction hypothesis, I_2 has canonical representation $(s_n^2)_{n\in\mathbb{N}}$ with respect to $\mathcal{A}_{\xi_{m_2}}$, since $1 \leq \xi_{m_2} < \xi$. Set $s_2 = \{m_2\} \cup s_1^2$. Of course $s_2 \in \mathcal{A}_\xi$ and $s_1 < s_2$. Set $m_3 = min(I \setminus (s_1 \cup s_2))$ and $I_3 = I \setminus (s_1 \cup s_2 \cup \{m_3\})$ and proceed analogously. In this way, we can construct a sequence $(s_n)_{n=1}^\infty$ in $\mathcal{A}_\xi$ such that $s_1 < s_2 < \ldots$ and $I = \bigcup_{n=1}^{\infty} s_n$ This representation of I with respect to $\mathcal{A}_\xi$ is unique, since $\mathcal{A}_\xi$ is a thin family (Proposition 2.2). Hence, I has canonical representation with respect to the family $\mathcal{A}_\xi$.

Now, let $\xi < \omega_1$ and $s = \{m_1, \ldots, m_\kappa\}$, be a non-empty, finite subset of $\mathbb{N}$ with $m_1 < \ldots < m_\kappa$. Set $m_{\kappa+i} = m_\kappa + i$ for every $i = 1, 2, \ldots$. The infinite set $I = (m_n)_{n=1}^\infty$ has canonical representation $(s_n)_{n=1}^\infty$ with respect to $\mathcal{A}_\xi$. Using this fact, it is easy to prove that s has canonical representation with respect to $\mathcal{A}_\xi$.

Corollary 2.5 Let ξ be a countable ordinal number. For every non-empty, finite set s of $\mathbb{N}$ exactly one of the following possibilities occurs:

either (i) s is a proper initial segment of some element of $\mathcal{A}_\xi$;

or (ii) there exists an element of $\mathcal{A}_\xi$ which is an initial segment of s.

Proof Let $s \in [\mathbb{N}]^{<\omega}, s \neq \emptyset$. According to Proposition 2.4, the case $t_{\mathcal{A}_\xi}(s) = 0$ gives equivalently (i), while the complementary case, $t_{\mathcal{A}_\xi}(s) \geq 1$, gives equivalently (ii).

In the following we will estimate the strong Cantor-Bendixson index of the families $\mathcal{A}_\xi$. This index (in Definition 2.6 below) is analogous to the well-known Cantor-Bendixson index ([B],[C1]) and has been defined in [A-M-T]. Our notation is different from the one used by these authors.

We will prove in Proposition 2.9 below, that the corresponding hereditary family of the thin Schreier family $\mathcal{A}_\xi$, for $\xi < \omega_1$ has strong Cantor-Bendixson index equal to $\xi + 1$, moreover for every $M \in [\mathbb{N}]$ the restricted family $(\mathcal{A}_\xi \cap [M]^{<\omega})_\star$ has also index equal to $\xi + 1$.

This is the reason we have called $(\mathcal{A}_\xi)_{\xi<\omega_1}$ **complete system.**

Definition 2.6 ([A-M-T]) Let $\mathcal{F}$ be a hereditary and pointwise closed family of finite subsets on $\mathbb{N}$. For $M \in [\mathbb{N}]$ we define the **strong Cantor-Bendixson derivatives** $(\mathcal{F})_M^\xi$ of $\mathcal{F}$ on M for every $\xi < \omega_1$ as follows:

$$(\mathcal{F})_M^1 = \{F \in \mathcal{F}[M] : F \text{ is a cluster point of } \mathcal{F}[F \cup L] \text{ for each } L \in [M]\};$$

(where, $\mathcal{F}[M] = \mathcal{F} \cap [M]^{<\omega}$).

If $(\mathcal{F})_M^\xi$ has been defined, then $(\mathcal{F})_M^{\xi+1} = \left((\mathcal{F})_M^\xi\right)_M^1$.

If ξ is a limit ordinal and $(\mathcal{F})_M^{\beta}$ have been defined for each $\beta < \xi$, then

$$(\mathcal{F})_M^{\xi} = \bigcap_{\beta<\xi} (\mathcal{F})_M^{\beta} .$$

The **strong Cantor-Bendixson index of $\mathcal{F}$ on M** is defined to be the smallest countable ordinal ξ such that $(\mathcal{F})_M^{\xi} = \emptyset$. We denote this index by $s_M(\mathcal{F})$.

Remark 2.7 (i) The strong Cantor-Bendixson index $s_M(\mathcal{F})$ of a hereditary and pointwise closed family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ on some $M \in [\mathbb{N}]$ is a countable successor ordinal and is less than or equal to the "usual" Cantor-Bendixson index $O(\mathcal{F})$ of $\mathcal{F}$ (see [K]).

(ii) If $\mathcal{F}_1, \mathcal{F}_2 \subseteq [\mathbb{N}]^{<\omega}$ are hereditary and pointwise closed families with $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $s_M(\mathcal{F}_1) \leq s_M(\mathcal{F}_2)$ for every $M \in [\mathbb{N}]$.

(iii) $s_M(\mathcal{F}) = s_M(\mathcal{F} \cap [M]^{<\omega})$ for every $M \in [\mathbb{N}]$.

(iv) For every $M \in [\mathbb{N}]$ and $s \in [M]^{<\omega}$, according to a remark in $[J]$, we have :

$$s \in (\mathcal{F})_M^1 \text{ if and only if the set } \{m \in M : s \cup \{m\} \notin \mathcal{F}\} \text{ is finite .}$$

(v) Using the previous remark (iv), it can be proved by induction that for every $L \in [M]$ and $\xi < \omega_1$ if $A \in (\mathcal{F})_M^{\xi}$, then $F \cap L \in (\mathcal{F})_L^{\xi}$. Hence, $s_L(\mathcal{F}) \geq s_M(\mathcal{F})$. (see also [A-M-T]).

(vi) If L is almost contained in M (i.e. the relative complement $L \setminus M$ of L in M is a finite set), then $s_L(\mathcal{F}) \geq s_M(\mathcal{F})$.

In the following we will calculate the strong Cantor-Bendixson index of the thin Schreier families $\mathcal{A}_\xi$.

Lemma 2.8 Let ξ be a countable ordinal, $L \in [\mathbb{N}]$ and $\mathcal{L}$ a family of finite subsets of $\mathbb{N}$ such that $\mathcal{L}_\star$ and $\mathcal{L}(n)_\star$ are pointwise closed for every $n \in L$.

(i) If $F \in \big(\mathcal{L}(n)_\star\big)_L^{\xi}$ for some $n \in L$, then $\{n\} \cup F \in (\mathcal{L}_\star)_L^{\xi}$.

(ii) If $F \neq \emptyset$ and $F \in (\mathcal{L}_\star)_L^{\xi}$, then there exist $l \in \mathbb{N}$ with $l \leq minF$ and $I \in [L]$ such that $F \setminus \{l\} \in (\mathcal{L}(l)_\star)_I^{\xi}$.

Proof (i) We use induction on ξ. Let $F \in \big(\mathcal{L}(n)_\star\big)_L^1$. Since $\{m \in M : F \cup \{m\} \in \mathcal{L}(n)_\star\} \subseteq \{m \in M : F \cup \{m\} \cup \{n\} \in \mathcal{L}_\star\}$, we have, according to Remark 2.7 (iv), that $F \cup \{n\} \in (\mathcal{L}_\star)_L^1$. Let $1 < \xi$. Suppose that the assertion holds for all ordinals ζ with $\zeta < \xi$. If $F \in (\mathcal{L}(n)_\star)_L^{\zeta+1}$, then, according to the induction hypothesis, $\{m \in M : F \cup \{m\} \in (\mathcal{L}(n)_\star)_M^{\zeta}\} \subseteq \{m \in M : F \cup \{m\} \cup \{n\} \in (\mathcal{L}_\star)_M^{\zeta}\}$. Hence $\{n\} \cup F \in (\mathcal{L}_\star)_L^{\zeta+1}$ (Remark 2.7 (iv)).
The case where ξ is a limit ordinal is trivial.

(ii) We use induction on ξ. Let $F \neq \emptyset$ and $F \in (\mathcal{L}_\star)^1_L$. According to Remark 2.7 (iv), the set $L_F = \{m \in L : F \cup \{m\} \in \mathcal{L}_\star$ with $minF \leq m\}$ is almost equal to L. For each $m \in L_F$ there exists $s_m \in \mathcal{L}$ such that $F \cup \{m\} \subseteq s_m$. Of course $1 \leq mins_m \leq minF$ for every $m \in L_F$. Set

$l = min\{n \in \mathbb{N} :$ the set $\{m \in L_F : min\ s_m = n\}$ is infinite$\}$; and

$I = \{m \in L_F : \ min\ s_m = l\} \cup F$.

Then, $\{l\} \leq F$, $I \in [L]$ and $F \setminus \{l\} \in (\mathcal{L}(l)_\star)^1_I$, as required.

Suppose now that the assertion holds for all ordinals β with $\beta < \xi$. Firstly we examine the case $\xi = \zeta + 1$. Let $F \neq \emptyset$ and $F \in (\mathcal{L}_\star)^{\zeta+1}_L$. According to Remark 2.7 (iv), the set $L_F = \{m \in L : F \cup \{m\} \in (\mathcal{L}_\star)^\zeta_L$ and $minF \leq m\}$ is almost equal to L. Let $m_1 = minL_F$. By the induction hypothesis there exist $l_1 \in \mathbb{N}$ with $l_1 \leq minF$ and $I_1 \in [L_F]$ such that $F \cup \{m_1\} \setminus \{l_1\} \in \big(\mathcal{L}(l_1)_\star\big)^\zeta_{I_1 \cup F}$, since $F \cup \{m_1\} \in (\mathcal{L}_\star)^\zeta_{L_F \cup F}$ (Remark 2.7 (v)). Choose $m_2 \in I_1$ and $m_2 > m_1$. Since $F \cup \{m_2\} \in (\mathcal{L}_\star)^\zeta_{I_1 \cup F}$,there exist $l_2 \in \mathbb{N}$ with $l_2 \leq minF$ and $I_2 \in [I_1]$ such that $F \cup \{m_2\} \setminus \{l_2\} \in \big(\mathcal{L}(l_2)_\star\big)^\zeta_{I_2 \cup F}$. We continue analogously choosing $m_3 \in I_2$ with $m_3 > m_2$ and so on. Hence, we construct an increasing sequence $(m_i)_{i=1}^\infty$ in L_F, a sequence $(l_i)_{i=1}^\infty$ in $\mathbb{N}$, with $1 \leq l_i \leq minF$ for every $i \in \mathbb{N}$, and a decreasing sequence $(I_i)_{i=1}^\infty$ in $[L_F]$ such that $F \cup \{m_i\} \setminus \{l_i\} \in (\mathcal{L}(l_i)_\star)^\zeta_{I_i \cup F}$ for every $i \in \mathbb{N}$. Let $l \in \mathbb{N}$ with $1 \leq l \leq minF$ such that the set $L_1 = \{i \in \mathbb{N} : l_i = l\}$ is infinite. Set $I = \{m_i : i \in L_1\} \cup F$. Then, $F \setminus \{l\} \in (\mathcal{L}(l)_\star)^{\zeta+1}_I$, as required.

In the case where ξ is a limit ordinal we fix a strictly increasing sequence $(\zeta_i)_{i=1}^\infty$ of ordinals with $\zeta_i < \xi$ for every $i \in \mathbb{N}$ and $sup_i\ \zeta_i = \xi$. Let $F \in (\mathcal{L}_\star)^\xi_L$ and $F \neq \emptyset$. Then $F \in (\mathcal{L}_\star)^{\zeta_i}_L$ for every $i \in \mathbb{N}$. According to the induction hypothesis, there exist $l_1 \in \mathbb{N}$ with $l_1 \leq minF$ and $I_1 \in [L \cap (minF, +\infty)]$ such that $F \setminus \{l_1\} \in (\mathcal{L}(l_1)_\star)^{\zeta_1}_{I_1 \cup F}$. Since $F \in (\mathcal{L}_\star)^{\zeta_2}_{I_2 \cup F}$ there exists $l_2 \in \mathbb{N}$ with $l_2 \leq minF$ and $I_2 \in [I_1]$ such that $I_2 \neq I_1$ and $F \setminus \{l_2\} \in (\mathcal{L}(l_2)_\star)^{\zeta_2}_{I_2 \cup F}$. In this way, we construct a sequence $(l_i)_{i=1}^\infty$ with $1 \leq l_i \leq minF$ and a strictly decreasing sequence $(I_i)_{i=1}^\infty$ in $[L]$ such that $F \setminus \{l_i\} \in (\mathcal{L}(l_i)_\star)^{\zeta_i}_{I_i \cup F}$, for every $i \in \mathbb{N}$.
Let $l \in \mathbb{N}$ with $1 \leq l \leq minF$ such that the set $L_1 = \{i \in \mathbb{N} : l_i = l\}$ is infinite. Set $I = \{minI_i : i \in L_1\} \cup F$. Then $F \setminus \{l\} \in (\mathcal{L}(l)_\star)^{\zeta_i}_I$ for every $i \in L_1$.Since $sup_{i \in L_1}\ \zeta_i = \xi$, we have that $F \setminus \{l\} \in \big((\mathcal{L}(l)_\star\big)^\xi_I$.

This completes the proof.

Proposition 2.9 (Cantor-Bendixson index of $\mathcal{A}_\xi$) Let M be an infinite subset of $\mathbb{N}$. Then

$$s_L((\mathcal{A}_\xi \cap [M]^{<\omega})_\star) = \xi + 1 \ \text{ for every } \xi < \omega_1 \quad \text{and } L \in [M].$$

Proof It is easily proved, by induction on ξ, that the family $(\mathcal{A}_\xi \cap [M]^{<\omega})_\star$ is pointwise closed for every $\xi < \omega_1$ and $M \in [\mathbb{N}]$. Also, the family $((\mathcal{A}_\xi \cap [M]^{<\omega})(m))_\star$

is pointwise closed for every $\xi < \omega_1$, $M \in [\mathbb{N}]$ and $m \in M$, since, according to Proposition 1.7, $(\mathcal{A}_\xi \cap [M]^{<\omega})(m) = \mathcal{A}_{\xi_m} \cap [M \setminus \{1,\dots,m\}]^{<\omega}$, where $\xi_m = \zeta$ if $\xi = \zeta + 1$ is a successor ordinal and $(\xi_m)_{m\in M}$ is a strictly increasing to ξ sequence, if ξ is a limit ordinal.

We will prove, by induction on ξ, that $s_L((\mathcal{A}_\xi \cap [M]^{<\omega})_\star) = \xi + 1$ for every $\xi < \omega_1, M \in [\mathbb{N}]$ and $L \in [M]$. Since $(\mathcal{A}_1 \cap [M]^{<\omega})_\star = \{\{m\} : m \in M\} \cup \{\emptyset\}$, we have $((\mathcal{A}_1 \cap [M]^{<\omega})_\star)^1_L = \{\emptyset\}$ and consequently that $s_L((\mathcal{A}_1 \cap [M]^{<\omega})_\star) = 2$ for every $M \in [\mathbb{N}]$ and $L \in [M]$.

Suppose that $\xi > 1$ and that the assertion holds for every ordinal ζ with $\zeta < \xi$. Let $M \in [\mathbb{N}]$ and $L \in [M]$. For every $m \in M$, using Proposition 1.7, Remark 2.7(vi) and the induction hypothesis we get that $s_L((\mathcal{A}_\xi \cap [M]^{<\omega})(m)_\star) = \xi_m + 1$ for every $m \in M$ and consequently that $\emptyset \in ((\mathcal{A}_\xi \cap [M]^{<\omega})(m)_\star)^{\xi_m}_L$ for every $m \in M$. In case $\xi = \zeta + 1$ be a successor ordinal, $\xi_m = \zeta$ for every $m \in M$ and, according to Lemma 2.8 (i), $\{l\} \in ((\mathcal{A}_\xi \cap [M]^{<\omega})_\star)^\zeta_L$ for every $l \in L$. Hence, $\emptyset \in ((\mathcal{A}_\xi \cap [M]^{<\omega})_\star)^\xi_L$ (Remark 2.7(iv)). On the other hand, in case ξ be a limit ordinal, according to Lemma 2.8 (i), $\emptyset \in ((\mathcal{A}_\xi \cap [M]^{<\omega})_\star)^{\xi_l}_L$ for every $l \in L$. Since $\xi_l < \xi$ and $sup_{l\in L}\xi_l = \xi$, we have also that $\emptyset \in ((\mathcal{A}_\xi \cap [M]^{<\omega})_\star)^\xi_L$.

In fact, $\{\emptyset\} = ((\mathcal{A}_\xi \cap [M]^{<\omega})_\star)^\xi_L$. Indeed, let $F \in ((\mathcal{A}_\xi \cap [M]^{<\omega})_\star)^\xi_L$ and $F \neq \emptyset$. Then, according to Lemma 2.8 (ii), there exist $n \in \mathbb{N}$ with $n \leq minF$ and $I \in [L]$ such that $F \setminus \{n\} \in ((\mathcal{A}_\xi \cap [M]^{<\omega})(n)_\star)^\xi_I$. This gives that $s_I((\mathcal{A}_\xi \cap [M]^{<\omega})(n)_\star) \geq \xi + 1 > \xi_n + 1$. A contradiction, according to Proposition 1.7 and the induction hypothesis. Hence, $\{\emptyset\} = ((\mathcal{A}_\xi \cap [M]^{<\omega})_\star)^\xi_L$ and consequently $s_L((\mathcal{A}_\xi \cap [M]^{<\omega})_\star) = \xi + 1$ for every $\xi < \omega_1$.

3. Strengthened Nash-Williams partition theorems

We now turn our attention to the strengthened forms of the Nash-Williams theorem. These, contained in Theorems 3.7(= Theorem B), 3.10(= Theorem B′), 3.11(= Theorem C) and 3.14, are consequences of the extended Ramsey Theorem 1.5 (= Theorem A), and of the tools contained in Section 2 (canonical representation 2.4, Cantor-Bendixson index 2.9)

Theorem 3.7(= Theorem B) can be considered as the extended Ramsey Theorem 1.5(= Theorem A), strengthened for the case that we restrict ourselves, not to arbitrary, but only to hereditary families, $\mathcal{F}$ of finite subsets of $\mathbb{N}$. As already remarked above it constitutes in reality a strengthened Nash-Willimas type partition theorem, if we keep in mind the Gowers reformulation of Nash-Williams theorem (mentioned in the introduction above).

Proposition 3.1, a consequence of Theorem A, using also the canonical representation (Proposition 2.4), has consequences (Corollaries 3.4, 3.5) regarding generalized Schreier families (defined is 3.3).

Proposition 3.1 Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$ which is a tree ($\mathcal{F} = \mathcal{F}^\star$), M an infinite subset of $\mathbb{N}$ and ξ a countable ordinal number. Then there exists $L \in [M]$ such that

$$\text{either} \quad \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \quad \text{or} \quad \mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi \ .$$

Proof According to the Ramsey partition theorem for the countable ordinal ξ there exists $L \in [M]$ such that either $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$ or $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$. Since $\mathcal{F}$ is a tree, we have $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ if and only if $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi$. Indeed, let $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ and $F \in \mathcal{F} \cap [L]^{<\omega}$. According to Corollary 2.5, either there exist $s \in \mathcal{A}_\xi$ such that F is a proper initial segment of s which gives that $F \in (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi$, as required, or there exists $t \in \mathcal{A}_\xi$ such that t is an initial segment of F. The second case is impossible. Indeed, since $\mathcal{F}$ is a tree and $F \in \mathcal{F} \cap [L]^{<\omega}$, we have $t \in \mathcal{A}_\xi \cap [L]^{<\omega} \cap \mathcal{F}$. This contrary to our assumption that $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$. Hence, $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi$.
It is obvious that if $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi$, then $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$.

Corollary 3.2 Let ξ_1, ξ_2 be countable ordinal numbers with $\xi_1 < \xi_2$. For every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that $(\mathcal{A}_{\xi_1})_\star \cap [L]^{<\omega} \subseteq (\mathcal{A}_{\xi_2})^\star \setminus \mathcal{A}_{\xi_2}$.

Proof Of course $(\mathcal{A}_{\xi_1})_\star$ is a tree. According to Proposition 3.1, for every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that

$$\text{either} \quad \mathcal{A}_{\xi_2} \cap [L]^{<\omega} \subseteq (\mathcal{A}_{\xi_1})_\star \quad \text{or} \quad (\mathcal{A}_{\xi_1})_\star \cap [L]^{<\omega} \subseteq (\mathcal{A}_{\xi_2})^\star \setminus \mathcal{A}_{\xi_2} \ .$$

The first alternative is impossible, since if $\mathcal{A}_{\xi_2} \cap [L]^{<\omega} \subseteq (\mathcal{A}_{\xi_1})_\star$, then $\xi_2 + 1 = s_L((\mathcal{A}_{\xi_2} \cap [L]^{<\omega})_\star) \leq s_L((\mathcal{A}_{\xi_1})_\star) = \xi_1 + 1$ (Proposition 2.9).
A contradiction; hence $(\mathcal{A}_{\xi_1})_\star \cap [L]^{<\omega} \subseteq (\mathcal{A}_{\xi_2})^\star \setminus \mathcal{A}_{\xi_2}$.

In the following, using Proposition 3.1, we indicate the close connection that exists between the generalized Schreier families $(\mathcal{F}_\alpha)_{\alpha<\omega_1}$ and the ω^α- thin Schreier families $\mathcal{A}_{\omega^\alpha} = \mathcal{B}_\alpha$ for $\alpha < \omega_1$. Firstly we will give the appropriate definitions.

Definition 3.3 (i) (Generalized Schreier families [S], [A-O],[A-A])

$\mathcal{F}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$;

$\mathcal{F}_{\alpha+1} = \{F \subseteq \mathbb{N} : F = \bigcup_{i=1}^{k} F_i \ , \{k\} \leq F_1 < \ldots < F_k, \text{ and } F_i \in \mathcal{F}_\alpha\} \cup \{\emptyset\}$;

If α is a limit ordinal choose and fix $(\alpha_n)_{n\in\mathbb{N}}$ strictly increasing to α and set

$\mathcal{F}_\alpha = \{F \subseteq \mathbb{N} : F \in \mathcal{F}_{\alpha_k} \text{ with } k \leq minF\} \cup \{\emptyset\}$.

(ii) For a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ and $L = (l_n)_{n=1}^\infty \in [\mathbb{N}]$ we set $\mathcal{F}(L) = \{(l_{n_1}, \ldots, l_{n_k}) \in [L]^{<\omega} : (n_1, \ldots, n_k) \in \mathcal{F}\}$.

Corollary 3.4 Let α be a countable ordinal. For every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that $\mathcal{F}_\alpha(L) \subseteq (\mathcal{B}_\alpha)^\star \subseteq \mathcal{F}_\alpha$.

Proof The family $\mathcal{F}_\alpha$ is hereditary, hence, according to Proposition 3.1, for every $M \in [\mathbb{N}]$ there exists $I \in [M]$ such that
either $\mathcal{A}_{\omega^\alpha+1} \cap [I]^{<\omega} \subseteq \mathcal{F}_\alpha$ or $\mathcal{F}_\alpha \cap [I]^{<\omega} \subseteq (\mathcal{A}_{\omega^\alpha+1})^\star \setminus \mathcal{A}_{\omega^\alpha+1}$.
The first alternative is impossible. Indeed, if $\mathcal{A}_{\omega^\alpha+1} \cap [I]^{<\omega} \subseteq \mathcal{F}_\alpha$, then
$\omega^\alpha + 2 = s_I((\mathcal{A}_{\omega^\alpha+1} \cap [I]^{<\omega})_\star) \leq s_I(\mathcal{F}_\alpha) = \omega^\alpha + 1$ (Proposition 2.9).
A contradiction; hence $\mathcal{F}_\alpha \cap [I]^{<\omega} \subseteq (\mathcal{A}_{\omega^\alpha+1})^\star \setminus \mathcal{A}_{\omega^\alpha+1}$.

Let $I = (i_n)_{n=1}^\infty$. We set $L = (i_n)_{n=3}^\infty = (l_n)_{n=1}^\infty$. We will prove that $\mathcal{F}_\alpha(L) \subseteq (\mathcal{B}_\alpha)^\star$. Indeed, let $(l_{n_1}, \ldots, l_{n_k}) \in \mathcal{F}_\alpha(L)$, with $(n_1, \ldots, n_k) \in \mathcal{F}_\alpha$. Then $(n_1{+}1, n_1{+}2, \ldots, n_k{+}2) \in \mathcal{F}_\alpha$ and consequently $(i_{n_1+1}, i_{n_1+2}, \ldots, i_{n_k+2}) \in \mathcal{F}_\alpha \cap [I]^{<\omega}$ (for the properties of $\mathcal{F}_\alpha$ see [A-M-T]). This gives that $(i_{n_1+1}, l_{n_1}, \ldots l_{n_k}) \in (\mathcal{A}_{\omega^\alpha+1})^\star$ and consequently that $(l_{n_1}, \ldots, l_{n_k}) \in (\mathcal{A}_{\omega^\alpha})^\star$, as required. Hence, $\mathcal{F}_\alpha(L) \subseteq (\mathcal{B}_\alpha)^\star$. It is obvious that $(\mathcal{B}_\alpha)^\star \subseteq \mathcal{F}_\alpha$.

R. Judd in [J] had provided, using Schreier games, that for every hereditary family $\mathcal{F}$ of finite subsets of $\mathbb{N}$, $\alpha < \omega_1$ and $M \in [\mathbb{N}]$, either there exists $L \in [M]$ such that $\mathcal{F}_\alpha(L) \subseteq \mathcal{F}$ or there exists $L \in [M]$ and $N \in [\mathbb{N}]$ such that $\mathcal{F} \cap [N]^{<\omega}(L) \subseteq \mathcal{F}_\alpha$.

As a corollary of Proposition 3.1 we will prove a stronger version of this result.

Corollary 3.5 For every family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ which is a tree, every countable ordinal α and $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that
either $\mathcal{F}_\alpha(L) \subseteq \mathcal{F}$ or $\mathcal{F} \cap [L]^{<\omega} \subseteq \mathcal{F}_\alpha$.

Proof According to Proposition 3.1 there exists $N \in [M]$ such that
either $\mathcal{B}_\alpha \cap [N]^{<\omega} \subseteq \mathcal{F}$ or $\mathcal{F} \cap [N]^{<\omega} \subseteq (\mathcal{B}_\alpha)^\star$;
If $\mathcal{B}_\alpha \cap [N]^{<\omega} \subseteq \mathcal{F}$, then, according to Corollary 3.4 and Proposition 2.4, there exists $L \in [N]$ such that $\mathcal{F}_\alpha(L) \subseteq (\mathcal{B}_\alpha)^\star \cap [L]^{<\omega} \subseteq \left(\mathcal{B}_\alpha \cap [N]^{<\omega}\right)^\star \subseteq \mathcal{F}$.
Hence, either $\mathcal{F}_\alpha(L) \subseteq \mathcal{F}$, or $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{B}_\alpha)^\star \subseteq \mathcal{F}_\alpha$.

Since we will study the hereditary families of finite subsets of $\mathbb{N}$ which in addition are closed in the pointwise topology we will give an elementary characterization of them.

Proposition 3.6 Let $\mathcal{F}$ be a non empty, family of finite subsets of $\mathbb{N}$.

(i) Let $\mathcal{F}$ be a tree. Then $\mathcal{F}$ is pointwise closed if and only if there does not exist an infinite sequence $(s_i)_{i=1}^{\infty}$ of elements of $\mathcal{F}$ with $s_1 \prec s_2 \prec \dots$.

(ii) Let $\mathcal{F}$ be hereditary. Then $\mathcal{F}$ is pointwise closed if and only if there does not exist $M \in [\mathbb{N}]$ such that $[M]^{<\omega} \subseteq \mathcal{F}$.

Proof (i) Let $\mathcal{F}$ be a tree. If $(s_i)_{i=1}^{\infty}$ is a sequence in $\mathcal{F}$ with $s_1 \prec s_2 \prec \dots$, then $(s_i)_{i=1}^{\infty}$ converges pointwise to an infinite subset s of $\mathbb{N}$. Since $s \notin \mathcal{F}$, $\mathcal{F}$ is not closed. We assume that there does not exist an infinite sequence $(s_i)_{i=1}^{\infty}$ of elements of $\mathcal{F}$ with $s_1 \prec s_2 \prec \dots$. Let $(t_n)_{n=1}^{\infty} \subseteq \mathcal{F}$ converges pointwise to some subset t of $\mathbb{N}$. If t is finite, then t is an initial segment of some t_{n_0} for some $n_0 \in \mathbb{N}$. Since $\mathcal{F}$ is a tree, $t \in \mathcal{F}$. If $t = (n_1, n_2, \dots)$ with $n_1 < n_2 < \dots$, then we set $s_i = (n_1, n_2, \dots, n_i)$ for every $i \in \mathbb{N}$. Of course $s_1 \prec s_2 \prec \dots$. Let $s_n^i = t_n \cap [0, n_i]$ for every $i \in \mathbb{N}$ and $n \in \mathbb{N}$. It is easy to see that the sequence $(s_n^i)_{n=1}^{\infty}$ in $\mathcal{F}$ converges pointwise to s_i. According to the previous case, $s_i \in \mathcal{F}$, for every $i \in \mathbb{N}$. A contradiction to our assumption, so t is finite and $t \in \mathcal{F}$. Hence, $\mathcal{F}$ is pointwise closed.

(ii) It is easily proved, using (i).

Now, using Propositions 3.1, 3.6 and the concept of the strong Cantor-Bendixson index (Proposition 2.9) we state and prove the stronger form of the Nash-Williams partition theorem for hereditary families of finite subsets of $\mathbb{N}$.

Theorem 3.7 (=Theorem B, Stronger form of Nash-Williams partition theorem for hereditary families) Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}$ and M an infinite subset of $\mathbb{N}$. We have the following cases:

[**Case 1**] The family $\mathcal{F} \cap [M]^{<\omega}$ is not pointwise closed.
Then, there exists $L \in [M]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$.

[**Case 2**] The family $\mathcal{F} \cap [M]^{<\omega}$ is pointwise closed. Then, setting

$$\xi_M^{\mathcal{F}} = sup\{s_L(\mathcal{F}) : L \in [M]\} ,$$

which is a countable ordinal, the following subcases obtain:

2(i) If $\xi_M^{\mathcal{F}} > \xi + 1$, then there exists $L \in [M]$ such that

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} ;$$

2(ii) if $\xi_M^{\mathcal{F}} < \xi + 1$, then for every $I \in [M]$ there exists $L \in [I]$

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F} \ ; \ (\text{ equivalently such that} \mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi) \ ; \text{ and,}$$

2(iii) if $\xi_M^{\mathcal{F}} = \xi + 1$, then there exists $L \in [M]$ such that

$$\text{either } \ \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \ \text{ or } \ \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}.$$

Both alternatives in 2(iii) may materialize.

Proof [Case 1] If the family $\mathcal{F} \cap [M]^{<\omega}$ is not pointwise closed, then there exists $L \in [M]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$, according to Proposition 3.6.
[Case 2] Let $\mathcal{F} \cap [M]^{<\omega}$ be pointwise closed. Then $\xi_M^{\mathcal{F}}$ is a countable ordinal. Indeed, since the Cantor-Bendixson index $O(\mathcal{F})$ of $\mathcal{F}$ (see [K]) is a countable ordinal (as the family of derived sets of $\mathcal{F}$ is countable) and since $s_I(\mathcal{F}) \leq O(\mathcal{F})$ for every $I \in [\mathbb{N}]$, we have $\xi_M^{\mathcal{F}} \leq O(\mathcal{F}) < \omega_1$.

2(i) Let $\xi + 1 < \xi_M^{\mathcal{F}}$. Then, there exists $I \in [M]$ such that $\xi + 1 < s_I(\mathcal{F})$. According to Theorem 1.5 and Proposition 3.1, there exists $L \in [I]$ such that
either $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$ or $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi \subseteq (\mathcal{A}_\xi)_\star$.
The second alternative is impossible. Indeed, if $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)_\star$, then, using Proposition 2.9 and Remark 2.7, we have
$s_I(\mathcal{F}) \leq s_L(\mathcal{F}) = s_L(\mathcal{F} \cap [L]^{<\omega}) \leq s_L((\mathcal{A}_\xi)_\star) = \xi + 1$.
This is a contradiction; hence $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$.

2(ii) Let $\xi_M^{\mathcal{F}} < \xi + 1$ and $I \in [M]$. According to the Ramsey partition type theorem for the countable ordinal ξ (Theorem 1.5), there exists $L \in [I]$ such that
either $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$ or $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$.
The first alternative is impossible. Indeed, if $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$, then, using Proposition 2.9 and Remark 2.7, we obtain $\xi + 1 = s_L((\mathcal{A}_\xi \cap [L]^{<\omega})_\star) \leq s_L(\mathcal{F}) \leq \xi_M^{\mathcal{F}}$.
This is a contradiction; hence, $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ and equivalently,
$\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi$, according to Proposition 3.1.

2(iii) That both alternatives in the case $\xi_M^{\mathcal{F}} = \xi + 1$ may materialize can be seen by considering two simple examples:

(1) $\mathcal{F} = \{s \in [\mathbb{N}]^{<\omega} : s \neq \emptyset \text{ and } |s| = 2min\ s + 1\}$,
where $|s|$ denotes the cardinality of s.
(It is easy to see that $\mathcal{F}(n) = [\mathbb{N} \cap (n, +\infty)]^{2n} = \mathcal{A}_{2n} \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega}$ for every $n \in \mathbb{N}$. The family $\mathcal{F}_\star$ is pointwise closed and according to Lemma 2.8, $s_I(\mathcal{F}_\star) = \omega + 1$ for every $I \in [\mathbb{N}]$. Hence $\xi_M^{\mathcal{F}_\star} = \omega + 1$ for every $M \in [\mathbb{N}]$. It is now easy to verify that $\mathcal{A}_\omega \cap [L]^{<\omega} \subseteq \mathcal{F}_\star$ for every $L \in [M]$) ; and,

(2) $\mathcal{F} = \{s \in [M]^{<\omega} : s \neq \emptyset \text{ and } |s| = \frac{min\ s}{2}\}$,
where M stands for all non zero, even natural numbers.
(Since $\mathcal{F}(m) = \mathcal{A}_{\frac{m}{2}-1} \cap [M \cap (m, +\infty)]^{<\omega}$ for every $m \in M$, from Lemma 2.8 we get that $s_I(\mathcal{F}_\star) = \omega + 1$ for every $I \in [M]$. Thus $\xi_M^{\mathcal{F}_\star} = \omega + 1$. It is now easy to verify

that $\mathcal{F}_\star \cap [L]^{<\omega} \subseteq (\mathcal{A}_\omega)^\star \setminus \mathcal{A}_\omega$ for every $L \in [M]$ and, according to Proposition 2.1, that $\mathcal{A}_\omega \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}_\star$ for every $L \in [M]$.)

As a corollary of Theorem 3.7 we have the following result of Argyros, Merkourakis and Tsarpalias ([A-M-T]).

Corollary 3.8 Let $\mathcal{F}$ be a hereditary and pointwise closed family of finite subsets of $\mathbb{N}$, If there exists $M \in [\mathbb{N}]$ such that $s_M(\mathcal{F}) \geq \omega^\alpha$, then there exists $L \in [M]$ such that $\mathcal{F}_\alpha(L) \subseteq \mathcal{F}$.

Proof If $s_M(\mathcal{F}) > \omega^\alpha + 1$, then, according to Theorem 3.7, there exists $N \in [M]$ such that $\mathcal{B}_\alpha \cap [N]^{<\omega} \subseteq \mathcal{F}$ and according to Corollary 3.4 and Proposition 2.4 there exists $L \in [N]$ such that $\mathcal{F}_\alpha(L) \subseteq (\mathcal{B}_\alpha)^\star \cap [L]^{<\omega} \subseteq (\mathcal{B}_\alpha \cap [N]^{<\omega})^\star \subseteq \mathcal{F}$.

Now, if $s_M(\mathcal{F}) = \omega^\alpha + 1$, then we set $\mathcal{L} = \{\{m\} \cup s : s \in \mathcal{F}, m \in M \text{ and } \{m\} < s\}$. It is easy to see that $s_M(\mathcal{L}) > \omega^\alpha + 1$. So applying the previous case to the family $\mathcal{L}$ we can find $N = (n_i)_{i=1}^\infty \in [M]$ such that $\mathcal{F}_\alpha(N) \subseteq \mathcal{L}$. Setting $L = (n_i)_{i=3}^\infty$ we have that $\mathcal{F}_\alpha(L) \subseteq \mathcal{F}$, as required.

The version of Theorem B for trees is given below.

Definition 3.9 Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$. We set

(i) $\mathcal{F}_h = \{s \in \mathcal{F} :$ every non- empty subset of s belongs to $\mathcal{F}\} \cup \{\emptyset\}$, and

Of course, $\mathcal{F}_h$ is the largest subfamily of $\mathcal{F}$ which is hereditary.

Theorem 3.10 (= Theorem B′, Stronger form of Nash-Williams partition theorem for trees) Let $\mathcal{F}$ be a tree of finite subsets of $\mathbb{N}$ and M an infinite subset of $\mathbb{N}$. We have the following cases:

[Case 1] The family $\mathcal{F}_h \cap [M]^{<\omega}$ is not pointwise closed.
Then, there exists $L \in [M]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$.

[Case 2] The family $\mathcal{F}_h \cap [M]^{<\omega}$ is pointwise closed. Then setting

$$\zeta_M^{\mathcal{F}} = sup\{s_L(\mathcal{F}_h) : L \in [M]\} = \xi_M^{\mathcal{F}_h} ,$$

which is a countable ordinal, the following subcases obtain:

2(i) If $\zeta_M^{\mathcal{F}} > \xi + 1$, then there exists $L \in [M]$ such that

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} ;$$

2(ii) if $\zeta_M^{\mathcal{F}} < \xi$, then for every $I \in [M]$ there exists $L \in [I]$ such that

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F} \ ; \ (\text{ equivalently } \mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi) \ ; \text{ and,}$$

2(iii) if $\zeta_M^{\mathcal{F}} = \xi + 1$ or $\zeta_M^{\mathcal{F}} = \xi$, then there exists $L \in [M]$ such that

$$\text{either } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F} .$$

Proof [Case 1] If the hereditary family $\mathcal{F}_h \cap [M]^{<\omega}$ is not pointwise closed, then there exists $L \in [M]$ such that $[L]^{<\omega} \subseteq \mathcal{F}_h \subseteq \mathcal{F}$, according to Proposition 3.6.

[Case 2] Let $\mathcal{F}_h \cap [M]^{<\omega}$ be pointwise closed. Then $\zeta_M^{\mathcal{F}}$ is a countable ordinal, according to Theorem 3.7.

2(i) Let $\xi + 1 < \zeta_M^{\mathcal{F}}$. Then $\xi + 1 < \xi_M^{\mathcal{F}_h}$. According to Theorem 3.7 (subcase 2(i)) there exists $L \in [M]$ such that $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}_h \subseteq \mathcal{F}$.

2(ii) Let $\zeta_M^{\mathcal{F}} < \xi$ and $I \in [M]$. Then, according to Theorem 3.7 (subcase 2(ii)) there exists $M_1 \in [I]$ such that

$$(\star) \qquad \mathcal{A}_{\zeta_M^{\mathcal{F}}} \cap [M_1]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}_h .$$

Using the Ramsey partition theorem for the countable ordinal ξ (Theorem 1.5), there exists an infinite subset L of M_1 such that

either $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$, or $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$.

We claim that the first alternative does not hold. Indeed, let $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$. Then $(\mathcal{A}_\xi \cap [L]^{<\omega})^\star \subseteq \mathcal{F}^\star = \mathcal{F}$. Using the canonical representation of every infinite subset of $\mathbb{N}$ with respect to $\mathcal{A}_\xi$ (Proposition 2.4), it is easy to check that $(\mathcal{A}_\xi)^\star \cap [L]^{<\omega} = (\mathcal{A}_\xi \cap [L]^{<\omega})^\star$. Hence, $(\mathcal{A}_\xi)^\star \cap [L]^{<\omega} \subseteq \mathcal{F}$.

Since $\xi > \zeta_M^{\mathcal{F}}$ and according to Corollary 3.2, there exists $L_1 \in [L]$ such that $(\mathcal{A}_{\zeta_M^{\mathcal{F}}})_\star \cap [L_1]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \cap [L]^{<\omega} \subseteq \mathcal{F}$ and consequently $(\mathcal{A}_{\zeta_M^{\mathcal{F}}})_\star \cap [L_1]^{<\omega} \subseteq \mathcal{F}_h$. This is a contradiction to $(\star)$; hence, $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ and equivalently $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi$, according to Proposition 3.1

2(iii) In the cases $\zeta_M^{\mathcal{F}} = \xi + 1$ or $\zeta_M^{\mathcal{F}} = \xi$ we use Theorem 1.5.

Corollary 3.11 (=Theorem C, Stronger form of Nash-Williams theorem in Gowers reformulation) Let $\mathcal{F}$ be a tree of finite subsets of $\mathbb{N}$. Then there exists an infinite subset L of $\mathbb{N}$, such that

either (i) $[L]^{<\omega} \subseteq \mathcal{F}$;

or (ii) there is a countable ordinal ξ_0, such that for every infinite subsets I of L, there exists an initial segment s of I which belongs to $[\mathbb{N}]^{<\omega} \setminus \mathcal{F}$, and which is that unique initial segment of I that belongs to $\mathcal{A}_{\xi_0}$.

Proof We apply Theorem 3.10 (=Theorem B′ in $\mathcal{F}$).

If [Case 1] of Theorem 3.10 holds, then there exists $L \in [\mathbb{N}]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$.

If [Case 2] of Theorem 3.10 holds, then there is a countable ordinal $\xi_0 = \zeta_{\mathbb{N}}^{\mathcal{F}} + 1$ and $L \in [\mathbb{N}]$ such that $\mathcal{A}_{\xi_0} \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$. According to Proposition 2.4 every infinite subset I of L, has unique canonical representation with respect to $\mathcal{A}_{\xi_0}$, hence for every $I \in [L]$ there exists a unique initial segment $s_{\xi_0,I}$ of I that belongs to $\mathcal{A}_{\xi_0}$ and consequently to $[\mathbb{N}]^{<\omega} \setminus \mathcal{F}$.

Remark 3.12 Theorem C is indeed a stronger form of the classical Nash-Williams partition theorem, because it implies the Gowers reformulation of the Nash-Williams partition theorem (as given in the introduction of this paper). To see that indeed Theorem C implies Gowers reformulation, let $\mathcal{F}$ be any family of finite subsets of $\mathbb{N}$. We set

$\mathcal{F}_t = \{s \in \mathcal{F} :$ every non empty initial segment of s belongs to $\mathcal{F}\} \cup \{\emptyset\}$.

The family $\mathcal{F}_t$ is a tree contained in $\mathcal{F}$. We apply Theorem C on $\mathcal{F}_t$. It follows that there exists an infinite subset L of $\mathbb{N}$ such that.

either (i)$[L]^{<\omega} \subseteq \mathcal{F}_t$ (and consequently $[L]^{<\omega} \subseteq \mathcal{F}$)

or (ii) for every infinite subset I of $\mathbb{N}$ there exists an initial segment u of I which belongs to $[\mathbb{N}]^{<\omega} \setminus \mathcal{F}_t = (\mathcal{F} \setminus \mathcal{F}_t) \cup ([\mathbb{N}]^{<\omega} \setminus \mathcal{F}$. Thus either $u \in [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ (in which case we set $s = u$), or $u \in \mathcal{F} \setminus \mathcal{F}_t$ (in which case, by the definition of $\mathcal{F}_t$, there is a non empty initial segment s of u so that $s \in [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$). Hence, in any of the two cases in (ii), for every infinite subset I of $\mathbb{N}$, there is an initial segment s of I which belongs to $[\mathbb{N}]^{<\omega} \setminus \mathcal{F}$, proving the Gowers reformulation of Nash-Williams theorem.

Remark 3.13 Gowers notices in [G], that if the first alternative (i) of the reformation of Nash-Williams's theorem does not hold, then $[\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ is large in an obvious sense and Nash-Williams's theorem asserts that if $[\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ is a large subset of $[\mathbb{N}]^{<\omega}$, then there is an infinite subset L of $\mathbb{N}$ for which $[\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ has a stronger largeness property (alternative (ii)). Theorem 3.10 (=Theorem B$'$) is stronger than the Nash-Williams's theorem in the part that in the second alternative (ii) the initial segments are located (uniformly for all infinite subsets) in the family $\mathcal{A}_\xi$ and consequently $[\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ has a much stronger largeness property than the given by Nash-Williams's theorem.

Finally we state our strengthening of the Nash-Williamsn [N-W] partition theorem in its original formulation the one concerning of pointwise closed families of infinite subsets of $\mathbb{N}$.

Firstly, we will give the necessary definitions.

Definition 3.14 Let M be an infinite subset of $\mathbb{N}$, s a finite subset of $\mathbb{N}$ and ξ a countable ordinal. We set

(i) $[s, M] = \{s \cup L : L \in [M]$ and $s < L\}$, $\quad [\emptyset, M] = [M]$.

(ii) $s_{\xi,M}$ is the unique initial segment of M which is an element of $\mathcal{A}_\xi$ (according to Proposition 2.4); note that $s_{0,M} = \emptyset$.

Theorem 3.15 (Stronger form of Nash-Williams's theorem) Let $\mathcal{U}$ be a pointwise closed family of infinite subsets of $\mathbb{N}$ and M an infinite subset of $\mathbb{N}$. Then

either (i) there exists $L \in [M]$ such that $[L] \subseteq \mathcal{U}$;

or (ii) there exists a countable ordinal $\zeta_M^{\mathcal{U}}$ such that for every countable ordinal ξ with $\xi > \zeta_M^{\mathcal{U}}$ and every $M_1 \in [M]$ there exists $L \in [M_1]$ such that for every infinite subset I of L the unique initial segment $s_{\xi,I}$ of I that belongs to $\mathcal{A}_\xi$ satisfies the relation $[s_{\xi,I}, \mathbb{N}] \subseteq [\mathbb{N}] \setminus \mathcal{U}$.

Proof Let $\mathcal{F} = \{s \in [\mathbb{N}]^{<\omega}\colon\ [s, \mathbb{N}] \cap \mathcal{U} \neq \emptyset\}$. Of course $\mathcal{F}$ is a tree. We use Theorem 3.10.

If [Case 1] of Theorem 3.10 holds, then there exists $L \in [M]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$. Then, $[s, \mathbb{N}] \cap \mathcal{U} \neq \emptyset$ for every $s \in [L]^{<\omega}$. This gives that $[L] \subseteq \mathcal{U}$, since $\mathcal{U}$ is a pointwise closed family.

If [Case 2] of Theorem 3.10 holds, then setting $\zeta_M^{\mathcal{U}} = \zeta_M^{\mathcal{F}}$ we have $\zeta_M^{\mathcal{U}} < \omega_1$ and for every $\xi > \zeta_M^{\mathcal{U}}$ and every $M_1 \in [M]$ there exists $L \in [M_1]$ such that $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$. For every $I \in [L]$ let $s_{\xi,I}$ be the unique initial segment of I which is an element of $\mathcal{A}_\xi$ (Proposition 2.4). Then $s_{\xi,I} \in [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ for every $I \in$ [L]. Hence, $[s_{\xi,I}, \mathbb{N}] \subseteq [\mathbb{N}] \setminus \mathcal{U}$ for every $I \in [L]$.

Immediate consequence of Theorem 3.15 is the classical Nash-Williams partition theorem:

Corollary 3.15 (Nash-Williams [N-W]) Let $\mathcal{U}$ be a pointwise closed family of infinite subsets of $\mathbb{N}$ and M an infinite subset of $\mathbb{N}$. Then

either (i) there exists $L \in [M]$ such that $[L] \subseteq \mathcal{U}$;

or (ii) there exists $L \in [M]$ such that $[L] \subseteq [\mathbb{N}] \setminus \mathcal{U}$, equivalently, such that for every infinite subset I of L there exists an initial segment s of I such that $[s, \mathbb{N}] \subseteq \setminus \mathcal{U}$.

4. The derivation of Ellentuck's theorem

We finally show that our Theorem 3.10(= Theorem B′) implies, using the simple argument contained in Theorem 4.6, Ellentuck's theorem (and hence, Galvin-Prikry's and Silver's).

We recall the definition of the completely Ramsey families, given initially in [G,P] and [S].

Definition 4.1 A family $\mathcal{U}$ of infinite subsets of $\mathbb{N}$ is called **completely Ramsey** if for every $\alpha \in [\mathbb{N}]^{<\omega}$ and $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that

either (i) $[\alpha, L] \subseteq \mathcal{U}$;

or (ii) $[\alpha, L] \subseteq [\mathbb{N}] \setminus \mathcal{U}$.

Theorem 4.2 Let $\mathcal{U}$ be a pointwise close family of infinite subsets of $\mathbb{N}$, α a finite subset of $\mathbb{N}$ with cardinality m and M an infinite subset of $\mathbb{N}$. Then

either (i) there exists $L \in [M]$ such that $[\alpha, L] \subseteq \mathcal{U}$;

or (ii) there exists a countable ordinal $\zeta^{\mathcal{U}}_{\alpha,M}$ such that for every countable ordinal ξ with $\xi > \zeta^{\mathcal{U}}_{M}$ and every $M_1 \in [M]$ there exists $L \in [M_1]$ such that $[\alpha \cup s_{\xi,I}, \mathbb{N}] \subseteq [\mathbb{N}] \setminus \mathcal{F}$ for every infinite subset I of L.

Proof Let $\mathcal{F} = \{s \in [\mathbb{N}]^{<\omega}\colon\ [\alpha \cup s, \mathbb{N}] \cap \mathcal{U} \neq \emptyset\}$. Of course $\mathcal{F}$ is a tree. We use Theorem 2.10.

Corollary 4.3 (Galvin-Prikry [G-P]) Every pointwise closed (resp. pointwise open) family of infinite subsets of $\mathbb{N}$ is completely Ramsey.

Definition 4.4 Ellentuck's topology on $[\mathbb{N}]$ is the topology which has base the family of all sets $[\alpha, M]$, where $\alpha \in [\mathbb{N}]^{<\omega}$ and $M \in [\mathbb{N}]$. Of course Ellentuck's topology is weaker than the topology of pointwise convergence.
We denote by $\hat{\mathcal{U}}$ and $\mathcal{U}^{\diamond}$ the closure and the interior respectively of a subset $\mathcal{U}$ of $\mathbb{N}$ in the Ellentuck's topology. Then, it is easy to see that

(i) $\hat{\mathcal{U}} = \{L \in [\mathbb{N}] : [s_{n,L}, L] \cap \mathcal{U} \neq \emptyset$ for every $n \in \mathbb{N}$; and

(ii) $\mathcal{U}^{\diamond} = \{L \in [\mathbb{N}] :$ there exists $n \in \mathbb{N}$ such that $[s_{n,L}, L] \subseteq \mathcal{U}\}$.

Lemma 4.5 Let $\mathcal{L} \subseteq \{[s, I] :\quad s \in [\mathbb{N}]^{<\omega}$ and $I \in [\mathbb{N}]\}$ with the following two properties:

(i) For every $(\alpha, M) \in [\mathbb{N}]^{<\omega} \times [\mathbb{N}]$ there exists $I \in [M]$ such that $[\alpha, I] \in \mathcal{L}$; and

(ii) if $[s, I] \in \mathcal{L}$, then $[s, L] \in \mathcal{L}$ for every $L \in [I]$.

Then for every $(\alpha, M) \in [\mathbb{N}]^{<\omega} \times [\mathbb{N}]$ there exists $L \in [\alpha, M]$ such that $[\alpha \cup \beta, L] \in \mathcal{L}$ for every $\beta \in [L]^{<\omega}$.

Proof Let $(\alpha, M) \in [\mathbb{N}]^{<\omega} \times [\mathbb{N}]$ and let m be the cardinality of α. We can assume that $\alpha \prec M$. Set $L_0 = M$. According to property (i) of $\mathcal{L}$ there exists $L_1 \in [M]$ such that $[\alpha, L_1] \in \mathcal{L}$. Set $L_1 = I$. Let $L_1 \subseteq \ldots \subseteq L_n$ have been constructed and let $\{s_1, \ldots, s_r\} = \{s \in [L_n]^{<\omega} : s \subseteq s_{m+n,L_n}$ and $s \not\subseteq s_{m+n-1,L_n}\}$. According to property (i) of $\mathcal{L}$ there exists $I^1_{n+1} \in [L_n]$ such that $[s_1, I^1_{n+1}] \in \mathcal{L}$. Setting $L^1_{n+1} = I^1_{n+1} \cup s_{m+n,L_n}$ we have that $L^1_{n+1} \in [s_{m+n,L_n}, L_n]$ and that $[s_1, L^1_{n+1}] =$

$[s_1, I^1_{n+1}] \in \mathcal{L}$. Analogously, we can choose $L^2_{n+1} \in [s_{m+n,L_n}, L^1_{n+1}]$ such that $[s_2, L^2_{n+1}] \in \mathcal{L}$ and so on. Set $L_{n+1} = L^r_{n+1}$.

Since $L_{n+1} \in [s_{m+n,L_n}, L_n]$ for every $n \in \mathbb{N}$ there exists $L \in [M]$ such that $s_{m+n,L} = s_{m+n,L_n}$ for every $n \in \mathbb{N}$. Hence, L has the desired property, according to property (ii) of $\mathcal{L}$.

Theorem 4.6 Let $\mathcal{U}$ be a family of infinite subsets of $\mathbb{N}$, M an infinite subset of $\mathbb{N}$ and α a finite subset of $\mathbb{N}$. Then there exists $L \in [M]$ such that

either (i) $[\alpha, L] \subseteq \hat{\mathcal{U}}$;

or (ii) $[\alpha, L] \subseteq [\mathbb{N}] \setminus \mathcal{U}$.

Proof Let $(\alpha, M) \in [\mathbb{N}]^{<\omega} \times [\mathbb{N}]$. Set
$\mathcal{L}^{\mathcal{U}} = \{[s, I] : \text{either } [s, I] \cap \mathcal{U} = \emptyset \quad \text{or } [s, I_1] \cap \mathcal{U} \neq \emptyset \text{ for every } I_1 \in [I]\}$.
It is easy, to check that $\mathcal{L}^{\mathcal{U}}$ satisfies the assumptions (i) and (ii) of Lemma 3.9, hence there exists $I \in [\alpha, M]$ such that $[\alpha \cup \beta, I] \in \mathcal{L}^{\mathcal{U}}$ for every $\beta \in [I]^{<\omega}$. We assume that $[\alpha, I_1] \cap \mathcal{U} \neq \emptyset$ for every $I_1 \in [I]$.

Set $\mathcal{F} = \{\beta \in [I]^{<\omega} : \alpha < \beta \text{ and } [\alpha \cup \beta, I_1] \cap \mathcal{U} \neq \emptyset \text{ for every } I_1 \in [I]\}$. The family $\mathcal{F}$ is a tree. We use Theorem 3.10.

If [Case 1] of Theorem 3.10 holds, then there exists $L \in [I]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$. Then $[\alpha, L] \subseteq \hat{\mathcal{U}}$.

[Case 2] of Theorem 3.10 does not occur. Let $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ for some $\xi < \omega_1$. Then $[\alpha, L] \cap \mathcal{U} = \emptyset$. Indeed, let $L_2 \in [\alpha, L] \cap \mathcal{U}$, $L_2 = \alpha \cup L_1$, $L_1 \in [L]$ and $\alpha < L_1$. Then $L_2 \in [\alpha \cup s_{\xi, L_1}, I] \cap \mathcal{U}$ and consequently $s_{\xi, L_1} \in \mathcal{A}_\xi \cap [L]^{<\omega} \cap \mathcal{F}$. A contradiction; hence $[\alpha, L] \cap \mathcal{U} = \emptyset$. This is a contradiction to our assumption that $[\alpha, I_1] \cap \mathcal{U} \neq \emptyset$ for every $I_1 \in [I]$.

Hence, either there exists $L \in [M]$ such that $[\alpha, L] \subseteq [\mathbb{N}] \setminus \mathcal{U}$ or there exists $L \in [M]$ such that $[\alpha, L] \subseteq \hat{\mathcal{U}}$.

Corollary 4.7 Every family of infinite subsets of $\mathbb{N}$ which is closed (resp. is open) in the Ellentuck's topology, is completely Ramsey.

Corollary 4.8 Let $\mathcal{U}$ be a family of infinite subset of $\mathbb{N}$ which is a meager set in the Ellentuck's topology, $M \in [\mathbb{N}]$ and $\alpha \in [\mathbb{N}]^{<\omega}$. Then there exists $L \in [M]$ such that $[\alpha, L] \subseteq [\mathbb{N}] \setminus \mathcal{U}$.

Proof Let $\mathcal{U} = \bigcup\limits_{n=0}^{\infty} \mathcal{U}_n$ where $(\hat{\mathcal{U}_n})^\diamond = \emptyset$ for every $n = 0, 1, \ldots$. According to Theorem 3.10, there exists $L \in [M]$ such that

either (i) $[\alpha, L] \subseteq \hat{\mathcal{U}}$;

or (ii) $[\alpha, L] \subseteq [\mathbb{N}] \setminus \mathcal{U}$.

We will prove that the first alternative is impossible. Let $L \in [M]$. Set $\mathcal{L} = \{[s, I] : s \in [\mathbb{N}]^{<\omega}, I \in [\mathbb{N}]$ and $[s, I] \cap \mathcal{U}_\kappa = \emptyset$ for every $\kappa \in \mathbb{N}$ with $\kappa \leq |s|\}$.

The family $\mathcal{L}$ satisfies the assumptions (i) and (ii) of Lemma 3.9, according to Theorem 3.10. Hence there exists $L_1 \in [\alpha, L]$ such that $[\alpha \cup \beta, L_1] \cap \mathcal{U}_\kappa = \emptyset$ for every $\beta \in [L_1]^{<\omega}$ and $\kappa \in \mathbb{N}$ with $\kappa \leq |\alpha \cup \beta|$. Then $L_1 \notin \hat{\mathcal{U}}$, since $[\alpha, L_1] \cap \mathcal{U} = \emptyset$. Indeed, if $L_2 \in [\alpha, L_1] \cap \mathcal{U}$, then $L_2 \in \mathcal{U}_\kappa$ for some $\kappa = 0, 1, \dots$ and choosing an initial segment β of L_2 such that $|\alpha \cup \beta| \geq \kappa$ we have $L_2 \in [\alpha \cup \beta, L_2] \cap \mathcal{U}_\kappa = \emptyset$. A contradiction; hence $I_1 \notin \hat{\mathcal{U}}$ and consequently $[\alpha, L] \not\subseteq \hat{\mathcal{U}}$.

Corollary 4.9 (Ellentuck [E]) A family $\mathcal{U}$ of infinite subsets of $\mathbb{N}$ is completely Ramsey if and only if $\mathcal{U}$ has the Baire property in Ellentuck's topology.

Proof Let $\mathcal{U}$ has the Baire property in Ellentuck's topology. Then, setting $\mathcal{C}^c = [\mathbb{N}] \setminus \mathcal{C}$ for every $\mathcal{C} \subseteq [\mathbb{N}]$, we have $\mathcal{U} = \mathcal{B} \triangle \mathcal{C} = (\mathcal{B} \cap \mathcal{C}^c) \cup (\mathcal{C} \cap \mathcal{B}^c)$ where $\mathcal{B}$ is a closed set and $\mathcal{C}$ a meager set in Ellentuck's topology. According to Corollary 3.12, there exists $L_1 \in [\mathbb{N}]$ such that $[\alpha, L_1] \subseteq \mathcal{C}^c$. According to Theorem 3.10, there exists $L \in [L_1]$ such that

either (i) $[\alpha, L] \subseteq \mathcal{B} \cap \mathcal{C}^c \subseteq \mathcal{U}$;

or (ii) $[\alpha, L] \subseteq \mathcal{B}^c \cap \mathcal{C}^c \subseteq [\mathbb{N}] \setminus \mathcal{U}$

Hence $\mathcal{U}$ is completely Ramsey.

On the other hand, if $\mathcal{U}$ is completely Ramsey, then $\mathcal{U}$ has the Baire property in Ellentuck's topology, since $\mathcal{U} = \mathcal{U}^\diamond \cup (\mathcal{U} \setminus \mathcal{U}^\diamond)$ and $\mathcal{U} \setminus \mathcal{U}^\diamond$ is a meager set in Ellentuck's topology.

Remark 4.10 (i) (Galvin-Prikry [G-P]) Every family of finite subsets of $\mathbb{N}$ which is a Borel set in the topology of pointwise convergence is completely Ramsey, since every Borel set has the Baire property.

(ii) (Silver [S]) Every family of finite subsets of $\mathbb{N}$ which is an analytic set in the topology of pointwise convergence is completely Ramsey, since every analytic set has the Baire property.

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