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Systems of Ramsey families (**).

Abstract. – *A system of Ramsey families on an infinite subset M of the set $\mathbb{N}$ of all natural numbers is a collection $(\mathcal{A}_\xi)_{\xi<\omega_1}$ of families of finite subsets of M (where, $\mathcal{A}_k$ consists of all k-element subsets of M, for $k \in \mathbb{N}$) such that the Cantor-Bendixson index, defined for $\mathcal{A}_\xi$, is equal to $\xi+1$ for every $\xi<\omega_1$. Using that notion we establish a far-reaching generalization of the classical Ramsey theorem (which corresponds to the finite ordinal indices) for every countable order. Indeed, for an arbitrary family $\mathcal{F}$ of finite subsets of $\mathbb{N}$, we obtain the following:*
(1) *For every infinite subset M of $\mathbb{N}$ and every countable ordinal ξ, there is an infinite subset L of M such that either $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$ or $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$; (where $[L]^{<\omega}$ denotes the family of all finite subsets of L).*
(2) *If in addition $\mathcal{F}$ is hereditary and pointwise closed, then for every infinite subset M of $\mathbb{N}$ there is a countable ordinal number $\xi_M^{\mathcal{F}}$ such that:* (a) *for ξ with $\xi+1<\xi_M^{\mathcal{F}}$ there is an infinite subset L of M such that $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$; (b) for ξ with $\xi_M^{\mathcal{F}}<\xi+1$ there is an infinite subset L of M such that $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ and equivalently $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi$; ($(\mathcal{A}_\xi)^\star$ denotes the family of all initial segments of elements of $\mathcal{A}_\xi$)* (c) *for $\xi+1=\xi_M^{\mathcal{F}}$, both alternatives* ((a) *and* (b)) *may materialize.*
We note that an analogous Ramsey principle has been developed in a previous paper of the author, using the related notion of the complete thin Schreier system.

Introduction.

In a previous paper ([F]) we developed a Ramsey-type principle for every countable ordinal (the classical Ramsey theorem corresponds to the finite ordinals). The vehicle for developing this principle was the

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complete thin Schreier system consisting of families of finite subsets of $\mathbb{N}$ (Definition 1.4). In the present paper we extend this result proving the analogous Ramsey dichotomies, using the concept of a system of Ramsey families, a wider concept than the complete thin Schreier system.

A system of Ramsey families is a collection $(\mathcal{A}_\xi)_{\xi<\omega_1}$ of families of finite subsets of $\mathbb{N}$ (for $k\in\mathbb{N}$, $\mathcal{A}_k=[\mathbb{N}]^k$ the set of all k-element subsets of $\mathbb{N}$) with the following properties: (i) each $\mathcal{A}_\xi$ is thin (i.e it does not contain proper initial segments of any of its elements) (Proposition 1.7); (ii) the Cantor Bendixson index defined for $\mathcal{A}_\xi$ is precisely equal to $\xi+1$ and does not decrease, but on the contrary is stable, when we restrict ourselves to any infinite subset of $\mathbb{N}$ (Theorem 1.12); and (iii) every finite subset of $\mathbb{N}$ has a «canonical representation» with respect to $\mathcal{A}_\xi$ for $\xi<\omega_1$ (Theorem 1.14).

Every system of Ramsey families is characterized by the choice for each countable limit ordinal number ξ of a strictly increasing sequence $(\xi_n)_{n\in\mathbb{N}}$ with $\sup_n \xi_n=\xi$. With suitable choices one can define such systems that are useful for theoretical purposes or for applications. The complete thin Schreier system is the simpliest system of Ramsey families where the families $\mathcal{A}_{\omega^\alpha}$, for $\alpha<\omega_1$, are defined using only the familes $\mathcal{A}_{\omega^\beta}$ for $\beta<\alpha$ and not all the previous families (Theorem 1.6).

Using the preceding properties of the families of a system of Ramsey families we state a far-reaching generalization of the classical Ramsey theorem, for every countable ordinal index. Here is the statement of the theorem (Theorem 2.5) using the notation mentioned below.

Refined ξ-Ramsey type theorem: Let M be an infinite subset of $\mathbb{N}$ and $(\mathcal{A}_\xi)_{\xi<\omega_1}$ a system of Ramsey families on M. For an arbitrary family $\mathcal{F}$ of finite subsets of M we obtain the following:

1. For every countable ordinal ξ there exists an infinite subset L of M such that

$$\text{either } \mathcal{A}_\xi\cap[L]^{<\omega}\subseteq\mathcal{F} \text{ or } \mathcal{A}_\xi\cap[L]^{<\omega}\subseteq[\mathbb{N}]^{<\omega}\setminus\mathcal{F}.$$

2. If the family $\mathcal{F}$ is hereditary and pointwise closed, then setting

$$\xi_M^{\mathcal{F}}=\sup\{s_L(\mathcal{F}): L\in[M]\};$$

(where $s_L(\mathcal{F})$ is the strong Cartor-Bendixson index on L of $\mathcal{F}$ defined in Definition 1.8 below) the following obtained:

2(i) For every countable ordinal ξ with $\xi + 1 < \xi_M^{\mathcal{F}}$ there exists $L \in \in [M]$ such that

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}.$$

2(ii) For every countable ordinal ξ with $\xi_M^{\mathcal{F}} < \xi + 1$ there exists $L \in [M]$ such that

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F};$$

and equivalently,

$$\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi.$$

2(iii) If $\xi_M^{\mathcal{F}} = \xi + 1$, then both alternatives ((i) and (ii)) may materialize.

NOTATION. We denote by $\mathbb{N} = \{1, 2, \ldots\}$ the set of all natural numbers. For an infinite subset M of $\mathbb{N}$ we denote by $[M]^{<\omega}$ the set of all finite subsets of M, for $k \in \mathbb{N}$ we denote by $[M]^k$ the set of all k-element subsets of M, and by $[M]$ the set of all infinite subsets of M (considering them as strictly increasing sequences).

If s, t are finite subsets of $\mathbb{N}$, then $s \preccurlyeq t$ means that s is an initial segment of t, while $s \prec t$ means that s is a proper initial segment of t. We write $s \leqslant t$ if $\max s \leqslant \min t$, while $s < t$ if $\max s < \min t$.

Identifying every subset of $\mathbb{N}$ with its characteristic function, we topologize the set of all subsets of $\mathbb{N}$ by the topology of pointwise convergence.

Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$.

(i) $\mathcal{F}$ is **thin** if there are no elements s, $t \in \mathcal{F}$ with s a proper initial segment (in the order of the natural numbers) of t.

(ii) $\mathcal{F}^\star = \{t \in [\mathbb{N}]^{<\omega}$: t is an initial segment of some $s \in \mathcal{F}\}$.

(iii) $\mathcal{F}_\star = \{t \in [\mathbb{N}]^{<\omega}$: t is a subset of some $s \in \mathcal{F}\}$.

(iv) $\mathcal{F}$ is **hereditary** if $\mathcal{F}_\star = \mathcal{F}$.

(v) For $n \in \mathbb{N}$ we set $\mathcal{F}(n) = \{s \in [\mathbb{N}]^{<\omega}$: $\{n\} < s$ and $\{n\} \cup s \in \mathcal{F}\}$.

1. – Systems of Ramsey families.

DEFINITION 1.1. Let M be an infinite subset of $\mathbb{N}$. For every non-zero, limit countable ordinal ξ let $(\xi_m)_{m\in M}$ be a strictly increasing sequence of ordinals with $\sup_{m\in M} \xi_m = \xi$. A **system of Ramsey families on** M is a collection $\mathcal{A} = (\mathcal{A}_\xi)_{\xi<\omega_1}$ of finite subsets of M defined recursively as follows:

$$\mathcal{A}_0 = \{\emptyset\};$$

for every countable ordinal ζ

$$\mathcal{A}_{\zeta+1} = \{s \in [M]^{<\omega}: s = \{m\} \cup s, \text{ where } m \in M, \quad \{m\} < s \text{ and } s \in \mathcal{A}_\zeta\};$$

and for every non zero, limit countable ordinal ξ

$$\mathcal{A}_\xi = \{s \in [M]^{<\omega}: s \in \mathcal{A}_{\xi_m+1}, \text{ where } m = \min s\}.$$

REMARK 1.2. Let $M \in [\mathbb{N}]$ and $(\mathcal{A}_\xi)_{\xi<\omega_1}$ a system of Ramsey families on M. Then

(i) For every $L \in [M]$, the collection $(\mathcal{A}_\xi \cap [L]^{<\omega})_{\xi<\omega_1}$ is a system of Ramsey families on L.

(ii) $\emptyset \notin \mathcal{A}_\xi$ for every $1 \leqslant \xi < \omega_1$.

(iii) $\mathcal{A}_k = [M]^k$ for every $k \in \mathbb{N}$.

(iv) For every $k \in \mathbb{N}$ and $1 \leqslant \xi < \omega_1$, it is easy to prove by induction on k, that

$$\mathcal{A}_{\xi+k} = \{s \in [M]^{<\omega}: s = s_1 \cup s_2, \text{ where } s_1 < s_2,\ s_1 \in [M]^k \text{ and } s_2 \in \mathcal{A}_\xi\}.$$

(v) For every non zero, countable ordinal ξ and $m \in M$ we have

$$\mathcal{A}_\xi(m) = \mathcal{A}_\zeta \cap [\mathbb{N} \cap (m, +\infty)]^{<\omega}, \quad \text{if } \xi = \zeta + 1; \text{ and}$$

$$\mathcal{A}_\xi(m) = \mathcal{A}_{\xi_m} \cap [\mathbb{N} \cap (m, +\infty)]^{<\omega}, \quad \text{if } \xi \text{ is a limit ordinal},$$

where $(\xi_m)_{m\in M}$ is the related to ξ sequence, according to Definition 1.1.

There are as many systems of Ramsey families on $\mathbb{N}$ as the multitude of all the choices of strictly increasing sequences $(\xi_n)_{n\in\mathbb{N}}$ with $\sup_n \xi_n = \xi$, for each countable limit ordinal ξ. With suitable choices of sequences one can define interesting systems of Ramsey families.

The complete thin Schreier type system $\mathcal{A}=(\mathcal{A}_\xi)_{\xi<\omega_1}$, defined in [F], is the simpliest system of Ramsey families. This system in the ω^α-position (for every $\alpha<\omega_1$) has the family $\mathcal{B}_\alpha=\mathcal{A}_{\omega^\alpha}$ (Definition 1.4 below) which is similar to the Schreier set $\mathcal{F}_\alpha$ defined in [S], [A-O] and [A-A].

PROPOSITION 1.3 (*Representation of ordinals*, [C] [L]). Let α be a non-zero, countable ordinal. For every limit ordinal ξ, so that $\omega^\alpha<\xi<$ $<\omega^{\alpha+1}$, there exist a unique natural number $m\geqslant 0$, a sequence of ordinals $\alpha>\alpha_1\ldots>\alpha_m>0$ and natural numbers $p, p_1, \ldots, p_m\geqslant 1$ (so that either $p>1$ or $p=1$ and $m\geqslant 1$), such that

$$\xi=p\omega^\alpha+\sum_{i=1}^m p_i\omega^{\alpha_i}.$$

DEFINITION 1.4 ([F]) (*The complete thin Schreier system* $(\mathcal{A}_\xi)_{\xi<\omega_1}$). For every non zero, limit ordinal λ we choose and fix a strictly increasing sequence (λ_n) of successor ordinals smaller than λ with $\sup_n \lambda_n=\lambda$.

The system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ is defined as follows:

(1) [Case $\xi=0$]

$$\mathcal{A}_0=\{\emptyset\};$$

(2) [Case $\xi=\zeta+1$]

$$\mathcal{A}_\xi=\mathcal{A}_{\zeta+1}=\{s\subseteq\mathbb{N}: s=\{n\}\cup s_1, \text{ where } n\in\mathbb{N}, \{n\}<s_1 \text{ and } s_1\in\mathcal{A}_\zeta\};$$

(3) [Case $\xi=\omega^{\beta+1}$, β countable ordinal]

$$\mathcal{A}_\xi=\mathcal{A}_{\omega^{\beta+1}}=\left\{s\subseteq\mathbb{N}: s=\bigcup_{i=1}^n s_i \text{ with } n=\min s_1,\ s_1<\ldots<s_n,\right.$$

$$\left.\text{and } s_1, \ldots, s_n\in\mathcal{A}_{\omega^\beta}\right\};$$

(4) [Case $\xi=\omega^\lambda$, λ non-zero, countable limit ordinal]

$$\mathcal{A}_\xi=\mathcal{A}_{\omega^\lambda}=\{s\subseteq\mathbb{N}: s\in\mathcal{A}_{\omega^{\lambda_n}} \text{ with } n=\min s\},$$

(where (λ_n) is the sequence of ordinals, converging to λ, fixed above); and

(5) [Case ξ limit, $\omega^\alpha<\xi<\omega^{\alpha+1}$ for some $0<\alpha<\omega_1$].

Let $\xi = p\omega^{\alpha} + \sum_{i=1}^{m} p_i \omega^{\alpha_i}$ be the above representation (Proposition 1.3).

$$\mathscr{A}_{\xi} = \Big\{ s \subseteq \mathbb{N} : s = s_0 \cup \Big(\bigcup_{i=1}^{m} s_i \Big) \text{ with } s_m < \ldots < s_1 < s_0,$$

$$s_0 = s_1^0 \cup \ldots \cup s_p^0 \text{ with } s_1^0 < \ldots < s_p^0,\ s_j^0 \in \mathscr{A}_{\omega^{\alpha}},\ 1 \leqslant j \leqslant p,$$

$$s_i = s_1^i \cup \ldots \cup s_{p_i}^i, \text{ with } s_1^i < \ldots < s_{p_i}^i,\ s_j^i \in \mathscr{A}_{\omega^{\alpha_i}},\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant p_i \Big\}.$$

We set

$$\mathscr{B}_{\alpha} = \mathscr{A}_{\omega^{\alpha}} \text{ for each } 1 \leqslant \alpha < \omega_1.$$

REMARK 1.5 (i) $\mathscr{B}_1 = \mathscr{A}_{\omega} = \{ s \in [\mathbb{N}]^{<\omega} : s = (n_1 < \ldots < n_k) \text{ with } n_1 = k \}$. Thus $\mathscr{A}_{\omega}$ is a modification of the **classical Schreier family** ([S])

$$\mathscr{F}_1 = \{ s \subseteq \mathbb{N} : s = (n_1 < \ldots < n_k) \text{ with } n_1 \geqslant k \}.$$

In this sense $\mathscr{A}_{\omega}$ is a thin Schreier family.

(ii) $\mathscr{B}_{\alpha} = \mathscr{A}_{\omega^{\alpha}}$, for $\alpha < \omega_1$, is defined using only $\mathscr{B}_{\beta} = \mathscr{A}_{\omega^{\beta}}$ for $\beta < \alpha$, and not using all previously defined families $\mathscr{A}_{\xi}$, $\xi < \omega^{\alpha}$. $\mathscr{B}_k$, for $k \in \mathbb{N}$, is a modification of the generalized Schreier families defined by Alspach-Odell ([A-O]); and more generally $\mathscr{B}_{\alpha}$, for $\alpha < \omega_1$, is a modification of the families $\mathscr{F}_{\alpha}$, defined by Alspach-Argyros ([A-A]).

The complete thin Schreier system is a system of Ramsey families. The arguments for the proof is included in [F]. For completeness we give the proof analytically.

THEOREM 1.6. *The complete thin Schreier system $(\mathscr{A}_{\xi})_{\xi < \omega_1}$ is a system of Ramsey families on $\mathbb{N}$.*

PROOF. We will prove, by recursion on ξ, that for every $\xi < \omega_1$ there exists a sequence $(\xi(n))_{n \in \mathbb{N}}$ such that

$$\mathscr{A}_{\xi}(n) = \mathscr{A}_{\xi(n)} \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega} \text{ for every } n \in \mathbb{N},$$

where $\xi(n) = \xi - 1$ for every $n \in \mathbb{N}$ in case ξ is a successor ordinal and $(\xi(n), n \in \mathbb{N})$ is a strictly increasing sequence of ordinals with $\sup_n \xi(n) = \xi$, in case ξ is a limit ordinal.

(1) **[Case** $\xi = 1$] For every $n \in \mathbb{N}$ we have

$$\mathcal{A}_1(n) = \{s \subseteq \mathbb{N} : \{n\} < s \text{ and } \{n\} \cup s \in \mathcal{A}_1\}$$

$$= \{s \subseteq \mathbb{N} : \{n\} < s \text{ and } \{n\} \cup s \in [\mathbb{N}]^1\}$$

$$= \{\emptyset\} = \mathcal{A}_0 \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega}.$$

(2) **[Case** $\xi = \zeta + 1$] For every $n \in \mathbb{N}$ we have

$$\mathcal{A}_\xi(n) = \mathcal{A}_{\zeta+1}(n) = \{s \subseteq \mathbb{N} : \{n\} < s \text{ and } \{n\} \cup s \in \mathcal{A}_{\zeta+1}\}$$

$$= \{s \subseteq \mathbb{N} : \{n\} < s \text{ and } s \in \mathcal{A}_\zeta\} = \mathcal{A}_\zeta \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega}.$$

(3) **[Case** $\xi = \omega^{\beta+1}$ for $0 \leqslant \beta < \omega_1$] For every $n \in \mathbb{N}$ we have

$$\mathcal{A}_\xi(n) = \mathcal{A}_{\omega^{\beta+1}}(n) = \{s \subseteq \mathbb{N} : \{n\} < s \text{ and } \{n\} \cup s \in \mathcal{A}_{\omega^{\beta+1}}\}$$

$$= \left\{s \subseteq \mathbb{N} : \{n\} < s, \{n\} \cup s = \bigcup_{i=1}^{n} s_i,\ s_1 < \ldots < s_n \text{ and } s_1, \ldots, s_n \in \mathcal{A}_{\omega^\beta}\right\}$$

$$= \left\{s \subseteq \mathbb{N} : \{n\} < s,\ s = s_0 \cup \left(\bigcup_{i=2}^{n} s_i\right) \text{ with } s_0 \in \mathcal{A}_{\omega^\beta}(n),\ s_2, \ldots, s_n \in \mathcal{A}_{\omega^\beta} \text{ and } s_0 < s_2 < \ldots < s_n\right\}.$$

So, according to the induction hypothesis,

$$\mathcal{A}_\xi(n) = \left\{s \subseteq \mathbb{N} : s = s_0 \cup \left(\bigcup_{i=2}^{n} s_i\right) \text{ with } s_0 \in \mathcal{A}_{\omega^\beta(n)} \cap [\mathbb{N} \cap (n + \infty)]^{<\omega},\ s_0 < s_2 < \ldots < s_n \text{ and } s_2, \ldots, s_n \in \mathcal{A}_{\omega^\beta}\right\}$$

$$= \mathcal{A}_{(n-1)\omega^\beta + \omega^\beta(n)} \cap [\mathbb{N} \cap (n, +\infty)]^{<\omega}.$$

Hence, $\xi(n) = \omega^{\beta+1}(n) = (n-1)\omega^\beta + \omega^\beta(n)$ for every $n \in \mathbb{N}$ and obviously $\sup_n \xi(n) = \omega^{\beta+1}$. We note that in case $\xi = \omega$ we have $\xi(n) = n - 1$, since $\omega^0 = 1$.

(4) **[Case** $\xi = \omega^\lambda$ for λ non-zero, countable limit ordinal]. For every $n \in \mathbb{N}$ we have

$$\mathcal{A}_\xi(n) = \mathcal{A}_{\omega^\lambda}(n) = \{s \subseteq \mathbb{N} : \{n\} < s \text{ and } \{n\} \cup s \in \mathcal{A}_{\omega^\lambda}\}$$

$$= \{s \subseteq \mathbb{N} : \{n\} < s \text{ and } \{n\} \cup s \in \mathcal{A}_{\omega^{\lambda_n}}\},$$

where (λ_n) is the sequence of successor ordinals converging to λ fixed in the definition of the Schreier system $(\mathcal{A}_\xi)_{\xi<\omega_1}$. Hence,

$$\mathcal{A}_\xi(n)=\{s\subseteq\mathbb{N}: s\in\mathcal{A}_{\omega^{\lambda_n}}(n)\}=\mathcal{A}_{\omega^{\lambda_n(n)}}\cap[\mathbb{N}\cap(n,+\infty)]^{<\omega},$$

according to the induction hypothesis. If $\lambda_n=\mu_n+1$ for every $n\in\mathbb{N}$, then

$$\xi(n)=\omega^\lambda(n)=\omega^{\lambda_n}(n)=\omega^{\mu_n+1}(n)=(n-1)\,\omega^{\mu_n}+\omega^{\mu_n}(n) \text{ for every } n\in\mathbb{N}.$$

Of course $\sup_n \xi(n)=\omega^\lambda$, since $\omega^{\mu_n}\leqslant(n-1)\omega^{\mu_n}\leqslant(n-1)\,\omega^{\mu_n}+\omega^{\mu_n}(n)$.

(5) [**Case** ξ limit, $\omega^\alpha<\xi<\omega^{\alpha+1}$ for some $0<\alpha<\omega_1$]. In this case, according to Proposition 1.3, ξ has a unique representation of ordinals as follows: $\xi=p\omega^\alpha+\sum_{i=1}^m p_i\omega^{\alpha_i}$, where $m\in\mathbb{N}$, $\alpha>\alpha_1>\ldots>\alpha_m>0$ are ordinal numbers and $p,p_1,\ldots p_m\geqslant1$ are natural numbers, so that either $p>1$ or $p=1$ and $m\geqslant1$.

For simplicity we examine firstly the case $m=0$, namely the case $\xi=p\omega^\alpha$ for $p>1$. For every $n\in\mathbb{N}$ we have

$$\mathcal{A}_\xi(n)=\mathcal{A}_{p\omega^\alpha}(n)=\{s\subseteq\mathbb{N}:\{n\}<s \text{ and } \{n\}\cup s\in\mathcal{A}_{p\omega^\alpha}\}=$$

$$=\Big\{s\subseteq\mathbb{N}:\{n\}<s \text{ and } \{n\}\cup s=\bigcup_{i=1}^p s_i \text{ with } s_1<\ldots<s_p$$

$$\text{and } s_i\in\mathcal{A}_{\omega^\alpha} \text{ for } 1\leqslant i\leqslant p\Big\}$$

$$=\Big\{s\subseteq\mathbb{N}: s=s_0\cup\Big(\bigcup_{i=2}^p s_i\Big) \text{ with } \{n\}<s_0<s_2<\ldots<s_p,$$

$$s_0\in\mathcal{A}_{\omega^\alpha}(n) \text{ and } s_2,\ldots,s_p\in\mathcal{A}_{\omega^\alpha}\Big\}$$

$$=\Big\{s\subseteq\mathbb{N}: s=s_0\cup\Big(\bigcup_{i=2}^p s_i\Big) \text{ with } \{n\}<s_0<s_2<\ldots<s_p,$$

$$s_0\in\mathcal{A}_{\omega^{\alpha(n)}} \text{ and } s_2,\ldots,s_p\in\mathcal{A}_{\omega^\alpha}\Big\}$$

$$=\mathcal{A}_{(p-1)\omega^\alpha+\omega^\alpha(n)}\cap[\mathbb{N}\cap(n,+\infty)]^{<\omega}.$$

Thus $\xi(n)=p\omega^\alpha(n)=(p-1)\,\omega^\alpha+\omega^\alpha(n)$ for every $n\in\mathbb{N}$. Of course, $\sup_n \xi(n)=p\omega^\alpha=\xi$, since $\sup_n \omega^\alpha(n)=\omega^\alpha$.

Now, let $m\geqslant1$. In this case $\xi=\beta+p_m\omega^{\alpha_m}$, where $\beta=p\omega^\alpha+$

$+\sum_{i=1}^{m-1} p_i\omega^{\alpha_i}$. Of course $\beta<\xi$. Then for every $n\in\mathbb{N}$ we have

$\mathcal{A}_\xi(n)=\{s\subseteq\mathbb{N}:\{n\}<s \text{ and } \{n\}\cup s\in\mathcal{A}_\xi\}$

$$=\{s\subseteq\mathbb{N}:\{n\}<s,\ \{n\}\cup s=s_1\cup s_2,\ s_1<s_2,\ s_1\in\mathcal{A}_{p_m}\omega^{\alpha_m} \text{ and } s_2\in\mathcal{A}_\beta\}$$

$$=\{s\subseteq\mathbb{N}:s=s_0\cup s_2 \text{ with } \{n\}<s_0<s_2, s_0\in\mathcal{A}_{p_m}\omega^{\alpha_m}(n) \text{ and } s_2\in\mathcal{A}_\beta\}$$

$$=\{s\subseteq\mathbb{N}:s=s_0\cup s_2 \text{ with } \{n\}<s_0<s_2,\ s_0\in\mathcal{A}_{p_m}\omega^{\alpha_m}(n) \text{ and } s_2\in\mathcal{A}_\beta\}$$

$$=\mathcal{A}_{\beta+p_m\omega^{\alpha_m(n)}}\cap[\mathbb{N}\cap(n,+\infty)]^{<\omega}.$$

Hence, $\xi(n)=\beta+p_m\omega^{\alpha_m}(n)=p\omega^\alpha+\sum_{i=1}^{m-1}p_i\omega^{\alpha_i}+(p_m-1)\,\omega^\alpha+\omega^\alpha(n)$ for every $n\in\mathbb{N}$. Of course, $\sup_n \xi(n)=\xi$.

This finishes the proof.

In the following we will prove that a system of Ramsey families $(\mathcal{A}_\xi)_{\xi<\omega_1}$ on M, for some $M\in[\mathbb{N}]$, is a collection of families with remarkable properties:

(i) each family $\mathcal{A}_\xi$ is **thin** (i.e. does not contain proper initial segments of any of its elements)

(ii) the **Cantor-Bendixson index** defined for $\mathcal{A}_\xi$ is precisely equal to $\xi+1$ and does not decrease, but on the contrary is stable, when we restrict to any infinite subset of M; and

(iii) every finite subset of M has a (unique) **canonical representation** with respect to each family $\mathcal{A}_\xi$.

PROPOSITION 1.7. Let $M\in[\mathbb{N}]$ and $(\mathcal{A}_\xi)_{\xi<\omega_1}$ a system of Ramsey families on M. Every family $\mathcal{A}_\xi$, for $\xi<\omega_1$, is thin.

PROOF. It follows easily by induction on ξ.

DEFINITION 1.8 ([A-M-T], [F]). Let $\mathcal{F}$ be a hereditary and pointwise closed family of finite subsets on $\mathbb{N}$. For $M\in[\mathbb{N}]$ we define the **strong Cantor-Bendixson derivatives** $(\mathcal{F})_M^\xi$ of $\mathcal{F}$ on M for every $\xi<\omega_1$ as follows:

$(\mathcal{F})_M^1=\{F\in\mathcal{F}[M]: F \text{ is a cluster point of } \mathcal{F}[\mathrm{F}\cup\mathrm{L}] \text{ for each } L\in[M]\}$;

(where, $\mathcal{F}[M]=\mathcal{F}\cap[M]^{<\omega}$).

If $(\mathcal{F})_M^\xi$ has been defined, then

$$(\mathcal{F})_M^{\xi+1}=((\mathcal{F})_M^\xi)_M^1.$$

If ξ is a limit ordinal and $(\mathcal{F})_M^\beta$ have been defined for each $\beta < \xi$, then

$$(\mathcal{F})_M^\xi = \bigcap_{\beta < \xi} (\mathcal{F})_M^\beta .$$

The **strong Cantor-Bendixson index of** $\mathcal{F}$ **on** M is defined to be the smallest countable ordinal ξ such that $(\mathcal{F})_M^\xi = \emptyset$. We denote this index by $s_M(\mathcal{F})$.

REMARK 1.9 (i) The strong Cantor-Bendixson index $s_M(\mathcal{F})$ of a hereditary and pointwise closed family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ on some $M \in [\mathbb{N}]$ is a successor countable ordinal and is less or equal to the «usual» Cantor-Bendixson index of $\mathcal{F}$.

(ii) If $\mathcal{F}_1, \mathcal{F}_2 \subseteq [\mathbb{N}]^{<\omega}$ are hereditary and closed families with $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then

$$s_M(\mathcal{F}_1) \leq s_M(\mathcal{F}_2) \text{ for every } M \in [\mathbb{N}].$$

(iii) $s_M(\mathcal{F}) = s_M(\mathcal{F} \cap [M]^{<\omega})$ for every $M \in [\mathbb{N}]$, and

(iv) If L is almost contained in M, then

$$s_L(\mathcal{F}) \geq s_M(\mathcal{F}).$$

In [F] we proved the precise relation between the strong Cantor-Bendixson derivatives of the corresponding hereditary family $\mathcal{L}_\star$ of a given family $\mathcal{L} \subseteq [\mathbb{N}]^{<\omega}$ and the derivatives of the corresponding families $(\mathcal{L}(n))_\star$ for every $n \in \mathbb{N}$. Using these results we will calculate the strong Cantor-Bendixson index of a family belonging to a system of Ramsey families.

LEMMA 1.10. ([F]). Let $\mathcal{L} \subseteq [\mathbb{N}]^{<\omega}$ such that $\mathcal{L}_\star$ and $\mathcal{L}(n)_\star$ are pointwise closed for every $n \in \mathbb{N}$ and $M \in [\mathbb{N}]$.

(i) If there exists $L \in [M]$ such that $s_M(\mathcal{L}(n)_\star) = \xi$ for every $n \in L$, then $s_L(\mathcal{L}_\star) \geq \xi + 1$.

(ii) Let $\xi_n = s_M(\mathcal{L}(n)_\star)$ for every $n \in \mathbb{N}$. If there exists $L \in [M]$ such that $\xi = \sup_{n \in L} \xi_n < \omega_1$ and $\xi_n < \xi$ for every $n \in L$, then $s_M(\mathcal{L}_\star) \geq \xi + 1$.

(iii) If $\xi = \sup \{ s_L(\mathcal{L}(n)_\star) : n \in \mathbb{N} \text{ and } L \in [M] \}$, then $(\mathcal{L}_\star)_M^\xi \subseteq \{\emptyset\}$ and therefore $s_M(\mathcal{L}_\star) \leq \xi + 1$.

LEMMA 1.11. Let $M \in [\mathbb{N}]$ and $(\mathcal{A})_{\xi < \omega_1}$ a system of Ramsey families on M. Then (i) $(\mathcal{A}_\xi)_\star$ is pointwise closed family for every $\xi < \omega_1$.

(ii) $(\mathcal{A}(m))_\star$ is pointwise closed for every $\xi < \omega_1$ and $m \in M$.

PROOF. (i) This is easily proved by induction on ξ, using the fact that if a sequence of characteristic functions (χ_{s_n}) convergences pointwise to χ_s, where (s_n) is a sequence in $(\mathcal{A}_\xi)_\star$ and s is a non empty subset of $\mathbb{N}$, then $(\chi_{s_n \setminus \{k\}})$ convergences to $x_{s\setminus\{k\}}$, where $k = \min s$.

(ii) It follows from (i), since for some $m \in M$ the family $\mathcal{A}_\xi(m) = \mathcal{A}_{\xi_m} \cap [M_1]^{<\omega}$ where $M_1 = M \cap (m, +\infty)$, and the collection $(\mathcal{A}_\xi \cap [M_1]^{<\omega})_{\xi<\omega_1}$ is a system of Ramsey families on M_1.

THEOREM 1.12. Let $M \in [\mathbb{N}]$ and $(\mathcal{A}_\xi)_{\xi<\omega_1}$ a system of Ramsey families on M. Then for every $L \in [M]$ we have $((\mathcal{A}_\xi)_\star)_L^\xi = \{\emptyset\}$ and consequently $s_L((\mathcal{A}_\xi)_\star) = \xi + 1$ for every $\xi < \omega_1$.

PROOF. We use induction on ξ. For every $M \in [\mathbb{N}]$ and every system of Ramsey families $(\mathcal{A}_\xi)_{\omega_1}$ on M we have $(\mathcal{A}_1)_\star = \{\{m\} : m \in M\} \cup \{\emptyset\}$, hence $((\mathcal{A}_1)_\star)_L^1 = \{\emptyset\}$ and therefore $s_L((\mathcal{A}_1)_\star) = 2$ for every $L \in [M]$.

Suppose that $1 < \xi$ and the assertion holds for every ordinal ζ with $\zeta < \xi$. Let $M \in [\mathbb{N}]$ and $(\mathcal{A}_\xi)_{\xi<\omega_1}$ a system of Ramsey families on M. For some $m \in M$, we have that $\mathcal{A}_\xi(m) = \mathcal{A}_{\xi_m} \cap [M_1]^{<\omega}$, where $M_1 = M \cap (m, +\infty)$ and $\xi_m = \xi - 1$ in case ξ is a successor ordinal and in case ξ is a limit ordinal $(\xi_m)_{m\in M}$ is the strictly increasing to ξ sequence fixing in the definition of $(\mathcal{A}_\xi)_{\xi<\omega_1}$.

Since $(\mathcal{A}_\xi \cap [M_1]^{<\omega})_{\xi<\omega_1}$ is a system of Ramsey families on M_1, according to the induction hypothesis, Remark 1.2 (i) and Remark 1.9 (iv) we have for every $L \in [M]$

$$s_L((\mathcal{A}_\xi(m))_\star) = s_{L_1}((\mathcal{A}_{\xi_m} \cap [M_1]^{<\omega})_\star) = \xi_m + 1\,;$$

where $L_1 = L \cap (m, +\infty)$.

From Lemma 1.10 (i) and (ii) we have $s_L((\mathcal{A}_\xi)_\star) \geq \xi + 1$ and from Lemma 1.10 (iii) we have that $((\mathcal{A}_\xi)_\star)_L^\xi = \{\emptyset\}$ and finally that $s_L((\mathcal{A}_\xi)_\star) = \xi + 1$. In the following we will prove that every finite subset of $\mathbb{N}$ has a canonical representation with respect to each family of a system of Ramsey families.

DEFINITION 1.13. Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$. A non-empty, finite subset F of $\mathbb{N}$ has **canonical representation** $R_{\mathcal{F}}(F) = \{s_1, \ldots, s_n, s_{n+1}\}$ with type $t_{\mathcal{F}}(F) = n$ with respect to $\mathcal{F}$, if there exist unique $n \in \mathbb{N}, s_1, \ldots, s_n \in \mathcal{F}$ and s_{n+1} a proper initial segment of some element of $\mathcal{F}$ with $s_1 < \ldots < s_n < s_{n+1}$ and such that $F = \bigcup_{i=1}^{n+1} s_i$.

THEOREM. 1.14. Let M be an infinite subset of $\mathbb{N}$, $(\mathcal{A}_\xi)_{\xi<\omega_1}$ a system of Ramsey families on M and ξ a countable ordinal number. Every non-empty, finite subset of M has (unique) canonical representation with respect to the family $\mathcal{A}_\xi$ for every $1 \leq \xi < \omega_1$.

PROOF. We proceed by induction on ξ. For every $M \in [\mathbb{N}]$ and every system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ of Ramsey families on M we have $\mathcal{A}_1 = \{\{m\} : m \in M\}$. If $F = \{m_1, \ldots, m_k\} \in [M]^{<\omega}$ with $m_1 < \ldots < m_k$ and $1 \leq k$, then $R_{\mathcal{A}_1}(F) = \{\{m_1\}, \ldots, \{m_k\}\}$ and $t_{\mathcal{A}_1}(F) = k$.

Assume that $1 < \xi$ and the assertion holds for every $\zeta < \xi$. Let $M \in [\mathbb{N}]$ and $(\mathcal{A}_\xi)_{\xi<\omega_1}$ a system of Ramsey families on M. Then there exists a sequence $(\xi_m)_{m\in M}$ of ordinal numbers smaller than ξ such that $\mathcal{A}_\xi(m) = \mathcal{A}_{\xi_m} \cap [M_1]^{<\omega}$ for every $m \in M$.

Firstly we will prove that for every $F \in [M]^{<\omega}$, $F \neq \emptyset$ there exist $n \in \mathbb{N}$, $s_1, \ldots, s_n \in \mathcal{A}_\xi$, and $s_{n+1} \in (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi$ with $s_1 < \ldots < s_n < s_{n+1}$ such that $F = \bigcup_{i=1}^{n+1} s_i$. Indeed, let $F \in [M]^{<\omega}$, $F \neq \emptyset$. If $F \in (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi$, then we set $n = 0$ and $s_1 = F$ and if $F \in \mathcal{A}_\xi$ then we set $n = 1$ $s_1 = F$ and $s_2 = \emptyset$. Now we assume that $F \notin (\mathcal{A}_\xi)^\star$. If $m_1 = \min F$, then $F = \{m_1\} \cup t^1$ with $\{m_1\} < t^1$ and $t^1 \neq \emptyset$. According to the induction hypothesis, t^1 has canonical representation $R_{\mathcal{A}_\xi(m_1)}(t^1) = (t_1^1, \ldots, t_{n_1+1}^1)$, with type n_1, with respect to the family $\mathcal{A}_\xi(m_1)$. If $t^1 = t_1^1 \in (\mathcal{A}_\xi(m_1))^\star$ (case $n_1 = 0$ or $n_1 = 1$ and $t_2^1 = \emptyset$), then $F = \{m_1\} \cup t^1 \in (\mathcal{A}_\xi)^\star$, which contrary to our assuption for F; hence, $t_1^1 \in \mathcal{A}_\xi(m_1)$ and $t_2^1 \neq \emptyset$. Since t_1^1 is a proper initial segment of t^1, the set $s_1 = \{m_1\} \cup t_1^1$ is a proper initial segment of F. Hence,

$$F = s_1 \cup F_1 \text{ with } F_1 \neq \emptyset, \quad s_1 < F_1 \text{ and } s_1 \in \mathcal{A}_\xi.$$

We continue analogously treating F_1 in place of F. In detail the argument goes as follows: If $F_1 \in (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi$ we set $s_2 = F_1$ and finally $F = s_1 \cup s_2$; if $F_1 \in \mathcal{A}_\xi$, then we set $s_2 = F_1$, $s_3 = \emptyset$ and $F = s_1 \cup s_2 \cup s_3$. If $F_1 \notin (\mathcal{A}_\xi)^\star$, then there exists $s_2 \in \mathcal{L}$ and $F_2 \in [M]^{<\omega}$ with $F_2 \neq \emptyset$ such that $F = s_1 \cup s_2 \cup F_2$ with $s_1 < s_2 < F_2$. We continue in the same way.

The representation of every finite subset of M with respect to $\mathcal{A}_\xi$, for some $\xi < \omega_1$, is unique, since $\mathcal{A}_\xi$ is thin family (Proposition 1.7).

2. – Ramsey dichotomies.

We now, mimicuing the standard proof of the classical Ramsey theorem, establish the Ramsey character of the families of a system of Ramsey families.

We state a ξ-Ramsey type theorem, for ξ any countable ordinal, providing a far-reaching generalization of the classical Ramsey theorem which corresponds to the finite ordinals $\xi < \omega$. The arguments of the proof have been initially given in [F]. For completeness we include here the proof.

THEOREM 2.1 (**ξ-Ramsey type theorem**). Let M be an infinite subset of $\mathbb{N}$ and $(\mathcal{A}_\xi)_{\xi<\omega_1}$ a system of Ramsey families on M. For an arbitrary family $\mathcal{F}$ of finite subsets of M and a countable ordinal number ξ there exists an infinite subset L of M such that

$$\text{either } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}.$$

PROOF. We will prove it by induction on ξ. Let $\xi = 1$. For every $M \in \in [\mathbb{N}]$ and every system of Ramsey families $(\mathcal{A}_\xi)_{\xi<\omega_1}$ on M we have $\mathcal{A}_1 = = \{\{m\} : m \in M\}$. For every family $\mathcal{F} \subseteq [M]^{<\omega}$ we set

$$M_1 = \{m \in M : \{m\} \in \mathcal{F}\}; \text{ and}$$

$$M_2 = \{m \in M : \{m\} \in [\mathbb{N}]^{<\omega} \setminus \mathcal{F}\}.$$

In case M_1 is infinite we set $L = M_1$, otherwise we set $L = M_2$.

Let $1 < \xi$. Assume that the theorem in valid for all ordinals ζ less than ξ. Let $M \in [\mathbb{N}]$, $\mathcal{F} \subseteq [M]^{<\omega}$ and $(\mathcal{A}_\xi)_{\xi<\omega_1}$ a system of Ramsey families on M.

We set $m_1 = minM$ and $M_1 = M \cap (m_1, +\infty)$. Then $\mathcal{A}_\xi(m_1) = = \mathcal{A}_{\xi(m_1)} \cap [M_1]^{<\omega}$ with $\xi(m_1) < \xi$ (Remark 1.2 (v)). Hence $\mathcal{A}_\xi(m_1)$ is the $\xi(m_1)$ family of the system of Ramsey families $(\mathcal{A}_\xi \cap [M_1]^{<\omega})_{\xi<\omega_1}$ on M_1. Setting

$$\mathcal{F}_1 = \{s \subseteq M_1 : \{m_1\} \cup s \in \mathcal{F}\}$$

and using the induction hypothesis, we can find $L_1 \in [M_1]$ such that

$$\text{either } \mathcal{A}_\xi(m_1) \cap [L_1]^{<\omega} \subseteq \mathcal{F}_1 \text{ or } \mathcal{A}_\xi(m_1) \cap [L_1]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}_1.$$

Let $m_2 = \min L_1 > m_1$ and $M_2 = L_1 \cap (m_2, +\infty)$. Then, $\mathcal{A}_\xi(m_2) \cap \cap [M_2]^{<\omega} = \mathcal{A}_{\xi(m_2)} \cap [M_2]^{<\omega}$ (Remark 1.2(v)) with $\xi(m_2) < \xi$. Since $(\mathcal{A}_\xi \cap \cap [M_2]^{<\omega})_{\xi<\omega_1}$ is a system of Ramsey families on $[M_2]$ (Remark 1.2(i)), setting

$$\mathcal{F}_2 = \{s \subseteq M_2 : \{m_2\} \cup s \in \mathcal{F}\}$$

and applying the induction hypothesis, we can find $L_2 \in [M_2]$ such that

$$\text{either } \mathcal{A}_\xi(m_2) \cap [L_2]^{<\omega} \subseteq \mathcal{F}_2 \text{ or } \mathcal{A}_\xi(m_2) \cup [L_2]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}_2.$$

Set $m_3 = \min L_2 > m_2 > m_1$ and $M_3 = L_2 \cap (m_3, +\infty)$ and proceed analogously.

In this way we can construct a strictly increasing sequence $I = (m_n)_{n \in \mathbb{N}} \subseteq M$ and two decreasing sequences $(M_n)_{n \in \mathbb{N}}$ and $(L_n)_{n \in \mathbb{N}}$ of infinite subsets of M with the properties:

(i) $m_k \in L_n$ for every $k > n$;

(ii) $L_n \subseteq M_n$ for every $n \in \mathbb{N}$; and

(iii) if $\mathcal{F}_n = \{s \subseteq M_n : \{m_n\} \cup s \in \mathcal{F}\}$ for every $n \in \mathbb{N}$, then

$$\text{either } \mathcal{A}_\xi(m_n) \cap [L_n]^{<\omega} \subseteq \mathcal{F}_n \text{ or } \mathcal{A}_\xi(m_n) \cap [L_n]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}_n.$$

Set

$$I_1 = \{m_n \in I : \mathcal{A}_\xi(m_n) \cap [L_n]^{<\omega} \subseteq \mathcal{F}_n\} \text{ and}$$

$$I_2 = \{m_n \in I : \mathcal{A}_\xi(m_n) \cup [L_n]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}_n\}.$$

Since $I = I_1 \cup I_2$, either I_1 or I_2 is infinite. We will prove that $\mathcal{A}_\xi \cap [I_1]^{<\omega} \subseteq \mathcal{F}$; (analogously can be proved that $\mathcal{A}_\xi \cap [I_2]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$). Let $F \in \mathcal{A}_\xi \cap [I_1]^{<\omega}$. If $k = \min F$, then $k = m_n$ for some $m_n \in I_1$. Since $F \in \mathcal{A}_\xi$ we have $F = \{m_n\} \cup s$ for some $s \in \mathcal{A}_\xi(m_n)$ with $\{m_n\} < s$ and since $F \in [I_1]^{<\omega} \subseteq [I]^{<\omega}$ and $I \cap (m_n, +\infty) \subseteq L_n$ (property (i)) we have that $s \in [L_n]^{<\omega}$. Hence

$$s \in \mathcal{A}_\xi(m_n) \cap [L_n]^{<\omega} \subseteq \mathcal{F}_n (\text{since } m_n \in I_1),$$

and consequently

$$F = \{m_n\} \cup s \in \mathcal{A}_\xi (\text{since } s \subseteq L_n \subseteq M_n).$$

This finishes the proof of the theorem. Using the ξ-Ramsey theorem and Theorem 1.14 (the canonical representation of finite sets) we now prove a stronger dichotomy result for hereditary families.

THEOREM 2.2 (**ξ-Ramsey type theorem for hereditary families**). Let M be an infinite subset of $\mathbb{N}$ and $(\mathcal{A}_\xi)_{\xi < \omega_1}$ a system of Ramsey families on M. For a hereditary family $\mathcal{F}$ of finite subsets of M and a countable ordinal number ξ there exists an infinite subset L of M such that

$$\text{either } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi.$$

PROOF. According to the ξ-Ramsey type theorem (Theorem 2.1) there exists $L \in [M]$ such that

$$\text{either } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}.$$

We will prove that $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ if and only if $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi$. Indeed, let $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ and $F \in \mathcal{F} \cap [L]^{<\omega}$. According to Theorem 1.14, either there exists $s \in \mathcal{A}_\xi$ such that F is a proper initial segment of s which gives that $F \in (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi$, as required, or there exists $t \in \mathcal{A}_\xi$ such that t is an initial segment of F. The second case is impossible. Indeed, since $\mathcal{F}$ is a hereditary family and $F \in \mathcal{F} \cap [L]^{<\omega}$, we have $t \in \mathcal{F} \cap [L]^{<\omega}$. But $t \in \mathcal{A}_\xi$, so $t \in \mathcal{A}_\xi \cap [L]^{<\omega} \cap \mathcal{F}$. This contrary to our assumption that $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$.

It is obvious that if $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi$, then $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$.

COROLLARY 2.3. Let ξ_1, ξ_2 be countable ordinal numbers with $\xi_1 < \xi_2$. For every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that

$$\mathcal{A}_{\xi_1} \cap [L]^{<\omega} \subseteq (\mathcal{A}_{\xi_2})^\star \setminus \mathcal{A}_{\xi_2}.$$

PROOF. Let $\mathcal{F}$ be the corresponding hereditary family $(\mathcal{A}_{\xi_1})_\star$ of $\mathcal{A}_{\xi_1}$. According to Theorem 2.2, for every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that

$$\text{either } \mathcal{A}_{\xi_2} \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_{\xi_2})^\star \setminus \mathcal{A}_{\xi_2}.$$

The first alternative is impossible. Indeed, let $(\mathcal{A}_{\xi_2}) \cap [L]^{<\omega} \subseteq (\mathcal{A}_{\xi_1})_\star$. Since $s_L((\mathcal{A}_{\xi_2} \cap [L]^{<\omega})_\star) = \xi_2 + 1$ (Remark 1.2(i) and Theorem 1.12) and $s_L((\mathcal{A}_{\xi_1})_\star) = \xi_1 + 1$ we have

$$\xi_2 + 1 = s_L((\mathcal{A}_{\xi_2} \cap [L]^{<\omega})_\star) \leq s_L((\mathcal{A}_{\xi_1})_\star) = \xi_1 + 1.$$

A contradiction; hence $\mathcal{A}_{\xi_1} \cap [L]^{<\omega} \subseteq (\mathcal{A}_{\xi_2})^\star \setminus \mathcal{A}_{\xi_2}$. In the following theorem we will describe a criterion, with the help of the strong Cantor Bendixson index, in order a hereditary family of finite subsets of $\mathbb{N}$ to satisfy exactly one of the dichotomy conditions given in Theorem 2.2.

PROPOSITION 2.4 ([F]). Let $\mathcal{F}$ be a non empty, hereditary family of finite subsets of $\mathbb{N}$. The following are equivalent:

(i) $\mathcal{F}$ is not pointwise closed.

(ii) There exists an infinite sequence $(s_i)_{i=1}^\infty$ of elements of $\mathcal{F}$ with $s_1 < s_2 < \dots$.

(iii) There exists $M \in [\mathbb{N}]$ such that $[M]^{<\omega} \subseteq \mathcal{F}$.

THEOREM 2.5 (**Refined ξ-Ramsey type theorem**). Let M be an infinite subset of $\mathbb{N}$ and $(\mathcal{A}_\xi)_{\xi<\omega_1}$ a system of Ramsey families on M. For a pointwise closed and hereditary family $\mathcal{F}$ of finite subsets of M we set

$$\xi_M^{\mathcal{F}} = \sup\{s_I(\mathcal{F}) : I \in [M]\},$$

which is a countable ordinal.

For a countable ordinal ξ we obtain the following:

(i) If $\xi + 1 < \xi_M^{\mathcal{F}}$, then there exists $L \in [M]$ such that

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}.$$

(ii) If $\xi_M^{\mathcal{F}} < \xi + 1$, then there exists $L \in [M]$ such that

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}; \text{ and}$$

equivalently,

$$\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi.$$

(iii) If $\xi + 1 = \xi_M^{\mathcal{F}}$, then

$$\text{either } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi.$$

Both alternatives may materialize.

PROOF. (i) Let $\xi + 1 < \xi_M^{\mathcal{F}}$. Then there exists $I \in [M]$ such that $\xi + 1 < s_I(\mathcal{F})$. The collection $(\mathcal{A}_\xi \cap [I]^{<\omega})_{\xi<\omega_1}$ is a system of Ramsey families on I. According to Theorem 2.2, there exists $L \in [I]$ such that

$$\text{either } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi \subseteq (\mathcal{A}_\xi)_\star.$$

The second alternative is impossible. Indeed, let $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)_\star$. Using Theorem 1.12 and Remark 1.9 we have

$$s_M(\mathcal{F}) \leqslant s_L(\mathcal{F}) = s_L(\mathcal{F} \cap [L]^{<\omega}) \leqslant s_L((\mathcal{A}_\xi)_\star) = \xi + 1\,;$$

a contradiction to our assumption; hence $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$.

(ii) Let $\xi_M^{\mathcal{F}} < \xi + 1$. According to the ξ-Ramsey type theorem (Theorem 2.1), there exists $L \in [M]$ such that

$$\text{either } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \text{ or } \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}.$$

If $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$, then, using Remark 1.9 (ii) and Theorem 1.12, we obtain

$$\xi + 1 = s_L((\mathcal{A}_\xi \cap [L]^{<\omega})_\star) \leq s_L(\mathcal{F}).$$

A contradiction; hence, $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$ and equivalently, $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi)$ (according to Theorem 2.2).

(iii) In the limiting case $\xi_m^{\mathcal{F}} = \xi + 1$ both alternatives of the ξ-Ramsey type theorem for hereditary families (Theorem 2.2) may materialize. There are two simple examples in [F] which proves the statement.

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