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On spreading c_0-sequences in Banach spaces

by

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Abstract. We introduce and study the spreading-(s) and the spreading-(u) property of a Banach space and their relations. A space has the spreading-(s) property if every normalized weakly null sequence has a subsequence with a spreading model equivalent to the usual basis of c_0; while it has the spreading-(u) property if every weak Cauchy and non-weakly convergent sequence has a convex block subsequence with a spreading model equivalent to the summing basis of c_0. The main results proved are the following:

(a) A Banach space X has the spreading-(s) property if and only if for every subspace Y of X and for every pair of sequences (x_n) in Y and (x_n^*) in Y^*, with (x_n) weakly null in Y and (x_n^*) uniformly weakly null in Y^* (in the sense of Mercourakis), we have $x_n^*(x_n) \to 0$ (i.e. X has a hereditary weak Dunford–Pettis property).

(b) A Banach space X has the spreading-(u) property if and only if $B_1(X) \subseteq B_{1/4}(X)$ in the sense of the classification of Baire-1 elements of X^{**} according to Haydon–Odell–Rosenthal.

(c) The spreading-(s) property implies the spreading-(u) property.

Result (c), proved via infinite combinations, connects an internal condition on a Banach space with an external one.

0. Introduction. The notion of spreading model, introduced by Brunel and Sucheston in [B-S], proved fruitful in the study of Banach spaces as well as in the study of Baire-1 functions ([R1], [H-O-R], [F3] and others).

In this paper we introduce and study the spreading-c_0 properties of a Banach space, as they relate to the universal occurrence of sequences with spreading models equivalent to the usual or the summing basis of c_0.

In the first section we introduce the spreading-(s) property of a Banach space as follows: every normalized and weakly null sequence of the space has a subsequence with a spreading model equivalent to the usual basis of c_0, and we give equivalents and consequences of this property. An example of a space with the spreading-(s) property is the dual of Tsirelson's space (Example 1.9 below). The spreading-(s) property is the spread-theoretic analog of the stronger property (s), introduced earlier by Cembranos and Elton in [C]

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and [E], where they proved that property (s) is equivalent to the hereditary Dunford–Pettis property (i.e. for every pair of weakly null sequences (x_n) in a Banach space X and (x_n^*) in X^* we have $x_n^*(x_n) \to 0$). According to our result (Theorem 1.15 below), if we replace the weak convergence of (x_n^*) by the uniformly weak convergence (introduced earlier by Mercourakis in [M], where it is related to the Cesàro summability) we obtain an analogous equivalence for the spreading-(s) property.

On the other hand we introduce the spreading-(u) property by the condition that every weak Cauchy and non-weakly convergent sequence in a Banach space has a convex block subsequence with a spreading model equivalent to the summing basis of c_0. The predual of the space called the "Jamesification" of Tsirelson's space is a non-trivial example of a space with this property (Example 2.8 below). The spreading-(u) property is the spread-theoretic analog of the stronger property (u), introduced earlier by Pełczyński in [P].

In the second part of this paper we obtain several characterizations of the spreading-(u) property. One of them states that $B_1(X) \subseteq B_{1/4}(X)$ (Theorem 2.6 below) (i.e., the Baire-1 elements of the double dual space, considered as functions on the unit ball of the dual space with the w^*-topology, are uniform limits of sequences of differences of bounded semicontinuous functions, uniformly bounded in the D-norm), thus relating the spreading-(u) property to the class $B_{1/4}(K)$ for a compact space K introduced by Haydon–Odell–Rosenthal in [H-O-R] and studied in [F1], [F3], [R3].

Finally, we prove that the spreading-(s) property implies the spreading-(u) property (Theorem 2.10 below). So by looking at the weakly null sequences of a Banach space we obtain information on the Baire-1 elements of the second dual space.

1. The spreading-(s) property. We start with some definitions and remarks concerning spreading models.

DEFINITION 1.1. A basic sequence (y_n) in a Banach space Y is said to be *spreading* if it is 1-equivalent to all of its subsequences.

A basic sequence (x_n) in a Banach space X has a spreading sequence (y_n) as a *spreading model* if for every $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $m_k \in \mathbb{N}$ such that if $m_k < n_1 < \ldots < n_k \in \mathbb{N}$ then

$$(1-\varepsilon)\Big\|\sum_{i=1}^k a_i y_i\Big\| \le \Big\|\sum_{i=1}^k a_i x_{n_i}\Big\| \le (1+\varepsilon)\Big\|\sum_{i=1}^k a_i y_i\Big\|$$

for all scalars $a_1, \ldots, a_k$.

Every seminormalized basic sequence in a Banach space has a subsequence with a spreading model. For information on spreading models consult [B-S], [B-L], [G].

The *Schreier family* $\mathcal{F}_1$ is the family

$$\mathcal{F}_1 = \{(n_1, \ldots, n_k) : k \le n_1 < \ldots < n_k \in \mathbb{N}\}.$$

The following proposition connects the concept of spreading model with Schreier's family.

PROPOSITION 1.2 ([F1]). *Let (x_n) be a seminormalized basic sequence in a Banach space X, (y_n) a spreading model of (x_n), and (e_n) an arbitrary basic sequence. The following are equivalent:*

(i) *(y_n) is equivalent to (e_n).*

(ii) *There exists a subsequence (x'_n) of (x_n) and $0 < D < C$ such that*

$$D\Big\|\sum_{i=1}^k a_i e_i\Big\| \le \Big\|\sum_{i=1}^k a_i x'_{n_i}\Big\| \le C\Big\|\sum_{i=1}^k a_i e_i\Big\|$$

for all $(n_1, \ldots, n_k) \in \mathcal{F}_1$ and scalars $a_1, \ldots, a_k$.

For normalized weakly null sequences with a spreading model equivalent to the usual basis of c_0 we will need more detailed information provided by the following:

LEMMA 1.3. *Let (x_n) be a normalized weakly null sequence in a Banach space X. The following are equivalent:*

(i) *There exists $C > 0$ such that*

$$\Big\|\sum_{i=1}^k a_i x_{n_i}\Big\| \le C \sup_{1\le i\le k} |a_i|$$

for all $(n_1, \ldots, n_k) \in \mathcal{F}_1$ and scalars $a_1, \ldots, a_k$.

(ii) $\sup\{\sum_{i=1}^k |x^*(x_{n_i})| : (n_1, \ldots, n_k) \in \mathcal{F}_1\} < \infty$ *for every $x^* \in X^*$.*

(iii) *There exists $A > 0$ such that*

$$\Big\|\sum_{i=1}^k \varepsilon_i x_{n_i}\Big\| \le A$$

for all $(n_1, \ldots, n_k) \in \mathcal{F}_1$ and $\varepsilon_1, \ldots, \varepsilon_k \in \{-1, 1\}$.

(iv) *There exists $B > 0$ such that*

$$\Big\|\sum_{i=1}^k x_{n_i}\Big\| \le B$$

for all $(n_1, \ldots, n_k) \in \mathcal{F}_1$.

Proof. Obviously (i)⇒(iv).

(iv)⇒(iii). Let $(n_1,\dots,n_k)\in\mathcal{F}_1$ and $\varepsilon_1,\dots,\varepsilon_k\in\{-1,1\}$; set $I_1=\{i:1\le i\le k,\ \varepsilon_i=1\}$, $I_2=\{i:1\le i\le k,\ \varepsilon_i=-1\}$ and note that $\{n_i:i\in I_1\}$ and $\{n_i:i\in I_2\}$ are in $\mathcal{F}_1$. Thus

$$\Big\|\sum_{i=1}^k \varepsilon_i x_{n_i}\Big\|\le\Big\|\sum_{i\in I_1} x_{n_i}\Big\|+\Big\|\sum_{i\in I_2} x_{n_i}\Big\|\le 2B.$$

(iii)⇒(ii). Let $x^*\in X^*$ and $(n_1,\dots,n_k)\in\mathcal{F}_1$; then

$$\sum_{i=1}^k|x^*(x_{n_i})|=\sum_{i=1}^k\varepsilon_i x^*(x_{n_i})=x^*\Big(\sum_{i=1}^k\varepsilon_i x_{n_i}\Big)\le\|x^*\|\cdot\Big\|\sum_{i=1}^k\varepsilon_i x_{n_i}\Big\|\le A\|x^*\|,$$

where $\varepsilon_i\in\{-1,1\}$ and $|x^*(x_{n_i})|=\varepsilon_i x^*(x_{n_i})$.

(ii)⇒(i). This is a consequence of Baire's category theorem. Indeed, for every $n\in\mathbb{N}$ set

$$Z_n=\Big\{x^*\in X^*:\sup\Big\{\sum_{i=1}^k|x^*(x_{n_i})|:(n_1,\dots,n_k)\in\mathcal{F}_1\Big\}\le n\Big\},$$

and notice that $X^*=\bigcup_n Z_n$ and that $Z_n\subseteq X^*$ are norm closed for all $n\in\mathbb{N}$. Hence there exists $C=n_0\in\mathbb{N}$ such that $x^*\in Z_{n_0}$ for every $x^*\in X^*$ with $\|x^*\|\le 1$. Now, let $(n_1,\dots,n_k)\in\mathcal{F}_1$ and $a_1,\dots,a_k\in\mathbb{R}$; then there exists $x^*\in X^*$ with $\|x^*\|\le 1$ such that

$$\Big\|\sum_{i=1}^k a_i x_{n_i}\Big\|=x^*\Big(\sum_{i=1}^k a_i x_{n_i}\Big).$$

It follows that

$$\Big\|\sum_{i=1}^k a_i x_{n_i}\Big\|=\Big|\sum_{i=1}^k a_i x^*(x_{n_i})\Big|\le\sum_{i=1}^k|a_i|\cdot|x^*(x_{n_i})|\le C\sup_{1\le i\le k}|a_i|.$$

The proof is complete.

PROPOSITION 1.4. *A normalized weakly null sequence in a Banach space X has a subsequence with a spreading model equivalent to the usual basis of c_0 if and only if it has a subsequence which satisfies one of the equivalent conditions* (i) *to* (iv) *of Lemma* 1.3.

Proof. Let (z_n) be a normalized weakly null sequence in X and (x_n) a subsequence of (z_n) satisfying one of the conditions (i) to (iv) of Lemma 1.3. The sequence (x_n) has a basic subsequence (x'_n) with a spreading model (y_n) ([B-P], [B-S]). If a sequence satisfies one of the conditions (i) to (iv) of Lemma 1.3 then so does each of its subsequences. Hence (x'_n) satisfies (i) of Lemma 1.3 and according to Proposition 1.2 it has a spreading model equivalent to the usual basis of c_0.

On the other hand, if a normalized weakly null sequence (z_n) has a subsequence (x_n) with a spreading model equivalent to the usual basis of c_0 then (x_n) has a subsequence satisfying (i) of Lemma 1.3 according to Proposition 1.2.

DEFINITION 1.5. A weakly null sequence (x_n) in a Banach X is called a *null-coefficient sequence* iff whenever a sequence (α_n) of scalars satisfies

$$\sup\Big\{\Big\|\sum_{i=1}^k\alpha_{n_i}x_{n_i}\Big\|:(n_1,\dots,n_k)\in\mathcal{F}_1\Big\}<\infty$$

then $\alpha_n\to 0$.

For a normalized weakly null sequence the following dichotomy obtains:

PROPOSITION 1.6. *Let (x_n) be a normalized weakly null sequence in a Banach space X. Then either*

(i) *(x_n) has a subsequence with a spreading model equivalent to the usual basis of c_0, or*

(ii) *every subsequence of (x_n) is null-coefficient.*

Proof. Suppose there exists a subsequence (y_n) of (x_n), $(\alpha_n)\notin c_0$ and $C>0$ such that

$$\Big\|\sum_{i=1}^k\alpha_{n_i}y_{n_i}\Big\|\le C\quad\text{for every }(n_1,\dots,n_k)\in\mathcal{F}_1.$$

We can find an $\varepsilon>0$ and a subsequence (α_{n_m}) of (a_n) such that $a_{n_m}>\varepsilon$ for every $m\in\mathbb{N}$ (replacing, if necessary, a_n by $-a_n$).

We will prove that the subsequence $(y_{n_m})=(z_m)$ of (x_n) satisfies condition (ii) of Lemma 1.3 and thus, according to Proposition 1.4, (z_m) has a subsequence with a spreading model equivalent to the usual basis of c_0. Indeed, let $x^*\in X^*$ and $(m_1,\dots,m_k)\in\mathcal{F}_1$; then

$$\begin{aligned}\sum_{i=1}^k|x^*(z_{m_i})|&\le\sum_{i=1}^k a_{n_{m_i}}(1/\varepsilon)|x^*(z_{m_i})|=(1/\varepsilon)\sum_{i=1}^k a_{n_{m_i}}\varepsilon_i x^*(z_{m_i})\\&=(1/\varepsilon)x^*\Big(\sum_{\substack{i=1\\ \varepsilon_i=1}}^k a_{n_{m_i}}z_{m_i}\Big)+(1/\varepsilon)x^*\Big(\sum_{\substack{i=1\\ \varepsilon_i=-1}}^k a_{n_{m_i}}z_{m_i}\Big)\\&\le(2/\varepsilon)C\,\|x^*\|.\end{aligned}$$

Above we have set $|x^*(z_{m_i})|=\varepsilon_i x^*(z_{m_i})$, where $\varepsilon_i\in\{-1,1\}$ for $i=1,\dots,k$, and we have used the obvious fact that $B\in\mathcal{F}_1$ if $B\subseteq A$ and $A\in\mathcal{F}_1$.

The proof of the proposition is complete.

REMARK. It is easy to verify that the two alternatives of Proposition 1.6 are mutually exclusive.

We now wish to introduce and study (Definition 1.7 below) a class of Banach spaces which is characterized by a universal occurrence of weakly null sequences with spreading models equivalent to the c_0-basis. This is analogous to the universal occurrence of sequences equivalent to the c_0-basis in the class of Banach spaces with property (s) introduced by Cembranos [C] and Elton [E]. Specifically, a Banach space X has *property* (s) if every normalized weakly null sequence has a subsequence equivalent to the usual basis of c_0. However, property (s) is very strong and relates only to spaces that are close to c_0. We propose to study a more general property, where c_0-embeddings are replaced by spreading models of c_0 according to the following:

DEFINITION 1.7. A Banach space X has the *spreading*-(s) property if every normalized weakly null sequence in X admits a subsequence with a spreading model equivalent to the usual basis of c_0.

REMARKS 1.8. (a) If a Banach space X has property (s) then X has the spreading-(s) property; e.g. the space $c_0(\Gamma)$ or any space with the Schur property.

(b) The spreading-(s) property is hereditary: every closed subspace has the spreading-(s) property if the space does.

(c) The spaces ℓ_p, $1 < p < \infty$, and L^p, $2 < p < \infty$, do not have the spreading-(s) property since every normalized weakly null sequence in ℓ_p contains a subsequence equivalent to the ℓ_p-basis, and every subspace of L^p contains a smaller subspace isomorphic to ℓ_p or ℓ_2 ([K-P]).

EXAMPLE 1.9 (The Tsirelson space S). We consider S as the dual space of the space T, as described by Figiel and Johnson [F-J].

Let c_{00} be the linear space of all finitely supported real-valued functions on $\mathbb{N}$ and (e_n) the unit vector basis of c_{00}. For every $x = (\lambda_n) \in c_{00}$ and $m = 0, 1, 2, \ldots$ set

$$\|x\|_0 = \max_n |\lambda_n|,$$

$$\|x\|_{m+1} = \max\Big(\|x\|_m, \frac{1}{2}\max\Big\{\sum_{i=1}^{k}\Big\|\sum_{n=p_i+1}^{p_{i+1}} \lambda_n e_n\Big\|_m : k \le p_1 < \ldots < p_{k+1} \in \mathbb{N}\Big\}\Big).$$

Then $\|x\| = \lim_m \|x\|_m$ is a norm for c_{00}.

The space T is the completion of c_{00} with respect to the norm $\|\ \|$, and it is easy to see that

$$\|x\| = \max\Big(\|x\|_\infty, \frac{1}{2}\sup\Big\{\sum_{i=1}^{k}\Big\|\sum_{n=p_i+1}^{p_{i+1}} \lambda_n e_n\Big\| : k \le p_1 < \ldots < p_{k+1} \in \mathbb{N}\Big\}\Big).$$

The usual basis (e_n) is an unconditional basis of T and T is reflexive. Let (e_n^*) be the sequence of biorthogonal functionals of (e_n) and let $S = T^* = [(e_n^*)]$. Hence S is also a reflexive space with an unconditional basis.

Of course, since S is reflexive, it does not have property (s). However, we can prove the following:

CLAIM. *S has the spreading-(s) property.*

Proof. Let $(u_i)_{i\in\mathbb{N}}$ be a bounded block sequence of (e_i^*) (that is, $u_i = \sum_{n=p_i+1}^{p_{i+1}} \lambda_n e_n^*$ with $p_i < p_{i+1}$ for every $i \in \mathbb{N}$) with $\|u_i\| \le M$ for every $i \in \mathbb{N}$. Then for every $x = \sum_{n=1}^{\infty} a_n e_n \in T$, $k \le n_1 < \ldots < n_k \in \mathbb{N}$ and $n_{k+1} = n_k + 1$, we have

$$\sum_{i=1}^{k} |u_{n_i}(x)| = \sum_{i=1}^{k}\Big|u_{n_i}\Big(\sum_{n=p_{n_i}+1}^{p_{n_i+1}} a_n e_n\Big)\Big| \le \sum_{i=1}^{k}\|u_{n_i}\|\Big\|\sum_{n=p_{n_i}+1}^{p_{n_i+1}} a_n e_n\Big\|$$
$$\le M\sum_{i=1}^{k}\Big\|\sum_{n=p_{n_i}+1}^{p_{n_i+1}} a_n e_n\Big\| \le 2M\|x\|$$

since $k \le n_1 \le p_{n_1} < \ldots < p_{n_k} < p_{n_k+1}$.

Hence, according to Proposition 1.4, every seminormalized bounded block subsequence of (e_n^*) has a subsequence with a spreading model equivalent to the usual basis of c_0.

Finally, S has the spreading-(s) property since every normalized weakly null sequence in S admits a subsequence equivalent to a block subsequence of (e_n^*).

A weaker property than property (s) is the *weak Banach–Saks property* (or *Banach–Saks–Rosenthal property*) defined by the condition that every weakly null sequence contains a Cesàro summable subsequence (the sequence of arithmetic means converges in norm). We now prove that this property is in fact weaker than the spreading-(s) property.

PROPOSITION 1.10. *If a Banach space X has the spreading-(s) property then it has the weak Banach–Saks property.*

Proof. Let (x_n) be a normalized weakly null sequence in X. According to Proposition 1.4 it has a subsequence (y_n) satisfying, for some $B > 0$,

$$\Big\|\sum_{i=1}^{k} y_{n_i}\Big\| \le B \quad \text{for every } (n_1,\ldots,n_k)\in\mathcal{F}_1.$$

Then $\|(1/k)\sum_{i=1}^{k} y_{n_i}\| \le B/k$ for every $(n_1,\ldots,n_k)\in\mathcal{F}_1$ and obviously

$$\lim_{k\to\infty}\Big[\sup_{k<n_1<\ldots<n_k}\Big\|\frac{1}{k}\sum_{i=1}^{k} y_{n_i}\Big\|\Big] = 0.$$

According to a result of Mercourakis [M] (Theorem 1.12 below) this gives that every subsequence of (y_n) is Cesàro summable to zero.

REMARK. The spreading-(s) property is not equivalent to the weak Banach–Saks property since the spaces $L^p([0,1])$, $2<p<\infty$, all have the weak Banach–Saks property, but not the spreading-(s) property.

In the following we will prove a characterization of the spreading-(s) property via a property weaker than the Dunford–Pettis property (introduced in Definition 1.14 below). We recall that a Banach space X has the *Dunford–Pettis property* if for every pair of weakly null sequences $(x_n)\subseteq X$ and $(x_n^*)\subseteq X^*$ we have $x_n^*(x_n)\to 0$. According to results independently proved by Cebranos [C] and Elton [E], property (s) is equivalent to the hereditary Dunford–Pettis property. It is interesting that a modification of the Dunford–Pettis property relating to the convergence of (x_n^*) yields an equivalence, analogous to Cembranos–Elton's, for the new spreading-(s) property.

The right type of convergence turns out to be the uniformly weak convergence, introduced earlier in another context, related to Cesàro summability, by Mercourakis [M].

DEFINITION 1.11 ([M]). A sequence (x_n) *converges uniformly weakly* to x in a Banach space X if for every $\varepsilon>0$ there is a natural number $N(\varepsilon)$ such that

$$|\{n\in\mathbb{N} : |x^*(x_n-x)|\ge\varepsilon\}| \le N(\varepsilon)$$

for every $x^*\in X^*$ with $\|x^*\|\le 1$.

If $x=0$, we sometimes say that (x_n) is *uniformly weakly null.*

It is clear that $x_n\to x$ uniformly weakly implies $x_n\to x$ weakly.

We will need the following description of uniformly weak convergence via the Schreier family (and also via Cesàro summability), due to Mercourakis [M] (Theorem 1.12), and also the following dichotomy (Theorem 1.13), which exists for weakly null sequences, between the existence of a uniformly weakly convergent subsequence and the existence of a subsequence with spreading model equivalent to the ℓ_1-basis, due to Rosenthal [R1] (cf. also [M]), and ultimately based on the Erdős–Magidor summability dichotomy [E-M].

THEOREM 1.12 ([M]). *Let (x_n) be a bounded sequence in a Banach space X and $x\in X$. The following are equivalent:*

(i) *$x_n\to x$ uniformly weakly in X.*

(ii) $\displaystyle\lim_{n\to\infty}\Big[\sup_{n\le k_1\le\ldots\le k_n}\Big\|\frac{1}{n}\sum_{i=1}^{n} x_{k_i}-x\Big\|\Big]=0.$

(iii) *Every subsequence of (x_n) is Cesàro summable in X to x.*

THEOREM 1.13 ([R1], [E-M]). *Let (x_n) be a weakly null sequence in a Banach space X. Then there exists a subsequence (x_{n_k}) of (x_n) such that either*

(i) *(x_{n_k}) is uniformly weakly null, or*

(ii) *(x_{n_k}) has a spreading model equivalent to the usual basis of ℓ_1.*

DEFINITION 1.14. A Banach space Y has the *weak Dunford–Pettis property* if for every pair of sequences (x_n) in Y and (x_n^*) in Y^*, with (x_n) weakly null and (x_n^*) uniformly weakly null, we have $x_n^*(x_n)\to 0$.

A Banach space X has the *hereditary weak Dunford–Pettis property* if every subspace of X has the same property.

We are now in a position to state and prove the main result of this section.

THEOREM 1.15. *A Banach space X has the spreading-(s) property if and only if it has the hereditary weak Dunford–Pettis property.*

Proof. $(\Rightarrow)$ Let X have the spreading-(s) property, Y a closed subspace of X, (x_n) a weakly null sequence in Y and (x_n^*) a uniformly weakly null sequence in Y^*. If either (x_n) or (x_n^*) is norm convergent to zero, then clearly $x_n^*(x_n)\to 0$. If this is not the case, suppose towards a contradiction that $(x_n^*(x_n))$ does not converge to zero. Then there exist a subsequence (x_{n_k}) of (x_n) and $\varepsilon>0$ such that $x_{n_k}^*(x_{n_k})>\varepsilon$ (replace x_n by $-x_n$ if necessary).

By hypothesis we can assume that (x_{n_k}) has a spreading model equivalent to the usual basis of c_0. Let $z_k=x_{n_k}$ and $z_k^*=x_{n_k}^*$ for every $k\in\mathbb{N}$. According to Proposition 1.4, we can assume that there exists $B>0$ such that

$$\Big\|\sum_{i=1}^{n} z_{k_i}\Big\|\le B \quad \text{for every } (k_1,\ldots,k_n)\in\mathcal{F}_1.$$

Since (x_n^*) converges uniformly weakly to zero the same is true for (z_n^*) and according to Theorem 1.12 there exists $n_0\in\mathbb{N}$ such that

$$\Big\|\sum_{i=1}^{n} z_{k_i}^*\Big\|\le\delta n \quad \text{for every } (k_1,\ldots,k_n)\in\mathcal{F}_1 \text{ with } n_0\le n,$$

where $\delta=\varepsilon/(4B)$.

It follows that for every $(k_1,\dots,k_n)\in\mathcal{F}_1$ with $n_0\le n$ we have

$$(*)\qquad \frac{1}{n}\Big|\Big(\sum_{i=1}^{n} z_{k_i}^*\Big)\Big(\sum_{i=1}^{n} z_{k_i}\Big)\Big|\le\frac{1}{n}\Big\|\sum_{i=1}^{n} z_{k_i}^*\Big\|\Big\|\sum_{i=1}^{n} z_{k_i}\Big\|\le B\delta=\frac{\varepsilon}{4}.$$

In order to arrive at a contradiction we define by recursion natural numbers $k_1=n_0,k_2,\dots,k_{n_0}$ such that the element $(k_1,\dots,k_{n_0})$ of $\mathcal{F}_1$ does not satisfy $(*)$. Set $k_1=n_0$. Let $1\le\lambda<k_1$ and assume that $k_1,\dots,k_\lambda$ have been defined. Since $\lim_{k\to\infty} z_{k_i}^*(z_k)=0$ and $\lim_{k\to\infty} z_k^*(z_{k_i})=0$ for every $i=1,\dots,\lambda$ there exists $k_{\lambda+1}\in\mathbb{N}$ with $k_{\lambda+1}>k_\lambda$ such that

$$z_{k_i}^*(z_{k_{\lambda+1}})<\frac{\varepsilon}{2n_0},\qquad z_{k_{\lambda+1}}^*(z_{k_i})<\frac{\varepsilon}{2n_0}\qquad\text{for every } i=1,\dots,\lambda.$$

Of course $(k_1,\dots,k_{k_1})\in\mathcal{F}_1$ and $n_0=k_1$ but

$$\frac{1}{n_0}\Big|\Big(\sum_{i=1}^{n_0} z_{k_i}^*\Big)\Big(\sum_{i=1}^{n_0} z_{k_i}\Big)\Big|\ge\frac{1}{n_0}\Big|\sum_{i=1}^{n_0} z_{k_i}^*(z_{k_i})\Big|-\frac{1}{n_0}\Big|\sum_{i=1}^{n_0} z_{k_i}^*\Big(\sum_{\substack{j=1\\ j\ne i}}^{n_0} z_{k_j}\Big)\Big|$$

$$>\varepsilon-\frac{1}{n_0}\sum_{i=1}^{n_0}\sum_{\substack{j=1\\ j\ne i}}^{n_0}|z_{k_i}^*(z_{k_j})|>\varepsilon-\frac{1}{n_0}\cdot n_0(n_0-1)\cdot\frac{\varepsilon}{2n_0}>\frac{\varepsilon}{2},$$

a contradiction, completing the proof of necessity.

($\Leftarrow$) In order to prove the sufficiency we assume that for every closed subspace Y of X, every weakly null sequence (z_λ) in Y and uniformly weakly null sequence (z_λ^*) in Y^* we have $z_\lambda^*(z_\lambda)\to 0$ and also that there exists a normalized weakly null sequence (x_n) in X having no subsequence with a spreading model equivalent to the usual basis of c_0. We may assume that (x_n) is basic (cf. [B-P]).

Since (x_n) has no subsequence equivalent to the usual basis of c_0, it has a subsequence $y_k=x_{n_k}$, for $k\in\mathbb{N}$, such that

$$\sup_k\Big\|\sum_{i=1}^{k} a_i y_i\Big\|=\infty\qquad\text{for every } (a_i)\notin c_0,$$

according to the dichotomy proved by Odell in [O-1] (a corollary to the "nearly unconditional theorem" of Elton [E]). Hence the sequence (y_k^*) of the biorthogonal functionals of (y_k) is weakly null in Y^*, where $Y=[(y_k)]$ is the closed subspace spanned by the sequence (y_k).

Furthermore, applying a result of Odell [O-2], we can assume by passing to a subsequence that (y_k^*) is M-Schreier unconditional for some $M>2$, that is,

$$\Big\|\sum_{i=1}^{n} a_{k_i} y_{k_i}^*\Big\|\le M\Big\|\sum_{i=1}^{\infty} a_i y_i^*\Big\|$$

for every $(k_1,\dots,k_n)\in\mathcal{F}_1$ and every finitely non-zero sequence (a_i) of scalars.

We now make use of Theorem 1.13 for the sequence (y_k^*). There exists a subsequence $z_\lambda=y_{k_\lambda}$, $\lambda\in\mathbb{N}$, of (y_k) such that either (z_λ^*) is uniformly weakly null in Y^*, or there exists $\delta>0$ such that

$$\delta\sum_{i=1}^{k}|a_i|\le\Big\|\sum_{i=1}^{k} a_i z_{\lambda_i}^*\Big\|$$

for $(\lambda_1,\dots,\lambda_k)\in\mathcal{F}_1$ and scalars $a_1,\dots,a_k$.

The first alternative does not occur for any subsequence (z_λ^*) of (y_k^*) since $z_\lambda^*(z_\lambda)=1$ for every $\lambda\in\mathbb{N}$.

The second alternative is also impossible, since it implies the existence of a subsequence of (x_n) with a spreading model equivalent to the usual basis of c_0. Indeed, if (z_λ^*) satisfies the second alternative then for every $(\lambda_1,\dots,\lambda_k)\in\mathcal{F}_1$ and $f=\sum_{i=1}^{\infty} b_i z_i^*\in[(z_\lambda^*)]$ we have

$$\Big|\sum_{i=1}^{k} z_{\lambda_i}\Big|(f)=\Big|\sum_{i=1}^{k} b_{\lambda_i}\Big|\le\sum_{i=1}^{k}|b_{\lambda_i}|$$

$$\le\frac{1}{\delta}\Big\|\sum_{i=1}^{k} b_{\lambda_i} z_{\lambda_i}^*\Big\|\le\frac{M}{\delta}\Big\|\sum_{i=1}^{\lambda k} b_i z_i^*\Big\|\le\frac{M}{\delta}K\|f\|,$$

where K is the basis constant of (z_λ^*).

So we have, for some $B>0$,

$$\Big\|\sum_{i=1}^{k} z_{\lambda_i}\Big\|\le B\qquad\text{for every } (\lambda_1,\dots,\lambda_k)\in\mathcal{F}_1$$

(see e.g. Singer [S], p. 115).

According to Proposition 1.4, (x_n) has a subsequence with a spreading model equivalent to the usual basis of c_0, a contradiction, completing the proof of the theorem.

The method of proof of the above theorem provides us with the following interesting dichotomy.

THEOREM 1.6. *For a normalized weakly null sequence (x_n) in a Banach space X, exactly one of the following alternatives holds:*

(i) *Every subsequence of (x_n) has a subsequence with a spreading model equivalent to the usual basis of c_0.*

(ii) *There exists a basic subsequence (y_n) of (x_n) such that the sequence (y_n^*) of its biorthogonal functionals is uniformly weakly null in $Y=[(y_n)]$, and hence every subsequence of (y_n^*) is Cesàro summable.*

Proof. Arguments analogous to the proof of the sufficiency of Theorem 1.15 show that for every normalized weakly null sequence (x_n) which does not have a subsequence with a spreading model equivalent to the usual basis of c_0 there exists a basic subsequence (y_n) such that the sequence (y_n^*) of its biorthogonal functionals is uniformly weakly null in $Y=[(y_n)]$.

The two alternatives are exclusive; in fact, if the sequence (y_n^*) of the biorthogonal functionals of a normalized weakly null sequence (y_n) is uniformly weakly null in $Y=[(y_n)]$, then no subsequence of (y_n) has a spreading model equivalent to the usual basis of c_0. This can be shown by using arguments analogous to those in the proof of the necessity of Theorem 1.15.

2. The spreading-(u) property. In this section we introduce and study a property of Banach spaces analogous to (in fact weaker than) the spreading-(s) property, relating to the weak Cauchy and non-weakly convergent sequences. It turns out that this property provides information on the Baire-1 elements of the second dual space.

Orginally, Pełczyński in [P] defined property (u) of a Banach space X in the following way: for each $x^{**}\in X^{**}\setminus X$ which is the w^*-limit of a sequence in X, there exists a sequence (x_n) in X w^*-converging to x^{**} and satisfying

$$\sum_{n=1}^{\infty}|x^*(x_{n+1}-x_n)|<\infty\quad\text{ for all } x^*\in X^*.$$

In the same paper he proved that every subspace of a Banach space with an unconditional basis has property (u).

Later, in [H-O-R] and [R2] an equivalent formulation of property (u) was given using the summing basis (s_n) of c_0 (i.e. $s_n=e_1+\ldots+e_n$ for every $n\in\mathbb{N}$). Namely, it was proved that a Banach space X has property (u) if and only if for every weak Cauchy and non-weakly convergent sequence (x_n) in X there exists a convex block subsequence (y_n) of (x_n) which is equivalent to the summing basis of c_0.

By analogy we define the spreading-(u) property.

DEFINITION 2.1. A Banach space X has the *spreading-(u)* property if every weak Cauchy and non-weakly convergent sequence in X admits a convex block subsequence with a spreading model equivalent to the summing basis of c_0.

Since we will be dealing with weak Cauchy and non-weakly convergent sequences having a spreading model equivalent to the summing basis (s_n) of c_0, we now give some information about them.

REMARKS 2.2. (i) Every weak Cauchy and non-weakly convergent sequence (x_n) in a Banach space X has a subsequence (y_n) which is basic and dominates the summing basis (s_n) of c_0, i.e. there exists $D>0$ such that

$$D\Big\|\sum_{i=1}^{k}\lambda_i s_i\Big\|_\infty\le\Big\|\sum_{i=1}^{k}\lambda_i y_i\Big\|$$

for all $k\in\mathbb{N}$ and scalars $\lambda_1,\ldots,\lambda_k$ (see [H-O-R] or [R2]).

(ii) According to remark (i) and Proposition 1.2, a weak Cauchy and non-weakly convergent sequence (x_n) in a Banach space X has a subsequence (resp. a convex block subsequence) with a spreading model equivalent to the summing basis of c_0 if and only if there exists a subsequence (resp. a convex block subsequence) (z_n) of (x_n) and $C>0$ such that

$$\Big\|\sum_{i=1}^{k}\lambda_i z_{n_i}\Big\|\le C\Big\|\sum_{i=1}^{k}\lambda_i s_i\Big\|_\infty$$

for all $(n_1,\ldots,n_k)\in\mathcal{F}_1$ and scalars $\lambda_1,\ldots,\lambda_k$.

LEMMA 2.3. *Let (z_n) be a sequence in a Banach space X. The following are equivalent*:

(i) *There exists $C>0$ such that*

$$\Big\|\sum_{i=1}^{k}\lambda_i z_{n_i}\Big\|\le C\Big\|\sum_{i=1}^{k}\lambda_i s_i\Big\|_\infty$$

for all $(n_1,\ldots,n_k)\in\mathcal{F}_1$ and scalars $\lambda_1,\ldots,\lambda_k$.

(ii) $\sup\{\sum_{i=1}^{k}|x^*(z_{n_i}-z_{n_{i-1}})|:x^*\in X^*,\ \|x^*\|\le 1$ *and* $(n_1,\ldots,n_k)\in\mathcal{F}_1\}<\infty$ (*where $n_0=0=z_0$*).

(iii) $\sup\{\sum_{i=1}^{k}|x^*(z_{n_i}-z_{n_{i-1}})|:(n_1,\ldots,n_k)\in\mathcal{F}_1\}<\infty$ *for every $x^*\in X^*$* (*where $n_0=0=z_0$*).

Proof. (i)–(ii) was proved in [F3]. (ii)$\Rightarrow$(iii) is obvious. (iii)$\Rightarrow$(ii) is a consequence of Baire's category theorem (cf. proof of Lemma 1.3, (ii)$\Rightarrow$(i)).

Using these remarks we now establish an equivalence.

PROPOSITION 2.4. *A weak Cauchy and non-weakly convergent sequence (x_n) in a Banach space X has a subsequence* (*resp. a convex block subsequence*) *with a spreading model equivalent to the summing basis of c_0 if and only if (x_n) has a subsequence* (*resp. a convex block subsequence*) *(z_n) which satisfies one of the conditions* (i) *to* (iii) *of Lemma* 2.3.

In Proposition 1.6 above we proved a dichotomy-type result for weakly null sequences. An analogous dichotomy was proved in [F3] for weak Cauchy and non-weakly convergent sequences. For completeness we state this result below.

PROPOSITION 2.5 ([F3]). *Let (x_n) be a weak Cauchy and non-weakly convergent sequence in a Banach space X. Then either*

(i) *(x_n) has a subsequence (resp. a convex block subsequence) with a spreading model equivalent to the summing basis of c_0, or*

(ii) *every subsequence (resp. every convex block subsequence) (z_n) of (x_n) is null-coefficient (i.e., whenever a sequence (λ_n) of scalars satisfies*

$$\sup\Big\{\Big\|\sum_{i=1}^{k}\lambda_{n_{2i}}(y_{n_{2i}}-y_{n_{2i-1}})\Big\| : (n_1,\ldots,n_{2k})\in\mathcal{F}_1\Big\}<\infty$$

then $\lambda_n\to 0$).

Of course the two alternatives are mutually exclusive.

The spreading-(u) property of a Banach space will be characterized in terms of the Baire-1 elements of its dual space (Theorem 2.6).

We first deal with some preliminaries.

Let K be a compact space. We denote by $C(K)$ the class of continuous real-valued functions on K and by $B_1(K)$ the class of Baire-1 functions of K (i.e. the pointwise limits of uniformly bounded sequences of continuous functions). We denote by $D(K)$ the class of differences of bounded semicontinuous functions on K. We have

$$D(K)=\{f:K\to\mathbb{R}:\text{there are bounded and lower semicontinuous functions } u,v\geq 0 \text{ with } f=u-v\}.$$

The class $D(K)$ is a Banach space with respect to the D-norm defined as follows:

$$|f|_D=\inf\{\|u+v\|_\infty : f=u-v,\ u,v\geq 0 \text{ are bounded and lower semicontinuous}\}.$$

Of course $\|f\|_\infty\leq|f|_D$ for all $f\in D(K)$; the two norms are not equivalent, in general.

The class $B_{1/4}(K)$, introduced in [H-O-R], is defined by

$$B_{1/4}(K)=\{f:K\to\mathbb{R}:\text{there exists } (F_n)\subseteq D(K) \text{ such that } \|F_n-f\|_\infty\to 0 \text{ and } \sup|F_n|_D<\infty\}.$$

The space $B_{1/4}(K)$ is a Banach space with respect to the norm

$$|f|_{1/4}=\inf\{\sup_n|F_n|_D : (F_n)\subseteq D(K) \text{ and } \|F_n-f\|_\infty\to 0\}.$$

As proved in [H-O-R], [F1] and [F3] these classes bear the following relations to the summing basis of c_0:

(i) $D(K)\supseteq\{f\in B_1(K)$: there exist $(f_n)\subseteq C(K)$ and $C>0$ such that $f_n\to f$ pointwise and $\|\sum_{i=1}^n\lambda_i f_i\|_\infty\leq C\|\sum_{i=1}^n\lambda_i s_i\|_\infty$ for all $n\in\mathbb{N}$ and scalars $\lambda_1,\ldots,\lambda_n\}$.

(ii) $B_{1/4}(K)\supseteq\{f\in B_1(K)$: there exist $(f_n)\subseteq C(K)$ and $C>0$ such that $f_n\to f$ pointwise and $\|\sum_{i=1}^k\lambda_i f_{n_i}\|_\infty\leq C\|\sum_{i=1}^k\lambda_i s_i\|_\infty$ for all $(n_1,\ldots,n_k)\in\mathcal{F}_1$ and scalars $\lambda_1,\ldots,\lambda_k\}$.

If K is a compact metric space then the inclusions in (i) and (ii) are both equalities. For more information about these classes, see [H-O-R], [R2], [R3], [F2], [F3].

For a Banach space X we define

$$B_1(X)=\{x^{**}\in X^{**}:\text{there exists } (x_n) \text{ in } X\ w^*\text{-converging to } x^{**}\},$$
$$B_{1/4}(X)=\{x^{**}\in X^{**}:x^{**}|K\in B_{1/4}(K)\},$$

where $K=(B_{X^*},w^*)$ is the unit ball of X^* endowed with the weak* topology.

After these definitions and remarks we can state the following equivalence.

THEOREM 2.6. *Let X be a Banach space.*

(i) *X has the spreading-(u) property if and only if $B_1(X)\subseteq B_{1/4}(X)$.*

(ii) *If X is separable, then X has the spreading-(u) property if and only if $B_1(X)=B_{1/4}(X)$.*

Proof. (i) Let X have the spreading-(u) property. For every $x^{**}\in B_1(X)\setminus X$, there exists a sequence (x_n) in X w^*-converging to x^{**}, so there exists a convex block subsequence (y_n) of (x_n) with a spreading model equivalent to the summing basis of c_0. According to Remark 2.2(ii) there exists a convex block subsequence (z_n) of (x_n) and $C>0$ such that

$$\Big\|\sum_{i=1}^k\lambda_i z_{n_i}\Big\|\leq C\Big\|\sum_{i=1}^k\lambda_i s_i\Big\|_\infty$$

for all $(n_1,\ldots,n_k)\in\mathcal{F}_1$ and scalars $\lambda_1,\ldots,\lambda_k$. Since (z_n) is w^*-converging to x^{**} it follows that $x^{**}\in B_{1/4}(X)$ (cf. [F1]). Of course $X\subseteq B_{1/4}(X)$.

Conversely, we assume that $x^{**}\in B_{1/4}(X)$ for every $x^{**}\in B_1(X)$ and let (x_n) be a weak Cauchy and non-weakly convergent sequence in X. Then (x_n) w^*-converges to an element $x^{**}\in B_1(X)\setminus X$. Hence $x^{**}|K\in B_{1/4}(K)$, where $K=(B_{X^*},w^*)$. Let $Y=[(x_n)]$ be the closed linear subspace generated by (x_n) and K_1 the unit ball of Y^* endowed with the weak* topology. Then K_1 is a compact metric space and we observe that $y^{**}|K_1\in B_{1/4}(K_1)$ where y^{**} is the w^*-limit of (x_n) in Y^{**}. According to [F1], (x_n) admits a convex block subsequence with a spreading model equivalent to the summing basis of c_0. This completes the proof of (i).

(ii) Let X be a separable Banach space. According to [O-R] in this case

$$B_1(X)=\{x^{**}\in X^{**}:x^{**}|K\in B_1(K)\}$$

where $K=(B_{X^*},w^*)$. Since K is a compact metric space we have $B_{1/4}(K)\subseteq B_1(K)$. Hence $B_{1/4}(X)\subseteq B_1(X)$. The rest of the proof is similar to the general case in (i).

As a corollary, we have equivalent formulations of the spreading-(u) property. One of them is analogous to the definition of property (u) given by Pełczyński in [P].

COROLLARY 2.7. *A Banach space X has the spreading-(u) property if and only if for every $x^{**}\in B_1(X)\setminus X$ there exists a sequence (z_n) in X w^*-converging to x^{**} and satisfying one of the conditions* (i) *to* (iii) *of Lemma* 2.3.

We exploit our Theorem 2.6 to obtain a non-trivial space having the spreading-(u) property but not property (u).

EXAMPLE 2.8. Let X be the "Jamesification" of the Tsirelson space T as described in [B-H-O], defined in the following way. For every finitely supported function $x:\mathbb{N}\to\mathbb{N}$ define

$$\|x\|=\sup\Big\{\Big\|\sum_{i=1}^{k}(S_{n_i}-S_{p_i-1})(x)e_{p_i}\Big\|_T:1\le p_1\le n_1\le\ldots\le p_k\le n_k\Big\}$$

where $S_n(x)=\sum_{i=1}^{n}x(i)$ for every $n\in\mathbb{N}$ and $S_0(x)=0$. The space X is the completion of the linear space of all finitely supported functions with this norm.

As shown in [B-H-O] the unit vectors e_n, $n\in\mathbb{N}$, form a boundedly complete normalized basis for X. Thus $X=Y^*$ where $Y=[(e_n^*)]$ and e_n^* are the biorthogonal functionals of (e_n).

As was proved in [H-O-R], $B_1(Y)=B_{1/4}(Y)$, hence according to Theorem 2.6 the space Y has the spreading-(u) property. Also, Y does not have property (u), since c_0 is not isomorphically embedded into Y.

We now prove that the spreading-(s) property implies the spreading-(u) property; and hence looking (internally) at the weakly null sequences of a Banach space one can obtain (external) information on the Baire-1 elements of the second dual space.

In the proof we make use of infinitary Ramsey properties:

For an infinite subsequence M of $\mathbb{N}$ we denote by $[M]$ the set of all (infinite) subsequences of M. A subset $\mathcal{A}$ of $[\mathbb{N}]$ is called *Ramsey* if for all $M\in[\mathbb{N}]$ there exists $L\in[M]$ such that either $[L]\subseteq\mathcal{A}$ or $[L]\subseteq[\mathbb{N}]\setminus\mathcal{A}$.

THEOREM 2.9 (Galvin–Prikry [G-P]). *If $\mathcal{A}$ is a Borel subset of $[\mathbb{N}]$ endowed with the pointwise topology (relative topology of $2^{\mathbb{N}}$ with the Cartesian topology), then $\mathcal{A}$ is Ramsey.*

THEOREM 2.10. *If a Banach space has the spreading-(s) property, then it has the spreading-(u) property.*

Proof. Let X be a Banach space with the spreading-(s) property, and let (x_n) be a weak Cauchy but non-weakly convergent sequence in X. According to Remark 2.2(i), (x_n) has a subsequence which is basic and dominates the summing basis (s_n) of c_0. Without loss of generality we may assume that (x_n) itself has these properties. So there exists $D>0$ such that

$$(*)\qquad D\Big\|\sum_{i=1}^{k}\lambda_i s_i\Big\|_\infty\le\Big\|\sum_{i=1}^{k}\lambda_i x_i\Big\|$$

for all $k\in\mathbb{N}$ and scalars $\lambda_1,\ldots,\lambda_k$.

Proposition 2.5 implies that (x_n) has a subsequence with a spreading model equivalent to the summing basis if (x_n) has a non-null-coefficient subsequence. So it is enough to construct a subsequence (z_n) of (x_n) such that for some $C>0$,

$$(**)\qquad \Big\|\sum_{i=1}^{k}(z_{n_{2i}}-z_{n_{2i-1}})\Big\|\le C\quad\text{for every }(n_1,\ldots,n_{2k})\in\mathcal{F}_1.$$

Set $y_n=x_{n+1}-x_n$ for some every $n\in\mathbb{N}$. According to $(*)$, the sequence (y_n) is seminormalized, basic, and obviously weakly null. By our assumption, (y_n) has a subsequence (y_{n_j}) with a spreading model equivalent to the unit basis of c_0; equivalently, by Proposition 1.4,

$$(***)\qquad \Big\|\sum_{j\in F}y_{n_j}\Big\|=\Big\|\sum_{j\in F}x_{n_j+1}-x_{n_j}\Big\|\le B$$

for some $B>0$ and every $F\in\mathcal{F}_1$. We may assume that $n_j+1<n_{j+1}$ for every $j\in\mathbb{N}$.

Set $m_1=n_1,m_2=n_1+1,\ldots,m_{2j-1}=n_j,m_{2j}=n_j+1,\ldots$ From $(***)$, the subsequence (x_{m_n}) of (x_n) has the property:

$$(****)\qquad \Big\|\sum_{j\in F}(x_{m_{2j}}-x_{m_{2j-1}})\Big\|\le B\quad\text{for every }F\in\mathcal{F}_1.$$

For every $k\in\mathbb{N}$, set

$$\mathcal{A}_k=\Big\{M=(m_n)\in[\mathbb{N}]:\Big\|\sum_{j\in F}(x_{m_{2j}}-x_{m_{2j-1}})\Big\|\le k\text{ for every }F\in\mathcal{F}_1\Big\}.$$

The sets $\mathcal{A}_k$ are pointwise closed subsets of $[\mathbb{N}]$, so $\mathcal{A}=\bigcup_{k=1}^{\infty}\mathcal{A}_k$ is Borel. Hence the sets $\mathcal{A}_k$, $k\in\mathbb{N}$, and $\mathcal{A}$ are Ramsey (Theorem 2.9).

Choose $M \in [\mathbb{N}]$ such that either $[M] \subseteq \mathcal{A}$ or $[M] \subseteq [\mathbb{N}] \setminus \mathcal{A}$. Using the same argument as above for the sequence $(x_i)_{i\in M}$ in place of (x_n), we conclude that $[M] \subseteq \mathcal{A}$.

Now, choose $M_1 = (m_n^1) \in [M]$ such that either $[M_1] \subseteq \mathcal{A}_1$ or $[M_1] \subseteq \mathbb{N} \setminus \mathcal{A}_1$, and by induction choose $M_{k+1} = (m_n^{k+1}) \in [M_k]$ such that either $[M_{k+1}] \subseteq \mathcal{A}_{k+1}$ or $[M_{k+1}] \subseteq [\mathbb{N}] \setminus \mathcal{A}_{k+1}$.

We claim that there exists $k \in \mathbb{N}$ such that $[M_k] \subseteq \mathcal{A}_k$. Indeed, let $m_k = m_k^k$ for $k \in \mathbb{N}$ and set $L = (m_k)$. Obviously $L \in [M] \subseteq \mathcal{A}$, so there exists $k_0 \in \mathbb{N}$ such that $L \in \mathcal{A}_{2k_0}$ (which is possible since $(\mathcal{A}_k)_{k\in\mathbb{N}}$ is increasing with k).

Set $L_k = (m_{k+i})_{i\in\mathbb{N}}$ for $k \in \mathbb{N}$. Then $L_k \in [M_k]$. We now prove that $L_{2k_0} \in \mathcal{A}_{2k_0}$, so that $[M_{2k_0}] \subseteq \mathcal{A}_{2k_0}$. Indeed, for every $F = (j_1, \ldots, j_l) \in \mathcal{F}_1$ we have $G = (k_0 + j_1, \ldots, k_0 + j_l) \in \mathcal{F}_1$, hence

$$\Big\| \sum_{j\in F} (x_{m_{2k_0+2j}} - x_{m_{2k_0+2j-1}}) \Big\| = \Big\| \sum_{j\in G} (x_{m_{2j}} - x_{m_{2j-1}}) \Big\| \le 2k_0,$$

since $L \in \mathcal{A}_{2k_0}$. But then $L_{2k_0} \in \mathcal{A}_{2k_0}$ and finally $[M_{2k_0}] \subseteq \mathcal{A}_{2k_0}$.

So we have constructed a subsequence (z_n) of (x_n) ($z_n = x_{m_n^{2k_0}}$, $n \in \mathbb{N}$) with the property that every subsequence (w_n) of (z_n) satisfies

$$\Big\| \sum_{j\in F} (w_{2j} - w_{2j-1}) \Big\| \le 2k_0 \quad \text{for every } F \in \mathcal{F}_1.$$

We claim that (z_n) satisfies $(**)$. Indeed, let $(n_1, \ldots, n_{2k}) \in \mathcal{F}_1$. We define a subsequence (w_n) of (z_n) as follows:

$$\begin{aligned} w_i &= z_i && \text{for } i = 1, \ldots, 2k-2, \\ w_{2k-2+i} &= z_{n_i} && \text{for } i = 1, \ldots, 2k, \\ w_{4k-2+i} &= z_{n_{2k}+i} && \text{for every } i \in \mathbb{N}. \end{aligned}$$

Then

$$\begin{aligned} \Big\| \sum_{i=1}^{k} (z_{n_{2i}} - z_{n_{2i-1}}) \Big\| &= \Big\| \sum_{i=1}^{k} (w_{2k-2+2i} - w_{2k+2i-3}) \Big\| \\ &= \Big\| \sum_{i=1}^{k} (w_{2(k-1+i)} - w_{2(k-1+i)-1}) \Big\| \\ &= \Big\| \sum_{j\in F} (w_{2j} - w_{2j-1}) \Big\| \le 2k_0 \end{aligned}$$

where $F = (k, \ldots, 2k-1) \in \mathcal{F}_1$.

Hence (z_n) is not null-coefficient and according to Proposition 2.5, (x_n) has a subsequence equivalent to the summing basis of c_0. The proof is now complete.

COROLLARY 2.11. *If every normalized weakly null sequence in a Banach space X has a subsequence with a spreading model equivalent to the usual basis of c_0, then $B_1(X) \subseteq B_{1/4}(X)$.*

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