Ramsey dichotomies with ordinal index

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Abstract

A system of uniform families on an infinite subset M of $\mathbb{N}$ is a collection $(\mathcal{A}_\xi)_{\xi<\omega_1}$ of families of finite subsets of $\mathbb{N}$ (where, $\mathcal{A}_k$ consists of all k–element subset of M, for $k \in \mathbb{N}$) with the properties that each $\mathcal{A}_\xi$ is thin (i.e. it does not contain proper initial segments of any of its element) and the Cantor–Bendixson index, defined for $\mathcal{A}_\xi$, is equal to $\xi + 1$ and stable when we restrict ourselves to any subset of M. We indicate how to extend the generalized Schreier families to a system of uniform families.

Using that notion we establish the correct (countable) ordinal index generalization of the classical Ramsey theorem (which corresponds to the finite ordinal indices). Indeed, for a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$, we obtain the following:

(i) For every infinite subset M of $\mathbb{N}$ and every countable ordinal ξ, there is an infinite subset L of M such that either $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$ or $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$;
(where $[L]^{<\omega}$ denotes the family of all finite subsets of L).

(ii) If, in addition $\mathcal{F}$ is hereditary and pointwise closed, then for every infinite subset M of $\mathbb{N}$ there is a countable ordinal number ξ such that:

 (a) For every ordinal number ζ with $\zeta + 1 < \xi$ there is an infinite subset L of M such that $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}$.

 (b) For every ordinal number ζ with $\xi < \zeta + 1$ there is an infinite subset L of M such that $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\zeta)^* \setminus \mathcal{A}_\zeta$; which gives $\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$;
 (where generally $\mathcal{A}^*$ denotes the family of all initial segments of elements of $\mathcal{A}$).

 (c) For $\zeta = \xi + 1$, both alternatives ((a) and (b)) may materialize.

(iii) If $\mathcal{F}$ is hereditary, then $\mathcal{F}$ is not closed if and only if there is an infinite subset M of $\mathbb{N}$ such that $[M]^{<\omega} \subseteq \mathcal{F}$.

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0 Introduction

Our aim, in the present paper, is to establish the proper context, and the correct (countable) ordinal (Cantor–Bendixson type) index generalization of the classical Ramsey theorem [R] (stating that for every family $\mathcal{F}$ of finite subsets of $\mathbb{N}$, every natural number k and every infinite subset M of $\mathbb{N}$, there is an infinite subset L of M, such that all subsets of M consisting of exactly k–elements are either in $\mathcal{F}$ or in the complement of $\mathcal{F}$).

In this ordinal index context, the index of the classical Ramsey theorem is a natural number, while the infinitary Galvin–Prikry theorem, or infinite Ramsey, as is sometimes loosely called, ([N–W], [G–P], [S] and [E]) corresponds to the limiting ω_1–ordinal index.

Using the notion of a uniform family given by Pudlák and Rödl in [P–R] we introduce the notion of a system of uniform families (Definition 1.3). A system of uniform families on M (for $M \in [\mathbb{N}]$) is a collection $(\mathcal{A}_\xi)_{\xi<\omega_1}$ of families of finite subsets of M (with $\mathcal{A}_k = [M]^k$ for $k \in \mathbb{N}$) with the properties: (i) each $\mathcal{A}_\xi$ is thin (i.e. it does not contain proper initial segments of any of its elements) and (ii) the Cantor–Bendixson index defined for $\mathcal{A}_\xi$ is precisely equal to $\xi + 1$ and does not decrease, but on the contrary is stable, when we restrict ourselves to any infinite subset of M.

Every system of uniform families on M is characterized by the choice, for each countable limit ordinal number ξ, of an increasing sequence $(\xi_m)_{m\in M}$ of ordinals, so that $\xi_m < \xi$ for $m \in M$ and $\sup\limits_{m\in M} \xi_m = \xi$. With suitable choices one can define such systems that are useful for theoretical purposes or for applications. In Theorem 1.6 we define a (Schreier type) system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ of uniform families, which in the ω^a–position for every $a < \omega_1$ has the family $\mathcal{B}_a = \mathcal{A}_{\omega^a}$ (Definition 1.5), a family similar to the generalized Schreier set $\mathcal{F}_a$ (see Corollary 3.2) defined by Alspach and Argyros in [A–A]. Use of the system $(\mathcal{F}_a)_{a<\omega_1}$ has proved fruitful, especially in connection with the theory of Banach spaces. However, the system $(\mathcal{F}_a)_{a<\omega_1}$ is very difficult to employ in inductive arguments owing mainly to lack of adequate interrelation of the families $\mathcal{F}_a$, $a < \omega_1$, as there are missing families not defined for all ordinals ξ with $\omega^a < \xi < \omega^{a+1}$, $a < \omega_1$. The introduction, in this paper, of the system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ provides us with the correct amount of leeway to confront analogous problems (see Section 3).

Our starting point is the following far–reaching generalization of the classical Ramsey theorem.

Theorem A If $\mathcal{F}$ is a family of finite subsets of $\mathbb{N}$, then for every countable ordinal ξ, every infinite subset M on $\mathbb{N}$ and every ξ–uniform family $\mathcal{L}$ on M there exists an infinite subset L of M such that either $\mathcal{L} \cap [L]^{<\omega} \subseteq \mathcal{F}$ or $\mathcal{L} \cap [L]^{<\omega} \subseteq [M]^{<\omega} \setminus \mathcal{F}$.

A proof directly from the definitions involved is given in Theorem 2.2; another proof, using the combinatorial theorems of Nash–Williams in [N–W] is given in [P–R].

For hereditary families of finite subsets of $\mathbb{N}$ we prove a stronger dichotomy result (Theorem 2.12, Th. B below). For the proof we introduce the notion of the “canonical representation” for every finite subset of $\mathbb{N}$ with respect to a ξ–uniform family for every $\xi < \omega_1$ (Proposition 2.7).

Theorem B If $\mathcal{F}$ is a hereditary family of finite subsets of $\mathbb{N}$, then for every countable ordinal ξ, every infinite subset of $\mathbb{N}$ and every ξ–uniform family $\mathcal{L}$ on M there exists an infinite subset L of M such that either $\mathcal{L} \cap [L]^{<\omega} \subseteq \mathcal{F}$, or $\mathcal{F} \cap [L]^{<\omega} \subseteq \mathcal{L}^* \setminus \mathcal{L}$ (where $\mathcal{L}^*$ is the family of all the initial segments of the elements of $\mathcal{L}$).

After that dichotomy result, with the help of the strong Cantor–Bendixson index defined in [A–M–T] and denoted by s_M, we describe when a hereditary and pointwise closed family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ satisfies one of the conditions given in Theorem B. A hereditary family $\mathcal{F}$ is pointwise closed if and only if no infinite subset M of $\mathbb{N}$ exists such that $[M]^{<\omega} \subseteq \mathcal{F}$ (Proposition 2.14). In fact, the following result is proved in Theorem 2.16 and Remark 2.17.

Theorem C If $\mathcal{F}$ is a hereditary and pointwise closed family of finite subsets of $\mathbb{N}$, then for every countable ordinal ξ, every infinite subset M of $\mathbb{N}$ and every ξ–uniform family $\mathcal{L}$ on M, the following hold:

(i) If $s_M(\mathcal{F}) > \xi + 1$ then there exists an infinite subset L of M such that

$$\mathcal{L} \cap [L]^{<\omega} \subseteq \mathcal{F},$$

(ii) If $s_M(\mathcal{F}) < \xi + 1$ then there exists an infinite subset L of M such that

$$\mathcal{F} \cap [L]^{<\omega} \subseteq \mathcal{L}^* \setminus \mathcal{L}.$$

(iii) If $s_M(\mathcal{F}) = \xi + 1$ then both alternatives ((i) and (ii)) may materialize.

A consequence of these theorems is the existence, for a hereditary and pointwise closed family $\mathcal{F}$ of finite subsets of $\mathbb{N}$, of a countable ordinal ξ such that, for every system $(\mathcal{A}_\zeta)_{\zeta<\omega_1}$ of uniform families the following obtain:

(i) For every ζ with $\zeta + 1 < \xi$ there exists an infinite subset L of M such that

$$\mathcal{A}_\zeta \cap [L]^{<\omega} \subseteq \mathcal{F}.$$

(ii) For every ζ with $\zeta < \zeta + 1$ there exists an infinite subset L of M such that

$$\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\zeta)^* \setminus \mathcal{A}_\zeta;$$

which gives that

$$\mathcal{A}_\zeta \cap [L]^{<\omega} \subseteq [N]^{<\omega} \setminus \mathcal{F}.$$

(iii) If $\xi = \zeta + 1$ both alternatives ((i) and (ii)) may materialize.

Finally, for every hereditary family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ there exists an infinite subset L of $\mathbb{N}$ such that either $[L]^{<\omega} \subseteq \mathcal{F}$, if $\mathcal{F}$ not closed, or $[L]^{<\omega} \subseteq ([\mathbb{N}]^{<\omega} \setminus \mathcal{F})_*$ (if $\mathcal{F}$ is closed) (Corollary 2.15). $\mathcal{F}_*$ denotes the corresponding hereditary family to $\mathcal{F}$).

Notation and terminology: We denote by $\mathbb{N}$ the set of all natural numbers. For an infinite subset M of $\mathbb{N}$ we denote by $[M]^{<\omega}$ the set of all finite subsets of M and by $[M]$ the set of all infinite subsets of M (considering them as strictly increasing sequences).

If s, t are finite subsets of $\mathbb{N}$ then $s \preceq t$ means that s is an initial segment of t, while $s \prec t$ means that s is a proper initial segment of t. We write $s \leq t$ if $\max a \leq \min t$, while $s < t$ if $\max s < \min t$.

Identifying every subset of $\mathbb{N}$ with its characteristic function, we topologize the set of all subsets of $\mathbb{N}$ by the topology of pointwise convergence.

The generalized Schreier system $(\mathcal{F}_a)_{a<\omega_1}$, mentioned before, has been defined in [A–A] as follows:

$$\mathcal{F}_0 = \{\{n\} : n \in \mathbb{N}\};$$

if $\mathcal{F}_\xi$ has been defined

$$\mathcal{F}_{\xi+1} = \left\{ \bigcup_{i=1}^{n} F_i : n \leq F_1 < \ldots < F_n \text{ and } F_i \in \mathcal{F}_\xi \right\};$$

if ξ is limit choose $(\xi_n)_{n\in\mathbb{N}}$ strictly increasing to ξ and set

$$\mathcal{F}_\xi = \{F : F \in \mathcal{F}_{\xi_n} \text{ and } n \leq \min F\}.$$

1 Systems of uniform families and Cantor–Bendixson index

The definition of a uniform family (consisting of finite subsets of $\mathbb{N}$), stated below, is given by Pudlák and Rödl in [P–R].

Definition 1.1 Let $M \in [\mathbb{N}]$ and $\mathcal{L}$ be a family of finite subsets of M.

(i) For every $m \in M$, set $\mathcal{L}(m) = \{s \in [M]^{<\omega} : \{m\} \cup s \in \mathcal{L} \text{ and } \{m\} < s\}$.

(ii) (Recursive definition of a uniform family)

1. $\mathcal{L}$ is **0–uniform on** M if $\mathcal{L} = \{\emptyset\}$;
2. if ξ is a successor, countable ordinal, $\xi = \zeta + 1$, then $\mathcal{L}$ is ξ–**uniform on** M if $\emptyset \notin \mathcal{L}$ and the family $\mathcal{L}(m)$ is ζ–uniform on $M \cap (m, +\infty)$ for every $m \in M$; and
3. if ξ is a non–zero, limit countable ordinal then $\mathcal{L}$ is ξ–**uniform on** M if $\emptyset \notin \mathcal{L}$ and there is an increasing sequence $(\xi_m)_{m\in M}$ of ordinal numbers, smaller than ξ, with $\sup_m \xi_m = \xi$ such that the family $\mathcal{L}(m)$ is ξ_m–uniform on $M \cap (m, +\infty)$ for every $m \in M$.

(iii) $\mathcal{L}$ is called **uniform on** M if $\mathcal{L}$ is ξ–uniform on M for some countable ordinal ξ.

(iv) $\mathcal{L}^* = \{t \in [M]^{<\omega} : t \text{ is an initial segment of some } s \in \mathcal{L}\}$.

(v) $\mathcal{L}_* = \{t \in [M]^{<\omega} : t \subseteq s \text{ for some } s \in \mathcal{L}\}$.

(vi) $\mathcal{L}$ is **hereditary** if $\mathcal{L}_* = \mathcal{L}$.

(vii) $\mathcal{L}$ is **Sperner** if there do not exist $s, t \in \mathcal{L}$ such that s is a proper subset of t.

(viii) $\mathcal{L}$ is **thin** if there do not exist $s, t \in \mathcal{L}$ such that s is a proper initial segment of t.

Every family $\mathcal{L} \subset [M]^{<\omega}$ determines a partition $\mathcal{L} = \bigcup_{m \in M} \{\{m\} \cup s : m < s, s \in \mathcal{L}(m)\}$; and $\mathcal{L}$ is a ξ–uniform family precisely when $\mathcal{L}(m)$ is ξ_m–uniform on $M \cap (m, +\infty)$, with $\xi_m = \zeta$ for every $m \in M$, if $\xi = \zeta + 1$, and $(\xi_m)_{m \in M}$ an increasing to ξ sequence if ξ is limit. For example, if $\mathcal{L}$ is 1–uniform on M then $\mathcal{L} = \{\{m\} : m \in M\}$ since $\mathcal{L}(m) = \{\emptyset\}$ (the only 0–uniform on $M \cap (m, +\infty)$).

Conversely, for every countable ordinal ξ and $M \in [\mathbb{N}]$ we can construct a ξ–uniform family $\mathcal{L}$ on M if we have for every $m \in M$ a ξ_m–uniform family $\mathcal{A}_m$ on $M \cap (m, +\infty)$, ($(\xi_m)_{m \in M}$ is as before). Indeed, the family $\mathcal{L} = \bigcup_{m \in M} \{\{m\} \cup s : s \in \mathcal{A}_m\}$ is ξ–uniform on M, since $\mathcal{L}(m) = \mathcal{A}_m$ for every $m \in M$.

Remarks 1.2 (i) For every $M \in [\mathbb{N}]$ and every natural number k there is exactly one k–uniform family on M, namely the family $[M]^k$ of all k–element subsets of M.

(ii) If $\mathcal{L}$ is a ξ–uniform family on M ($M \in [\mathbb{N}]$) and $L \in [M]$, then, as can be proved by induction on ξ, $\mathcal{L} \cap [L]^{<\omega}$ is ξ–uniform on L (cf. [P–R]).

(iii) Using (i) and (ii) we can describe a way of constructing uniform families.

If $\mathcal{L}_\xi$ is a ξ–uniform family on M (with $M \in [\mathbb{N}]$) and $k \in \mathbb{N}$, then it is easy to see by induction on k that the family

$$\mathcal{L}_{\xi+k} = \left\{s \in [M]^{<\omega} : s = s_1 \cup s_2 \text{ where } s_1 < s_2, s_1 \in [M]^k \text{ and } s_2 \in \mathcal{L}_\xi\right\}$$

is a $\xi + k$–uniform family on M.

If ξ is a limit ordinal and $\mathcal{L}_\beta$ is a β–uniform family for every $\beta < \xi$, then we choose an increasing sequence $(\xi_m)_{m \in M}$ of ordinal numbers smaller than ξ with $\sup_m \xi_m = \xi$ and set

$$\mathcal{L}_\xi = \{s \in [M]^{<\omega} : s = \{m\} \cup s_1 \text{ where } m \in M, \{m\} < s_1 \text{ and } s_1 \in \mathcal{L}_{\xi_m}\};$$

the family $\mathcal{L}_\xi$ is in fact ξ–uniform on M, since $(\mathcal{L}_\xi)(m) = \mathcal{L}_{\xi_m} \cap [(m, +\infty)]^{<\omega}$ for every $m \in M$.

(iv) Every uniform family $\mathcal{L}$ on $M \in [\mathbb{N}]$ is a maximal thin subset of $[M]^{<\omega}$, (for the proof see [P–R]).

(v) A uniform family $\mathcal{L}$ on $M \in [\mathbb{N}]$ is not necessarily Sperner (see Example 1.12 below). However, for every uniform family $\mathcal{L}$ on M there exists $L \in [M]$ such that $\mathcal{L} \cap [L]^{<\omega}$ is Sperner (Corollary 2.4 below).

Now, we will introduce the concept of a system of uniform families. A system of uniform families on M ($M \in [\mathbb{N}]$) is a collection $\mathcal{A} = (\mathcal{A}_\xi)_{\xi<\omega_1}$, where each $\mathcal{A}_\xi$ is ξ–uniform on M, constructed in the way described in Remark 1.2 (iii), from uniform families $\mathcal{A}_\beta, \beta < \xi$, belonging to $\mathcal{A}$. The definition provides the necessary path, through which uniform families are constructed and also gives the means of verification that a given family is uniform.

Definition 1.3 Let $M \in [\mathbb{N}]$ and $\mathcal{A}_\xi \subseteq [M]^{<\omega}$ for every countable ordinal ξ. The collection $\mathcal{A} = (\mathcal{A}_\xi)_{\xi<\omega_1}$ is a **system of uniform families on** M if:

(i) $\mathcal{A}_\xi$ is ξ–uniform family on M for every $\xi < \omega_1$, and

(ii) For every $m \in M$ and $1 \le \xi < \omega_1$

$$\mathcal{A}_\xi(m) = \mathcal{A}_{\xi_m} \cap [(m, +\infty)]^{<\omega},$$

where $\xi_m + 1 = \xi$ if ξ is a successor ordinal; and $(\xi_m)_{m\in M}$ is an increasing sequence of ordinals smaller than ξ, with $\sup_m \xi_m = \xi$, if ξ is a limit ordinal.

Remarks 1.4 (i) If $(\mathcal{A}_\xi)_{\xi<\omega_1}$ is a system of uniform families on M and $L \in [M]$, then $(\mathcal{A}_\xi \cap [L]^{<\omega})_{\xi<\omega_1}$ is a system of uniform families on L, according to Remark 1.2 (ii).

(ii) Every system of uniform families on M is characterized by the choices of the sequences $(\xi_m)_{m\in M}$ for every limit ordinal ξ. Indeed, if for every limit ordinal ξ an increasing sequence $(\xi_m)_{m\in M}$ is given with $\xi_m < \xi$ for every $m \in M$ and $\sup_m \xi_m = \xi$, then we can define exactly one system of uniform families using these sequences in the following way:

$$\mathcal{A}_0 = \{\emptyset\};$$
$$\mathcal{A}_{\zeta+1} = \bigcup_{m\in M} \{\{m\} \cup s : s \in \mathcal{A}_\zeta \cap [(m, +\infty)]^{<\omega}\} \text{ for } \zeta < \omega_1; \text{ and}$$
$$\mathcal{A}_\xi = \bigcup_{m\in M} \{\{m\} \cup s : s \in \mathcal{A}_{\xi_m} \cap [(m, +\infty)]^{<\omega}\},$$

for ξ limit, countable ordinal.

As we observed in Remark 1.2 (iii), for every ξ with $\omega \le \xi < \omega_1$, there are continuum many ξ–uniform families. Indeed, there are as many ω–uniform families on $\mathbb{N}$, as the multitude of all the increasing, unbounded sequences of natural numbers. Also, according to Remark 1.4 (ii) there are as many systems of uniform families on $\mathbb{N}$, as the multitude of all the choices of increasing sequences $(\xi_n)_{n\in\mathbb{N}}$, with $\xi_n < \xi$ for all $n \in \mathbb{N}$ and $\sup_n \xi_n = \xi$, for each countable limit ordinal ξ.

With suitable choices of sequences $(\xi_n)_{n\in\mathbb{N}}$ one can define interesting systems of uniform families. Below, in Theorem 1.6, we will define a Schreier type system $\mathcal{A} = (\mathcal{A}_\xi)_{\xi<\omega_1}$ of uniform families. This system in the ω^a–position has the uniform family $\mathcal{B}_a = \mathcal{A}_{\omega^a}$ (Definition 1.5 below) which is similar to the Schreier set $\mathcal{F}_a$ (for every $a < \omega_1$) defined in [A–A].

Definition 1.5 (Schreier type system of uniform families)
(1) We define inductively for every $a < \omega_1$ the families $\mathcal{B}_a \subseteq [\mathbb{N}]^{<\omega}$ as follows:

(i) $\mathcal{B}_0 = \{\{n\} : n \in \mathbb{N}\}$;

(ii) If the family $\mathcal{B}_a$ has been defined, let

$$\mathcal{B}_{a+1} = \{s \subseteq \mathbb{N} : s = \bigcup_{i=1}^{n} s_i \text{ where } n = \min s_1, s_1 < \ldots < s_n \text{ and } s_1, \ldots, s_n \in \mathcal{B}_a\}; \text{ and}$$

(iii) If a is a limit countable ordinal and the families $\mathcal{B}_\zeta$ have been defined for each $\zeta < a$, let

$$\mathcal{B}_a = \{s \subseteq \mathbb{N} : s \in \mathcal{B}_{a_n} \text{ with } n = \min s\},$$

where (a_n) is a fixed increasing sequence of ordinal numbers smaller than a with $\sup\limits_n a_n = a$.

(2) We set $\mathcal{A}_{\omega^a} = \mathcal{B}_a$ for all ordinals $a < \omega_1$, and we complete the system of uniform families as follows:

(i) $\mathcal{A}_0 = \{\emptyset\}$

(ii) if $\xi < \omega_1$, and the family $\mathcal{A}_\xi$ has been defined, then set

$$\mathcal{A}_{\xi+1} = \{s \subseteq \mathbb{N} : s = \{n\} \cup s_1 \text{ where } n \in \mathbb{N}, \{n\} < s_1 \text{ and } s_1 \in \mathcal{A}_\xi\}; \text{ and}$$

(iii) if ξ is a limit countable ordinal and the families $\mathcal{A}_\zeta$ have been defined for every $\zeta < \xi$ and if ξ has the form $\xi = \sum_{i=1}^{m} p_i \omega^{a_i}$, where $m, p_1, \ldots, p_m \in \mathbb{N}$ and $a_1 > \ldots > a_m > 0$ are ordinal numbers, then we set

$$\begin{aligned}\mathcal{A}_\xi \;=\; & \{s \subseteq \mathbb{N} : s = \bigcup_{i=1}^{m} s_i \text{ where } s_m < \ldots < s_1, s_i = F_1^i \cup \ldots \cup F_{p_i}^i \text{ with } F_1^i < \ldots < F_{p_i}^i \\ & \text{and} F_1^i, \ldots, F_{p_i}^i \in \mathcal{B}_{a_i} \text{ for every } 1 \leq i \leq m\}.\end{aligned}$$

Theorem 1.6 *The collection $(\mathcal{A}_\xi)_{\xi<\omega_1}$ is a system of uniform families on $\mathbb{N}$.*

Proof. $\mathcal{A}_0 = \{\emptyset\}$, so it is 0–uniform on $\mathbb{N}$. We assume that for every $\zeta < \xi$ the families $\mathcal{A}_\zeta$ are ζ–uniform on $\mathbb{N}$ and also that $\mathcal{A}_\zeta(n) = \mathcal{A}_{\zeta_n} \cap [(n, +\infty)]^{<\omega}$ for every $n \in \mathbb{N}$, where $\zeta_n + 1 = \zeta$ for every $n \in \mathbb{N}$, if ζ is a successor ordinal; and (ζ_n) is an increasing sequence of ordinals smaller than ζ with $\sup \zeta_n = \zeta$, if ζ is limit.

Let $\xi = \zeta + 1$ be a successor ordinal. According to the definition of $\mathcal{A}_{\zeta+1}$ (1.5) $\mathcal{A}_\xi(n) = \mathcal{A}_\zeta \cap [(n, +\infty)]^{<\omega}$ for every $n \in \mathbb{N}$. Hence, $\mathcal{A}_\xi$ is ξ–uniform on $\mathbb{N}$, since $\mathcal{A}_\zeta$ is ζ–uniform (Remark 1.2 (ii)).

Let ξ be a limit ordinal. We will check all particular cases:
(1) If $\xi = \omega$ then for every $n \in \mathbb{N}$

$$\mathcal{A}_\omega(n) = \mathcal{B}_1(n) = \{s : \{n\} \cup s \in \mathcal{B}_1 \text{ and } \{n\} < s\} = [(n, +\infty)]^{n-1} = \mathcal{A}_{n-1} \cap [(n, +\infty)]^{<\omega}.$$

Hence $\mathcal{A}_\omega$ is ω–uniform on $\mathbb{N}$ and $\omega_n = n - 1$ for every $n \in \mathbb{N}$.
(2) If $\xi = \omega^{a+1}$ then for every $n \in \mathbb{N}$

$$\begin{aligned}\mathcal{A}_{\omega^{a+1}}(n) &= \mathcal{B}_{a+1}(n) = \{s : s = s_1 \cup s_2 \text{ where } \{n\} < s_1 < s_2, s_1 \in \mathcal{B}_a(n) \text{ and} s_2 \in \mathcal{A}_{(n-1)\omega^a}\} = \\ & \mathcal{A}_{(n-1)\omega^a + (\omega^a)_n} \cap [(n, +\infty)]^{<\omega}.\end{aligned}$$

Hence $\mathcal{A}_{\omega^{a+1}}$ is ω^{a+1}–uniform on $\mathbb{N}$ and $(\omega^{a+1})_n = (n-1)\omega^a + (\omega^a)_n$ for every $n \in \mathbb{N}$.
(3) If $\xi = \omega^a$ for a limit ordinal a then $\mathcal{A}_{\omega^a} = \mathcal{B}_a$. Let (a_n) be the fixed sequence of ordinal numbers which is used in the definition of $\mathcal{B}_a$ (Definition 1.5). For every $n \in \mathbb{N}$ we have

$$\mathcal{A}_{\omega^a}(n) = \mathcal{A}_{(\omega^{a_n})_n} \cap [(n, +\infty)]^{<\omega}.$$

Hence, $\mathcal{A}_{\omega^a}$ is ω^a–uniform on $\mathbb{N}$ and $(\omega^a)_n = (\omega^{a_n})_n$ for every $n \in \mathbb{N}$.
(4) If $\xi = p\omega^a$, where $p \in \mathbb{N}$ and $0 < a < \omega_1$, then for every $n \in \mathbb{N}$

$$\begin{aligned}\mathcal{A}_{p\omega^a}(n) &= \{s : s = s_1 \cup s_2 \text{ where } \{n\} < s_1 < s_2, s_1 \in \mathcal{A}_{(\omega^a)_n} \text{ and} s_2 \in \mathcal{A}_{(p-1)\omega^a}\} = \\ & \mathcal{A}_{(p-1)\omega^a + (\omega^a)_n} \cap [(n, +\infty)]^{<\omega}.\end{aligned}$$

Hence, $\mathcal{A}_{p\omega^a}$ is $p\omega^a$–uniform on $\mathbb{N}$ and $(p\omega^a)_n = (p-1)\omega^a + (\omega^a)_n$ for every $n \in \mathbb{N}$.
(5) Finally, if $\xi = \sum_{i=1}^{m} p_i \omega^{a_i}$, where $m, p_1, \ldots, p_m \in \mathbb{N}$ and $a_1 > \ldots > a_m > 0$, then for every $n \in \mathbb{N}$

$$\begin{aligned}\mathcal{A}_\xi(n) &= \{s : s = s_1 \cup s_2 \text{ where } \{n\} < s_1 < s_2, s_1 \in \mathcal{A}_{(p_m\omega^{a_m})_n} \text{ and} s_2 \in \mathcal{A}_\beta\} = \\ &= \mathcal{A}_{\beta + (p_m\omega^{a_m})_n} \cap [(n, +\infty)]^{<\omega},\end{aligned}$$

where $\beta = \sum_{i=1}^{m-1} p_i \omega^{a_i}$.

Hence, $\mathcal{A}_\xi$ is ξ–uniform on $\mathbb{N}$ and $\xi_n = \sum_{i=1}^{m-1} p_i \omega^{a_i} + (p_m \omega^{a_m})_n$ for every $n \in \mathbb{N}$.

This completes the proof of the theorem.

Corollary 1.7 For every $M \in [\mathbb{N}]$, the collection $(\mathcal{A}_\xi^M)_{\xi < \omega_1}$, where $\mathcal{A}_\xi^M = \mathcal{A}_\xi \cap [M]^{<\omega}$ for every $\xi < \omega_1$, is a system of uniform families on M.

Proof. This is a consequence of Theorem 1.6 and Remark 1.4 (i).

It would be very complicated to prove directly that the family $\mathcal{B}_a$ is ω^a– uniform on $\mathbb{N}$ for every $a < \omega_1$, but using the notion of a system of uniform families the proof is immediate after Theorem 1.6.

Corollary 1.8 For every countable ordinal a and $M \in [\mathbb{N}]$ the family $\mathcal{B}_a \cap [M]^{<\omega}$ is ω^a–uniform on M.

Proof. We have $\mathcal{B}_a = \mathcal{A}_{\omega^a}$ for every $a < \omega_1$.

Remarks 1.9 (i) In the definition of the Schreier type system $\mathcal{A} = (\mathcal{A}_\xi)_{\xi<\omega_1}$ of uniform families we have choices of increasing sequences $(\xi_n)_{n\in\mathbb{N}}$ (in fact $\xi_n = \omega^{a_n}$ for every $n \in \mathbb{N}$) with $\xi_n < \xi$ for all $n \in \mathbb{N}$ and $\sup_n \xi_n = \xi$ only in the cases $\xi = \omega^a$, where a is a limit countable ordinal. In the other limit countable ordinals ξ we use concrete sequences depending on ξ and the previous choices.

(ii) It is easy to see that $\mathcal{B}_a \subseteq \mathcal{F}_a$ for every $a < \omega_1$. In general, the hereditary family $(\mathcal{B}_a)_*$ of all the subsets of the elements of $\mathcal{B}_a$ is not equal to $\mathcal{F}_a$. However, in Section 3 (Proposition 3.1) we will prove that for every $M \in [\mathbb{N}]$ there exists $L = (\ell_n)_{n\in\mathbb{N}} \in [M]$ such that $\mathcal{F}_a(L) \subseteq (\mathcal{B}_a)_* \cap [M]^{<\omega}$, where

$$\mathcal{F}_a(L) = \{(\ell_{n_1}, \dots, \ell_{n_k}) \subseteq L : (n_1, \dots, n_k) \in \mathcal{F}_a\}.$$

At this point the reader might think that the definition of a system of uniform families is unneccesarily cumbersome. It bears similarity to the various Schreier–type system $(\mathcal{F}_a)_{a<\omega_1}$ used in the literature (e.g. Alspach–Argyros ([A–A]), Argyros–Mercourakis–Tsarpalias ([A–M–T]), Farmaki ([F1], [F2]), Kyriakouli–Negrepontis ([M–N]), Odell–Tomczak–Wagner ([O–T–W]) and others). However, the system $(\mathcal{F}_a)_{a<\omega_1}$ is very difficult to employ in inductive arguments, owing on the one hand to the concrete and fixed nature of the definition of $\mathcal{F}_a, a < \omega_1$ and also, and more significantly on a rather more hidden aspect of their interrelation (for different ordinals). We can clarify the precise relation between the system $(\mathcal{F}_a)_{a<\omega_1}$ and the system of uniform families $(\mathcal{A}_\xi)_{\xi<\omega_1}$, if we think that each family $\mathcal{F}_a$ is related not to the family $\mathcal{A}_a$ but to the uniform family $\mathcal{A}_{\omega^a}$. In other words, the difficulty in employing Schreier–type systems in inductive arguments lies with the fact that, e.g. there are missing families, not defined for all ordinals ξ with $\omega^a < \xi < \omega^{a+1}, a < \omega_1$. This filling up of the intermediate gaps was in effect performed in a special case, arising in Banach space theory, in our earlier work in [F2].

Thus, returning to the difficulty in employing induction on the Schreier sets $\mathcal{A}_a, a < \omega_1$ owing to their fixed nature, it will be seen clearly in Section 2 below that the notion of a uniform family ($\mathcal{A}_\xi$ is uniform for every $\xi < \omega_1$) provides us with the correct amount of leeway, a leeway that is precisely missing from the system $(\mathcal{F}_a)_{a<\omega_1}$.

In the following we will estimate the strong Cantor–Bendixson index of a uniform family. This index (see Definition 1.10 below) is analogous to the well–known Cantor–Bendixson index ([B], [C]) and has been defined in [A–M–T]. Here, we will use a different notation in order to avoid some misinterpretations.

We will prove in Proposition 1.18 below that, for every $\xi < \omega_1$, $M \in [\mathbb{N}]$, the corresponding hereditary family $\mathcal{L}_*$ of a ξ–uniform family $\mathcal{L}$ on M has strong Cantor–Bendixson index on M equal to $\xi + 1$. Hence, if $(\mathcal{A}_\xi)_{\xi<\omega_1}$ is a system of uniform families, then the collection $((\mathcal{A}_\xi)_*)_{\xi<\omega_1}$ contains hereditary families of arbitrary index.

Definition 1.10 ([A–M–T]) Let $\mathcal{F}$ be a hereditary and pointwise closed family of finite subsets of $\mathbb{N}$. For $M \in [\mathbb{N}]$ we define the **strong Cantor – Bendixson derivatives** $(\mathcal{F})^{\xi}_{M}$ of $\mathcal{F}$ on M for every $\xi < \omega_1$ as follows:

$$(\mathcal{F})^1_M = \{A \in \mathcal{F}[M] : A \text{ is a cluster point of } \mathcal{F}[A \cup L] \text{ for each } L \in [M]\};$$

(where, $\mathcal{F}[M] = \mathcal{F} \cap [M]^{<\omega}$).
If $(\mathcal{F})^{\xi}_{M}$ has been defined, then

$$(\mathcal{F})^{\xi+1}_M = ((\mathcal{F})^{\xi}_M)^1_M.$$

If ξ is a limit ordinal and $(\mathcal{F})^{\beta}_M$ have been defined for each $\beta < \xi$, then

$$(\mathcal{F})^{\xi}_M = \bigcap_{\beta<\xi} (\mathcal{F})^{\beta}_M.$$

The **strong Cantor – Bendixson index of $\mathcal{F}$ on** M is defined to be the smallest countable ordinal ξ such that $(\mathcal{F})^{\xi}_M = \emptyset$. We denote this index by $s_M(\mathcal{F})$.

We define the **strong Cantor – Bendixson index** $s(\mathcal{F})$ of $\mathcal{F}$ to be $s(\mathcal{F}) = s_M(\mathcal{F})$, where $M = \{n \in \mathbb{N} : \{n\} \in \mathcal{F}\}$ is the support of $\mathcal{F}$.

Remarks 1.11 (i) Of course, the strong Cantor – Bendixson index is a successor ordinal.

(ii) If $\mathcal{F}_1, \mathcal{F}_2 \subseteq [\mathbb{N}]^{<\omega}$ are hereditary and closed and $\mathcal{F}_1 \subseteq \mathcal{F}_2$ then $s_M(\mathcal{F}_1) \leq s_M(\mathcal{F}_2)$ for every $M \in [\mathbb{N}]$.

(iii) $s_M(\mathcal{F}) = s_M(\mathcal{F}[M])$ for every $M \in [\mathbb{N}]$.

(iv) For every $M \in [\mathbb{N}]$ and $A \in [M]^{<\omega}$ according to a remark in [J] we have:
$A \in (\mathcal{F})^1_M$ if and only if the set $\{m \in M : A \cup \{m\} \notin \mathcal{F}\}$ is finite.

(v) Using the previous remark (iv) can be proved by induction that for every $L \in [M]$ and $\beta < \omega_1$

$$A \cap L \in (\mathcal{F})^{\beta}_L \text{ if } A \in (\mathcal{F})^{\beta}_M.$$

Hence, $s_L(\mathcal{F}) \geq s_M(\mathcal{F})$. (see also [A–M–T]).

(vi) If L is almost contained in M, then

$$s_L(\mathcal{F}) \geq s_M(\mathcal{F}).$$

(vii) For every $a < \omega_1$, let $\mathcal{F}_a$ be the Shreier family. Then for every $M \in [\mathbb{N}]$

$$s_M(\mathcal{F}_a) = \omega^a + 1.$$

(see [A–M–T]).

In the following we will give the precise relation between the strong Cantor–Bendixson derivatives of the corresponding hereditary family $\mathcal{L}_*$ of a given family $\mathcal{L} \subseteq [\mathbb{N}]^{<\omega}$ and the derivatives of the families $(\mathcal{L}(n))_*$ for every $n \in \mathbb{N}$. After that, we will calculate the strong Cantor–Bendixson index of a uniform family.

First of all we must notice that the families $(\mathcal{L}(n))_*$ and $\mathcal{L}_*(n)$ are in general different as we can see from the following example.

Example 1.12 For every $n \in \mathbb{N}$ choose the following member of the Schreier type system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ (Definition 1.5).

$\mathcal{L}_1 = [\mathbb{N}]^5 = \mathcal{A}_5$,

$\mathcal{L}_2 = \mathcal{A}_\omega$, and

$\mathcal{L}_n = \mathcal{A}_{\omega+n}$ for every $n > 2$.

Set

$$\mathcal{L} = \bigcup_{n\in\mathbb{N}} \{\{n\} \cup s : s \in \mathcal{L}_n \text{ and } \{n\} < s\}.$$

Then $\mathcal{L}$ is a 2ω–uniform family. Let $s = (2, 3, 4, 5, 6)$ and $t = (1, 2, 3, 4, 5, 6)$. Since $t \in \mathcal{L}$ we have that $s \in \mathcal{L}_*$ and consequently that $s_1 = (3, 4, 5, 6) \in \mathcal{L}_*(2)$. As we can see, $\mathcal{L}(2) = \mathcal{A}_\omega \cap [(2, +\infty)]^{<\omega}$ and of course $s_1 \notin (\mathcal{L}(2))_*$.

It is remarkable that the family $\mathcal{L}$ is not Sperner (see Remark 1.2 (v)), since $F = (2, 3, 4, 5) \in \mathcal{L}$ and $F \subseteq t$, $(t \in \mathcal{L})$.

Lemma 1.13 Let β be a countable ordinal and $\mathcal{L} \subseteq [\mathbb{N}]^{<\omega}$ such that $\mathcal{L}_*$ and $\mathcal{L}(n)_*$ are closed for every $n \in \mathbb{N}$. If $A \in (\mathcal{L}(n)_*)^\beta_M$ for some $n \in \mathbb{N}$ and $M \in [\mathbb{N}]$, then $\{n\} \cup A \in (\mathcal{L}_*)^\beta_M$.

Proof. We use induction on β. Let $A \in (\mathcal{L}(n)_*)^1_M$ for some $n \in \mathbb{N}$ and $M \in [\mathbb{N}]$. Since

$$\{m \in M : A \cup \{m\} \in \mathcal{L}(n)_*\} \subseteq \{m \in M : A \cup \{m\} \cup \{n\} \in \mathcal{L}_*\},$$

we have, according to Remark 1.11 (iv), that $A \cup \{n\} \in (\mathcal{L}_*)^\beta_M$.

Suppose that the assertion holds for all ordinals ζ with $\zeta < \beta$. If $A \in (\mathcal{L}(n)_*)^{\zeta+1}_M$ then $\{n\} \cup A \in (\mathcal{L}_*)^{\zeta+1}_M$, since

$$\{m \in M : A \cup \{m\} \in (\mathcal{L}(n)_*)^\zeta_M\} \subseteq \{m \in M : A \cup \{m\} \cup \{n\} \in (\mathcal{L}_*)^\zeta_M\};$$

according to the induction hypothesis.

The case where β is limit ordinal is trivial.

Proposition 1.14 Let $\mathcal{L} \subseteq [\mathbb{N}]^{<\omega}$ such that $\mathcal{L}_*$ and $\mathcal{L}(n)_*$ are closed for every $n \in \mathbb{N}$ and $M \in [\mathbb{N}]$.

(i) If there exists $L \in [M]$ such that $s_M(\mathcal{L}(n)_*) = \xi$ for every $n \in L$ then $s_L(\mathcal{L}_*) \geq \xi + 1$.

(ii) Let $\xi_n = s_M(\mathcal{L}(n)_*)$ for every $n \in \mathbb{N}$. If there exists $L \in [M]$ such that $\xi_n < \xi = \sup_{n\in L} \xi_n$ for every $n \in L$, then $s_M(\mathcal{L}_*) \geq \xi + 1$.

Proof.

(i) Let $\xi = \beta + 1$ and $L \in [M]$ such that $s_M(\mathcal{L}(n)_*) = \xi$ for every $n \in L$. Then $\emptyset \in (\mathcal{L}(n)_*)^{\beta}_M$ for every $n \in L$. According to Lemma 1.13 we have $\{n\} \in (\mathcal{L}_*)^{\beta}_M$ for every $n \in L$ and then $\{n\} \in (\mathcal{L}_*)^{\beta}_L$ for every $n \in L$ by Remark 1.11 (v). From Remark 1.11 (iv) we have $\emptyset \in (\mathcal{L}_*)^{\beta+1}_L$ and therefore $s_L(\mathcal{L}_*) \geq \xi + 1$.

(ii) In this case ξ is a limit ordinal. Since the ξ_n are successor ordinals we set $\xi_n = \beta_n + 1$ for every $n \in \mathbb{N}$. According to our hypothesis $\emptyset \in (\mathcal{L}(n)_*)^{\beta_n}_M$ for every $n \in L$. Hence $\emptyset \in (\mathcal{L}_*)^{\beta_n}_M$ for every $n \in L$. Since $\sup\limits_{n \in L} \beta_n = \xi$ and $\beta_n < \xi$ we have $\emptyset \in (\mathcal{L}_*)^{\xi}_M$ and therefore $s_M(\mathcal{L}_*) \geq \xi + 1$.

Lemma 1.15 Let β be a countable ordinal and $\mathcal{L} \subseteq [\mathbb{N}]^{<\omega}$ such that $\mathcal{L}_*$ and $\mathcal{L}(n)_*$ are closed for every $n \in \mathbb{N}$. If $A \neq \emptyset$ and $A \in (\mathcal{L}_*)^{\beta}_M$ for some $M \in [\mathbb{N}]$, then there exist $\ell \in \mathbb{N}$ with $\ell \leq \min A$ and $L \in [M]$ such that $A \setminus \{\ell\} \in (\mathcal{L}(\ell)_*)^{\beta}_L$.

Proof. We use induction on β. Let $A \neq \emptyset$ and $A \in (\mathcal{L}_*)^1_M$. According to Remark 1.11 (iv) the set $M_A = \{m \in M : A \cup \{m\} \in \mathcal{L}_* \text{ and } \min A \leq m\}$ is almost equal to M. For each $m \in M_A$ there exists $s_m \in \mathcal{L}$ such that $A \cup \{m\} \subseteq s_m$. Set

$$\ell = \min\{n \in \mathbb{N} : \text{ the set } \{m \in M_A : \min s_m = n\} \text{ is infinite }\}.$$

Of course $\ell \leq A$. Set $L = \{m \in M_A : \min s_m = \ell\} \cup A$. Then $L \in [M]$ and $A \setminus \{\ell\} \in (\mathcal{L}(\ell)_*)^1_L$, as required.

Suppose now that the assertion holds for all ordinals ζ with $\zeta < \beta$ and let $\beta = \zeta + 1$. If $A \neq \emptyset$ and $A \in (\mathcal{L}_*)^{\zeta+1}_M$ then according to Remark 1.11 (iv) the set $M_A = \{m \in M : A \cup \{m\} \in (\mathcal{L}_*)^{\zeta}_M \text{ and } \min A \leq m\}$ is almost equal to M. Let $m_1 = \min M_A$. By the induction hypothesis there exist $\ell_1 \in \mathbb{N}$ and $L_1 \in [M_A]$ with $\ell_1 \leq \min A$ such that $A \cup \{m_1\} \setminus \{\ell_1\} \in (\mathcal{L}(\ell_1)_*)^{\zeta}_{L_1 \cup A}$, since $A \cup \{m_1\} \in (\mathcal{L}_*)^{\zeta}_{M_A \cup A}$ (Remark 1.11 (v)). Let $m_2 \in L_1$ and $m_2 > m_1$. Since $A \cup \{m_2\} \in (\mathcal{L}_*)^{\zeta}_{L_1 \cup A}$ there exists $\ell_2 \in \mathbb{N}$ with $\ell_2 \leq \min A$ and $L_2 \in [L_1]$ such that $A \cup \{m_2\} \setminus \{\ell_2\} \in (\mathcal{L}(\ell_2)_*)^{\zeta}_{L_2 \cup A}$. We continue analogously setting $m_3 \in L_2$ with $m_3 > m_2$ and so on.

Hence we construct an increasing sequence $(m_i)_{i=1}^{\infty}$ in M_A, a sequence $(\ell_i)_{i=1}^{\infty}$ in $\mathbb{N}$, with $1 \leq \ell_i \leq \min A$ for every $i \in \mathbb{N}$, and a decreasing sequence $(L_i)_{i=1}^{\infty}$ in $[M_A]$ such that

$$A \cup \{m_i\} \setminus \{\ell_i\} \in (\mathcal{L}(\ell_i)_*)^{\zeta}_{L_i \cup A}$$

for every $i \in \mathbb{N}$.

We can find $\ell \in \mathbb{N}$ with $1 \leq \ell \leq \min A$ such that the set $I = \{i \in \mathbb{N} : \ell_i = \ell\}$ is infinite. Set $L = \{m_i : i \in I\} \cup A$. Then $A \setminus \{\ell\} \in (\mathcal{L}(\ell)_*)^{\zeta+1}_L$, as required.

In the case where β is a limit ordinal and $A \in (\mathcal{L}_*)^{\beta}_M$, $A \neq \emptyset$, we fix a strictly increasing sequence $(\zeta_i)_{i=1}^{\infty}$ of ordinals with $\zeta_i < \beta$ for every $i \in \mathbb{N}$ and $\sup_i \zeta_i = \beta$. Then $A \in (\mathcal{L}_*)^{\zeta_i}_M$ for every $i \in \mathbb{N}$. According to the induction hypothesis if $M_A = \{m \in M : \min A \leq m\}$ there exist $\ell_1 \in \mathbb{N}$ with $\ell_1 \leq \min A$ and $L_1 \in [M_A]$ such that $A \setminus \{\ell_1\} \in (\mathcal{L}(\ell_1)_*)^{\zeta_1}_{L_1 \cup A}$.

Since $A \in (\mathcal{L}_*)^{\zeta_2}_{L_1 \cup A}$ there exists $\ell_2 \in \mathbb{N}$ with $\ell_2 \leq \min A$ and $L_2 \in [L_1]$ such that $L_2 \neq L_1$ and

$$A \setminus \{\ell_2\} \in (\mathcal{L}(\ell_2)_*)^{\zeta_2}_{L_2 \cup A}$$

So we construct a sequence $(\ell_i)_{i=1}^{\infty}$ with $1 \leq \ell_i \leq \min A$ and a strictly decreasing sequence $(L_i)_{i=1}^{\infty}$ in $[M_A]$ such that

$$A \setminus \{\ell_i\} \in (\mathcal{L}(\ell_i)_*)^{\zeta_i}_{L_i \cup A}$$

for every $i \in \mathbb{N}$.

We can find ℓ with $1 \leq \ell \leq \min A$ such that the set $I = \{i \in \mathbb{N} : \ell_1 = \ell\}$ is infinite. Set $L = \{\min L_i : i \in I\} \cup A$. Then $A \setminus \{\ell\} \in (\mathcal{L}(\ell)_*)^{\zeta_i}_L$ for every $i \in I$. Since $\sup_{i \in I} \zeta_i = \beta$, we have that

$$A \setminus \{\ell\} \in (\mathcal{L}(\ell)_*)^{\beta}_L.$$

This completes the proof.

Proposition 1.16 Let $\mathcal{L} \subseteq [\mathbb{N}]^{<\omega}$ such that $\mathcal{L}_*$ and $\mathcal{L}(n)_*$ are closed for every $n \in \mathbb{N}$ and $M \in [\mathbb{N}]$. If $\xi = \sup\{s_L(\mathcal{L}(n)_*) : n \in \mathbb{N} \text{ and } L \in [M]\}$ then $(\mathcal{L}_*)^{\xi}_M \subseteq \{\emptyset\}$ and therefore $s_M(\mathcal{L}_*) \leq \xi + 1$.

Proof. Let $A \in (\mathcal{L}_*)^{\xi}_M$ and $A \neq \emptyset$. According to lemma 1.15 there exist $n \in \mathbb{N}$ and $L \in [M]$ such that $A \setminus \{n\} \in (\mathcal{L}(n)_*)^{\xi}_L$. Hence, $s_L(\mathcal{L}(n)_*) \geq \xi + 1$. A contradiction, which finishes the proof.

After Propositions 1.14 and 1.16 we will see in Theorem 1.18 that the definition of a uniform family is the most suitable and least complicated in order to ensure that every ξ–uniform family on M ($M \in [\mathbb{N}]$) is thin (this arises from the condition $\emptyset \notin \mathcal{L}$ for every ζ–uniform family with $1 \leq \zeta$ in Definition 1.1) and the corresponding hereditary family has strong Cantor–Bendixson index on L equal to $\xi + 1$, for every $L \in [M]$.

Lemma 1.17 Let $M \in [\mathbb{N}]$ and $\mathcal{L}$ a ξ–uniform family on M, for some $\xi < \omega_1$. Then $\mathcal{L}_*$ is closed.

Proof. This is easily proved by induction on ξ.

Theorem 1.18 *Let ξ be a countable ordinal, $M \in [\mathbb{N}]$ and $\mathcal{L}$ a ξ–uniform family on M. Then for every $L \in [M]$ we have $(\mathcal{L}_*)^{\xi}_L = \{\emptyset\}$ and $s_L(\mathcal{L}_*) = \xi + 1$.*

Proof. We use induction on ξ. Let $\xi = 1$. Then $\mathcal{L}_* = \{\{n\} : n \in M\} \cup \{\emptyset\}$. Hence $(\mathcal{L}_*)^1_L = \{\emptyset\}$ and therefore $s_L(\mathcal{L}_*) = 2$ for every $L \in [M]$.

Suppose the assertion holds for every ordinal number β with $\beta < \xi$. In case $\mathcal{L}$ is a $\zeta + 1$–uniform on M the families $\mathcal{L}(n)$ are ζ–uniform on $M_n = M \cap (n, +\infty)$ for every $n \in M$. Hence according to the induction hypothesis $s_L(\mathcal{L}(n)_*) = \zeta + 1 = \xi$ for every $L \in [M]$ (cf. Remark 1.11 (vi)). By Proposition 1.14 (i) we have $s_L(\mathcal{L}_*) \geq \xi + 1$ for every $L \in [M]$. Hence $(\mathcal{L}_*)^\xi_L \neq \emptyset$ for every $L \in [M]$. On the other hand, according to Proposition 1.16 we have $(\mathcal{L}_*)^\xi_L \subseteq \{\emptyset\}$ for every $L \in [M]$. Hence $(\mathcal{L}_*)^\xi_L = \{\emptyset\}$ and therefore $s_L(\mathcal{L}_*) = \xi + 1$ for every $L \in [M]$.

In the case where $\mathcal{L}$ is a ξ–uniform family on M for a limit ordinal ξ we have that $\mathcal{L}(n)$ are β_n–uniform on $M \cap (n, +\infty)$ for every $n \in M$, where (β_n) is a sequence of ordinals smaller than ξ with $\sup\limits_{n \in M} \beta_n = \xi$. According to the induction hypothesis and Remark 1.11 (vi) we have $s_L(\mathcal{L}(n)_*) = \beta_n + 1 = \xi_n$ for every $L \in [M]$ and $n \in M$. From Proposition 1.14 (ii) we have that $s_L(\mathcal{L}_*) \geq \xi + 1$ for every $L \in [M]$ hence $(\mathcal{L}_*)^\xi_L \neq \emptyset$ for every $L \in [M]$. On the other hand $(\mathcal{L}_*)^\xi_L \subseteq \{\emptyset\}$ for every $L \in [M]$ according to Proposition 1.16. Hence $(\mathcal{L}_*)^\xi_L = \{\emptyset\}$ and therefore $s_L(\mathcal{L}_*) = \xi + 1$ for every $L \in [M]$.

The proof is complete.

Corollary 1.19 Let $M \in [\mathbb{N}]$ and $(\mathcal{A}^M_\xi)_{\xi < \omega_1}$ a system of uniform families on M. Then $s_L\left((\mathcal{A}^M_\xi)_*\right) = \xi + 1$ for every $\xi < \omega_1$ and $L \in [M]$.

Hence a system of uniform families on M is an appropriate selection $(\mathcal{A}_\xi)_{\xi<\omega_1}$ of thin subfamilies of $[M]^{<\omega}$ with strong Cantor–Bendixson index on L each countable ordinal, for every $L \in [M]$.

2 Ramsey dichotomies with ordinal index

We start this section with an equivalent formulation of the classical Ramsey theorem ([R]).

Theorem 2.1 *(Ramsey) For any positive integers r and k if we partition the family $[M]^{<\omega}$ of all the finite subsets of an infinite set M into k–parts, then there is an infinite subset L of M, all r–tuples of which belong to the same class of the partition.*

In the following we will show how the concept of ξ–uniform families can be applied to provide a far–reaching generalization of the classical Ramsey theorem. This happens because general ξ–uniform families share with the family $[M]^r$ of all r–tuples of M occuring in the Ramsey theorem the following properties: (a) they are thin and (b) the Cantor–Bendixson index does not dicrease when we restrict ourselves to any infinite subset of M.

Our proof will be an elementary one, directly from the definitions involved. Another proof can be obtained, using the combinatorial theorem of Nash–Williams in [N–W] (see also [P–R]).

Theorem 2.2 *Let M be an infinite subset of $\mathbb{N}$, $\{P_1, P_2\}$ a partition of the set $[M]^{<\omega}$ of all finite subsets of M, ξ a countable ordinal number and $\mathcal{L}$ a ξ–uniform family on M. Then there exists an infinite subset L of M such that either $\mathcal{L} \cap [L]^{<\omega} \subseteq P_1$ or $\mathcal{L} \cap [L]^{<\omega} \subseteq P_2$.*

Proof. We will prove the theorem by induction on ξ.

Let $\xi = 1$. Then $\mathcal{L} = \{\{m\} : m \in M\}$. Set

$$\begin{aligned} M_1 &= \{m \in M : \{m\} \in P_1\} \text{ and} \\ M_2 &= \{m \in M : \{m\} \in P_2\}. \end{aligned}$$

If M_1 is infinite then the theorem holds for $L = M_1$, otherwise it holds for $L = M_2$.

Assume that the theorem is valid for every ordinal ζ with $\zeta < \xi$ and let $\mathcal{L}$ be a ξ–uniform family on M. Then, according to Definition 1.1, there exists a sequence $(\xi_m)_{m\in M}$ of ordinal numbers such that $\xi_m < \xi$ for every $m \in M$ and the family $\mathcal{L}(m)$ is ξ_m–uniform on $M \cap (m, +\infty)$.

Let $m_1 = \min M$ and $M_1 = M \cap (m_1, +\infty)$. Set

$$\begin{aligned} P_1^1 &= \{s \subseteq M : \{m_1\} \cup s \in P_1 \text{ and } \{m_1\} < s\} \text{ and} \\ P_2^1 &= \{s \subseteq M : \{m_1\} \cup s \in P_2 \text{ and } \{m_1\} < s\}. \end{aligned}$$

Then $\{P_1^1, P_2^1\}$ is a partition of $[M_1]^{<\omega}$. Since $\mathcal{L}(m_1)$ is ξ_{m_1}–uniform on M_1 and $\xi_{m_1} < \xi$, according to the induction hypothesis, there exists an infinite subset L_1 of M_1 such that

$$\mathcal{L}(m_1) \cap [L_1]^{<\omega} \subseteq P_{i_1}^1$$

for some $i_1 \in \{1, 2\}$.

Let $m_2 = \min L_1 > m_1$ and $M_2 = L_1 \cap (m_2, +\infty)$. Now set

$$\begin{aligned} P_1^2 &= \{s \subseteq M_2 : \{m_2\} \cup s \in P_1\} \text{ and} \\ P_2^2 &= \{s \subseteq M_2 : \{m_2\} \cup s \in P_2\}. \end{aligned}$$

It is easy to see that $\{P_1^2, P_2^2\}$ is a partition of M_2 and that $\mathcal{L}(m_2) \cap [M_2]^{<\omega}$ is ξ_{m_2}–uniform on M_2 according to Remark 1.2 (ii). Using the induction hypothesis we can find an infinite subset L_2 of M_2 and $i_2 \in \{1, 2\}$ such that

$$\mathcal{L}(m_2) \cap [L_2]^{<\omega} \subseteq P_{i_2}^2.$$

Set $m_3 = \min L_2$ and $M_3 = L_2 \cap (m_3, +\infty)$. We proceed inductively and define a strictly increasing sequence $(m_n)_{n=1}^\infty$ in M, two decreasing sequences $(M_n)_{n=1}^\infty, (L_n)_{n=1}^\infty$ in $[M]$ and a sequence $(i_n)_{n=1}^\infty$ in $\{1, 2\}$ such that for every $n \in \mathbb{N}$ we have

$$m_n = \min L_{n-1}, (L_0 = M), L_n \subseteq M_n, M_n = L_{n-1} \cap (m_n, +\infty), \text{ and}$$

$$\mathcal{L}(m_n) \cap [L_n]^{<\omega} \subseteq P_{i_n}^n,$$

where $P_{i_n}^n = \{s \subseteq M_n : \{m_n\} \cup s \in P_{i_n}\}$.

It is clear that there exists an infinite subset K of $\mathbb{N}$ such that the subsequence $(i_k)_{k\in K}$ of $(i_n)_n^\infty$ is constant; set $i_k = i$ for every $k \in K$, and

$$L = \{m_k : k \in K\}.$$

Then

$$\mathcal{L} \cap [L]^{<\omega} \subseteq P_i.$$

Indeed, let $F \in \mathcal{L} \cap [L]^{<\omega}$. Then $\min F = m_n$ for some $n \in K$. Since $F \in \mathcal{L}$, we can find $s \in \mathcal{L}(m_n)$, such that $m_n < \min s$ and $F = \{m_n\} \cup s$. Also, since $L \cap (m_n, +\infty) \subseteq L_n$ for every $n \in \mathbb{N}$, we have that $s \in [L_n]^{<\omega}$. Hence

$$s \in \mathcal{L}(m_n) \cap [L_n]^{<\omega} \subseteq P^n_{i_n}.$$

According to the definition of $P^n_{i_n}$ ($n \in \mathbb{N}$) we have that $F \in P_{i_n}$ and, since $n \in K$, that $F \in P_i$.

The proof is complete.

The following Corollary is the precise generalization of the classical Ramsey theorem.

Corollary 2.3 Let ξ be a countable ordinal, $M \in [\mathbb{N}]$ and $\mathcal{L}$ a ξ–uniform family on M. For every $N \in [M]$ and every partition $\{P_1, \ldots, P_k\}$ of $[N]^{<\omega}$ there exist $L \in [N]$ and $i \in \{1, \ldots, k\}$ such that:

$$\mathcal{L} \cap [L]^{<\omega} \subseteq P_i.$$

Proof. Let $k = 2$. For every $N \in [M]$ the family $\mathcal{L} \cap [N]^{<\omega}$ is ξ–uniform on N and $\{P_i \cap [N]^{<\omega} : 1 \leq i \leq 2\}$ is a partition of $[N]^{<\omega}$. Hence the proof is immediate by Theorem 2.1.

The general case follows by induction on k.

In the following corollary we will describe a condition in order for a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ to contain a uniform family.

Corollary 2.4 Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$, M an infinite subset of $\mathbb{N}$, ξ a countable ordinal and $\mathcal{L}$ a ξ–uniform family on M. If $\mathcal{L} \cap \mathcal{F} \cap [L]^{<\omega} \neq \emptyset$ for every $L \in [M]$, then for every $N \in [M]$ there exists $L \in [N]$ such that:

$$\mathcal{L} \cap [L]^{<\omega} \subseteq \mathcal{F}.$$

Proof. Let $N \in [M]$. We set $P_1 = \mathcal{F} \cap [N]^{<\omega}$ and $P_2 = [N]^{<\omega} \setminus P_1$. According to Theorem 2.2 there exists $L \in [N]$ such that either $\mathcal{L} \cap [L]^{<\omega} \subseteq \mathcal{F}$, as required, or $\mathcal{L} \cap [L]^{<\omega} \subseteq [N]^{<\omega} \setminus \mathcal{F}$, which is impossible from our hypothesis.

As we observed in Example 1.12 a uniform family $\mathcal{L}$ is not necessary Sperner. Using Theorem 2.1 it is easy to prove that for every uniform family $\mathcal{L}$ there exists $L \in [\mathbb{N}]$ such that $\mathcal{L} \cap [L]^{<\omega}$ is Sperner.

Corollary 2.5 Let ξ a countable ordinal with $1 \leq \xi$, $M \in [\mathbb{N}]$ and $\mathcal{L}$ a ξ–uniform family on M. Then there exists $L \in [M]$ such that $\mathcal{L} \cap [L]^{<\omega}$ is Sperner.

Proof. Let $\mathcal{F} = \{s \in \mathcal{L} : \text{there is no } t \in \mathcal{L} \text{ such that } t \subsetneq s\}$. It is easy to see that $\mathcal{L} \cap \mathcal{F} \cap [L]^{<\omega} \neq \emptyset$ for every $L \in [M]$. Hence according to the previous corollary there exists $L \in [M]$ such that $\mathcal{L} \cap [L]^{<\omega} \subseteq \mathcal{F}$. This gives that $\mathcal{L} \cap [L]^{<\omega}$ is Sperner.

In the following we will prove that every finite subset of $\mathbb{N}$ has a "canonical representation" with respect to a ξ–uniform family on $\mathbb{N}$, for every $1 \leq \xi < \omega_1$. Using this fact we will prove a dichotomy result (Theorem 2.11 below) for hereditary families which is stronger than Theorem 2.1.

Definition 2.6 Let $\mathcal{L}$ a family of finite subsets of $\mathbb{N}$ and A a non–empty finite subset of $\mathbb{N}$. We will say that A has **canonical representation** $R_{\mathcal{L}}(A) = (s_1, \ldots, s_n, s_{n+1})$, with **type** $t_{\mathcal{L}}(A) = n$, **with respect to** $\mathcal{L}$, if there exist unique $n \in \mathbb{N}$, $s_1, \ldots, s_n \in \mathcal{L}$ and s_{n+1} a proper initial segment of some element of $\mathcal{L}$, such that $A = \bigcup_{i=1}^{n+1} s_i$ and $s_1 < \ldots < s_n < s_{n+1}$.

Proposition 2.7 Let M be an infinite subset of $\mathbb{N}$, ξ a countable ordinal and $\mathcal{L}$ a ξ–uniform family on M. Every non–empty finite subset of M has canonical representation with respect to $\mathcal{L}$.

Proof. We will proceed by induction on ξ. For $\xi = 1$ we have $\mathcal{L} = \{\{m\} : m \in M\}$; if $A = \{m_1, \ldots, m_n\} \in [M]^{<\omega}$, with $m_1 < \ldots < m_n$, then $R_{\mathcal{L}}(A) = (\{m_1\}, \ldots, \{m_n\})$, $t_{\mathcal{L}}(A) = n$.

Assume that $1 < \xi$ and the assertion holds for every $\zeta < \xi$; and let $\mathcal{L}$ be a ξ–uniform family on M. Then there exists a sequence $(\xi_m)_{m \in M}$ of ordinal numbers smaller than ξ such that $\mathcal{L}(m)$ is a ξ_m–uniform family on $M \cap (m, +\infty)$ for every $m \in M$.

Firstly, we will prove that for every $A \in [M]^{<\omega}$, $A \neq \emptyset$ there exist $n \in \mathbb{N}$ and $s_1, \ldots, s_n, s_{n+1} \in [M]^{<\omega}$ such that $A = \bigcup_{i=1}^{n+1} s_i$, $s_1 < \ldots < s_n < s_{n+1}$, $s_i \in \mathcal{L}$ for every $1 \leq i \leq n$ and $s_{n+1} \prec s_0$ for some $s_0 \in \mathcal{L}$ (i.e. s_{n+1} as a proper initial segment of s_0).

Let $A \in [M]^{<\omega}$ with $A \neq \emptyset$. If $A \in \mathcal{L}^* \setminus \mathcal{L}$ then set $n = 0$ and $s_1 = A$. If $A \in \mathcal{L}$ then set $n = 1$ and $s_1 = A$, $s_2 = \emptyset$. So assume that $A \notin \mathcal{L}^*$; then $A = \{m_1\} \cup t^1$ with $t^1 \neq \emptyset$ and $\{m_1\} < t^1$. Since $t^1 \in \mathcal{L}(m_1)$, according to the induction hypothesis, t^1 has canonical representation $R_{\mathcal{L}(m_1)}(t^1) = (t^1_1, \ldots, t^1_{n_1+1})$ with type $t_{\mathcal{L}(m_1)}(t^1) = n_1$ with respect to $\mathcal{L}(m_1)$. In this case $n_1 \geq 1$. Indeed, if $n_1 = 0$, then $A \in \mathcal{L}^*$, contrary to our assumption. Set $s_1 = \{m_1\} \cup t^1_1$. Obviously, $s_1 \in \mathcal{L}$, $s_1 \prec A$ and $s_1 \neq A$.

We continue analogously setting $A_1 = A \setminus s_1$ and treating A_1 in place of A in order to define s_2. In detail the argument goes as follows: if $A_1 \in \mathcal{L}^* \setminus \mathcal{L}$ then set $n = 1$ and $s_2 = A_1$. If $A_1 \in \mathcal{L}$ then set $n = 2$ and $s_2 = A_1$, $s_3 = \emptyset$. Assume that $A_1 \notin \mathcal{L}^*$; then $A_1 = \{m_2\} \cup t^2$ with $t^2 \neq \emptyset$ and $t^2 \in \mathcal{L}(m_2)$. If $R_{\mathcal{L}(m_2)}(t_2) = (t^2_1, \ldots, t^2_{n_2}, t^2_{n_2+1})$ with $t_{\mathcal{L}(m_2)}(t_2) = n_2$, then $n_2 \geq 1$. So set $s_2 = \{m_2\} \cup t^2_1$ and obviously in this case $s_2 \in \mathcal{L}$, $s_1 \cup s_2 \prec A$ and $s_1 \cup s_2 \neq A$. Set $A_3 = A \setminus s_1 \cup s_2$, and continue in the same way.

Secondly, we prove that for every $A \in [M]^{<\omega}$, $A \neq \emptyset$ the choice of such $n \in \mathbb{N}$ and sets $s_1, \ldots, s_n, s_{n+1}$ is unique; so that in fact $t_{\mathcal{L}}(A) = n$ and $R_{\mathcal{L}}(A) = (s_1, \ldots, s_n, s_{n+1})$. Indeed, let

$A \in [M]^{<\omega}$, $A \neq \emptyset$ and $t_1, \ldots, t_m, t_{m+1} \in [M]^{<\omega}$ such that $A = \bigcup_{i=1}^{m+1} t_i$, $t_1 < \ldots < t_m < t_{m+1}$, $t_i \in \mathcal{L}$ for every $1 \leq i \leq m$ and $t_{m+1} \prec t_0$ for some $t_0 \in \mathcal{L}$. We will prove, by induction on m, that $m = n$ and $t_i = s_i$ for every $1 \leq i \leq n+1$.

Let $m = 0$. Then $A = t_1$ and there exists $t_0 \in \mathcal{L}$ such that $t_1 \prec t_0$ and $t_0 \neq t_1$. We claim that $n = 0$ and consequently $s_1 = A = t_1$. Indeed, if $n \geq 1$ then we have $s_1 \prec t_0$, $s_1 \neq t_0$ and $s_1, t_0 \in \mathcal{L}$, which is impossible since $\mathcal{L}$ is thin.

If $m = k+1$ and the assertion holds for $m = k$, then, since $m \geq 1$, we have, as in the case $m = 0$, that $n \geq 1$. Hence, since $t_1 \prec A$, $s_1 \prec A$, $t_1, s_1 \in \mathcal{L}$ and $\mathcal{L}$ is thin, we have that $t_1 = s_1$. Set $A_1 = A \setminus t_1$; then according to the induction hypothesis $m = n$ and $t_i = s_i$ for every $1 \leq i \leq m+1$.

Corollary 2.8 If $M \in [\mathbb{N}]$, $\mathcal{L}$ is a uniform family on M, $M_1 \in [M]$, A is a finite subset of M_1 and $\mathcal{L}_1 = \mathcal{L} \cap [M_1]^{<\omega}$, then $t_{\mathcal{L}}(A) = t_{\mathcal{L}_1}(A)$ and $R_{\mathcal{L}}(A) = R_{\mathcal{L}_1}(A)$.

Proof. This holds since the canonical representation of A with respect to $\mathcal{L}$ is unique.

The principal use of the canonical representation of a finite set of $\mathbb{N}$ in Ramsey theory is contained in the following important Corollary 2.9.

Corollary 2.9 Let $M \in [\mathbb{N}]$ and $\mathcal{L}$ a uniform family on M. For every finite, non empty subset A of M exact one of the following possibilities occurs:
either (i) there exists $s \in \mathcal{L}$ such that $A \prec s$ and $A \neq s$;
or (ii) there exists $s \in \mathcal{L}$ such that $s \prec A$.

Proof. If $A \in [M]^{<\omega}$, $A \neq \emptyset$, then according to Proposition 2.7, either $t_{\mathcal{L}}(A) = 0$ (which equivalently gives (i)) or $t_{\mathcal{L}}(A) \geq 1$ (which equivalently gives (ii)).

Corollary 2.10 Let $M \in [\mathbb{N}]$ and $\mathcal{L}$ a uniform family on M. If s is a proper initial segment of some element of $\mathcal{L}$, then for every $m \in M$ with $s < \{m\}$, the set $s \cup \{m\}$ is an initial segment of some element of $\mathcal{L}$.

Proof. For every $m \in M$, obviously $\{m\} \in \mathcal{L}^*$. Let $s \in \mathcal{L}^* \setminus \mathcal{L}$ with $s \neq \emptyset$ and $m \in M$ with $s < \{m\}$. Set $A = s \cup \{m\}$. According to Corollary 2.9, either there exists $s_1 \in \mathcal{L}$ such that $A \prec s_1$ and $A \neq s_1$ or there exists $s_2 \in \mathcal{L}$ such that $s_2 \prec A$. In the second case, we have $s_2 = A \in \mathcal{L}$, since $\mathcal{L}$ is thin. Hence, in both cases $A \in \mathcal{L}^*$.

According to Corollary 2.5 for every uniform family $\mathcal{L}$ on M ($M \in [\mathbb{N}]$) there exists $L \in [M]$ such that $\mathcal{L} \cap [L]^{<\omega}$ is a Sperner uniform family on L. For Sperner uniform families, we have in fact the following equalities.

Corollary 2.11 Let $M \in [\mathbb{N}]$ and $\mathcal{L}$ a Sperner uniform family on M. Then

(i) $\mathcal{L}^* = \mathcal{L}_*$;

(ii) $\mathcal{L}^* \cap [L]^{<\omega} = (\mathcal{L} \cap [L]^{<\omega})^*$, and $\mathcal{L}_* \cap [L]^{<\omega} = (\mathcal{L} \cap [L]^{<\omega})_*$ for every $L \in [M]$.

Proof.

(i) Obviously, $\mathcal{L}^* \subseteq \mathcal{L}_*$. Let $A \in \mathcal{L}_* \setminus \mathcal{L}$. Then, there exists $s_0 \in \mathcal{L}$ such that $A \subseteq s_0$ and $A \neq s_0$. According to Corollary 2.9, either there exists $s \in \mathcal{L}$ such that $A \prec s$ and $A \neq s$ so that $A \in \mathcal{L}^* \setminus \mathcal{L}$ ensues or there exists $s \in \mathcal{L}$ such that $s \prec A$, an imposibility, since $s \subseteq s_0$ and $\mathcal{L}$ is Sperner.

(ii) Obviously, $(\mathcal{L} \cap [L]^{<\omega})^* \subseteq \mathcal{L}^* \cap [L]^{<\omega}$, and according to (i) $(\mathcal{L} \cap [L]^{<\omega})_* = (\mathcal{L} \cap [L]^{<\omega})^*$, for every $L \in [M]$. Let $L \in [M]$ and $A \in \mathcal{L}^* \cap [L]^{<\omega} \setminus \mathcal{L}$. According to Corollary 2.10 there exists $s_0 \in \mathcal{L}$ such that $A \prec s_0$ and $s_0 \subseteq L$. Hence, $A \in (\mathcal{L} \cap [L]^{<\omega})^*$. This establishes the required equalities.

Using Corollary 2.9 (to the canonical representation of finite subsets of $\mathbb{N}$) and the general Ramsey theorem (Theorem 2.2) we now prove a stronger dichotomy result for hereditary families.

Theorem 2.12 *Let $M \in [\mathbb{N}]$, $\mathcal{F}$ a hereditary family of finite subsets of M and $\mathcal{L}$ a uniform family on M. Then for every $M_1 \in [M]$ there exists $L \in [M_1]$ such that either $\mathcal{L}_* \cap [L]^{<\omega} \subseteq \mathcal{F}$, or $\mathcal{F} \cap [L]^{<\omega} \subseteq \mathcal{L}^* \setminus \mathcal{L}$.*

Proof. According to Corollary 2.5, there exists $N \in [M_1]$ such that $\mathcal{L} \cap [N]^{<\omega}$ is Sperner. Hence, using Corollary 2.3 of the general Ramsey theorem (Theorem 2.2) we can find $L \in [N]$ such that

$$\text{either} \quad \mathcal{L} \cap [L]^{<\omega} \subseteq \mathcal{F}, \quad \text{or} \quad \mathcal{L} \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}.$$

If $\mathcal{L} \cap [L]^{<\omega} \subseteq \mathcal{F}$ then $(\mathcal{L} \cap [L]^{<\omega})_* \subseteq \mathcal{F}$, since $\mathcal{F}$ is hereditary. Hence, $\mathcal{L}_* \cap [L]^{<\omega} \subseteq \mathcal{F}$, according to Corollary 2.11.

If $\mathcal{L} \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$, we will prove that $\mathcal{F} \cap [L]^{<\omega} \subseteq \mathcal{L}^* \setminus \mathcal{L}$. Indeed, let $A \in \mathcal{F} \cap [L]^{<\omega}$. Since $\mathcal{L} \cap [L]^{<\omega}$ is uniform on L, according to Corollary 2.9, either there exists $s \in \mathcal{L} \cap [L]^{<\omega}$ such that $A \prec s$ and $A \neq s$, which gives that $A \in \mathcal{L}^* \setminus \mathcal{L}$, as required, or there exists $s \in \mathcal{L} \cap [L]^{<\omega}$ such that $s \prec A$. But this case is impossible, since then $s \in \mathcal{F} \cap \mathcal{L} \cap [L]^{<\omega}$. This completes the proof.

Corollary 2.13 Let $M \in [\mathbb{N}]$, $\mathcal{L}_1$ a ξ_1–uniform on M and $\mathcal{L}_2$ a ξ_2–uniform on M with $\xi_1 < \xi_2 < \omega_1$. Then there exists $L \in [M]$ such that

$$\mathcal{L}_1 \cap [L]^{<\omega} \subseteq (\mathcal{L}_2)^* \setminus \mathcal{L}_2.$$

Proof. According to Theorem 2.12 there exists $L \in [M]$ such that,

either $(\mathcal{L}_2)_* \cap [L]^{<\omega} \subseteq (\mathcal{L}_1)_*$, or $(\mathcal{L}_1)_* \cap [L]^{<\omega} \subseteq (\mathcal{L}_2)_* \setminus \mathcal{L}_2$.

The first alternative, is impossible; in fact, if (i) holds, then $s_L((\mathcal{L}_2)_* \cap [L]^{<\omega}) \leq s_L((\mathcal{L}_1)_* \cap [L]^{<\omega})$. On the other hand, using Remark 1.11 (iii) and Theorem 1.18, we have

$$s_L((\mathcal{L}_2)_* \cap [L]^{<\omega}) = s_L((\mathcal{L}_2)_*) = \xi_2 + 1$$

and

$$s_L((\mathcal{L}_1)_* \cap [L]^{<\omega}) = s_L((\mathcal{L}_1)_*) = \xi_1 + 1 < \xi_2 + 1.$$

A contradiction. Thus (ii) holds, as required.

In the following theorem (Theorem 2.15) we will describe, with the help of the strong Cantor Bendixson index, sufficient conditions in order a family of finite subsets of $\mathbb{N}$ to satisfy exactly one of the conditions given in the dichotomy of Theorem 2.12.

Since we will restrict to the hereditary and closed families firstly we will give a characterization of them.

Proposition 2.14 Let $\mathcal{F}$ be a non empty, hereditary family of finite subsets of $\mathbb{N}$. The following are equivalent:

(i) $\mathcal{F}$ is closed.

(ii) There does not exist an infinite sequence $(s_i)_{i=1}^{\infty}$ of elements of $\mathcal{F}$ with $s_1 \prec s_2 \prec \ldots$.

(iii) There does not exist $M \in [\mathbb{N}]$ such that $[M]^{<\omega} \subseteq \mathcal{F}$.

Proof. (i) $\Rightarrow$ (ii) If $s_i = (n_1, \ldots, n_{k_i}) \in \mathcal{F}$ with $n_1 < \ldots < n_{k_i}$, for every $i \in \mathbb{N}$ and $(k_i)_{i=1}^{\infty}$ is an increasing sequence of natural numbers, then $(s_i)_{i=1}^{\infty}$ converges pointwise to the infinite subset $s = (n_1, n_2, \ldots)$ of $\mathbb{N}$ which does not belong to $\mathcal{F}$.

(ii) $\Rightarrow$ (i) Let $(t_n)_{n=1}^{\infty}$ a sequence of elements of $\mathcal{F}$, converging pointwise to some subset t of $\mathbb{N}$. If t is finite, then there exists $n_0 \in \mathbb{N}$ such that $t \prec t_{n_0}$, hence $t \in \mathcal{F}$, as required.

Let t is infinite. Set $t = (n_1, n_2, \ldots)$ with $n_1 < n_2 < \ldots$ and $s_i = (n_1, \ldots, n_i)$ for every $i \in \mathbb{N}$. For every $i \in \mathbb{N}$ the sequence $(t_n \cap [0, n_i])_{n=1}^{\infty}$ converges pointwise to s_i. According to the previous case, we have $s_i \in \mathcal{F}$ for every $i \in \mathbb{N}$. A contradiction to the condition (ii).

(iii) $\Rightarrow$ (ii) Let $(s_i)_{i=1}^{\infty} \subseteq \mathcal{F}$ with $s_1 \prec s_2 \prec \ldots$. Set $M = \bigcup_{i=1}^{\infty} s_i$. If t is an arbitrary subset of M then $t \subseteq s_i$ for some $i \in \mathbb{N}$, hence $t \in \mathcal{F}$. So $[M]^{<\omega} \subseteq \mathcal{F}$, a contradiction.

(ii) $\Rightarrow$ (iii) Let $M = (m_1, m_2, \ldots) \subseteq \mathbb{N}$ with $m_1 < m_2 < \ldots$. If $[M]^{<\omega} \subseteq \mathcal{F}$ then $s_i = (m_1, \ldots, m_i) \in \mathcal{F}$ for every $i \in \mathbb{N}$. Hence, the condition (ii) does not hold.

After the previous proposition we can give a dichotomy result rather closed to the infinite Ramsey theorem (c.f. Nash–Williams [N–W], Galvin–Prikry [G–P], Silver [S]), and in many (especially Banach space-) applications it can be used in its place.

Corollary 2.15 Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}$. For every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that either $[L]^{<\omega} \subseteq \mathcal{F}$ or $[L]^{<\omega} \subseteq ([\mathbb{N}]^{<\omega} \setminus \mathcal{F})_*$.

Proof. According to Proposition 2.14, if $\mathcal{F} \cap [M]^{<\omega}$ is not closed then there exists $L \in [M]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$ and if $\mathcal{F} \cap [M]^{<\omega}$ is closed then, there is $L \in [M]$ such that $[L]^{<\omega} \subseteq ([\mathbb{N}]^{<\omega} \setminus \mathcal{F})_*$.

Theorem 2.16 *Let $\mathcal{F}$ be a pointwise closed and hereditary family of finite subsets of $\mathbb{N}$, $M \in [\mathbb{N}]$, ξ a countable ordinal and $\mathcal{L}$ a ξ–uniform family on M.*

(i) If $\xi + 1 < s_M(\mathcal{F})$, then there exists $L \in [M]$ such that

$$\mathcal{L}_* \cap [L]^{<\omega} \subseteq \mathcal{F}; \text{ and,}$$

(ii) If $s_M(\mathcal{F}) < \xi + 1$, then there exists $L \in [M]$ such that

$$\mathcal{F} \cap [L]^{<\omega} \subseteq \mathcal{L}^* \setminus \mathcal{L}.$$

Proof. Using Theorem 2.12 for the family $\mathcal{F} \cap [M]^{<\omega}$, (at least) one of the following possibilities occurs: either there exists $L \in [M]$ such that $\mathcal{L}_* \cap [L]^{<\omega} \subseteq \mathcal{F}$ or there exists $L \in [M]$ such that $\mathcal{F} \cap [L]^{<\omega} \subseteq \mathcal{L}^* \setminus \mathcal{L}$.

(i): If $\xi + 1 < s_M(\mathcal{F})$, then the second case cannot occur, since then, we would have $\mathcal{F} \cap [L]^{<\omega} \subseteq \mathcal{L}^* \cap [L]^{<\omega}$, and consequently, according to Theorem 1.18,

$$\xi + 1 = s_L(\mathcal{L}^*) = s_L(\mathcal{L}^* \cap [L]^{<\omega}) \geq s_L(\mathcal{F} \cap [L]^{<\omega}) = s_L(\mathcal{F}),$$

a contradiction.

(ii): If $s_M(\mathcal{F}) < \xi + 1$ then the first case can not occur, since then

$$\xi + 1 = s_L(\mathcal{L}^*) \leq s_L(\mathcal{F}),$$

a contradiction to our hypothesis.

Remarks 2.17 It should be noted that in the limiting case $s_M(\mathcal{F}) = \xi + 1$ of Theorem 2.16 both alternatives may materialize. Indeed, we have the following two simple examples:

Example 1. Let

$$\mathcal{L} = \{s \in [\mathbb{N}]^{<\omega} : |s| = 2 \min s + 1\} \text{ and}$$
$$\mathcal{R} = \{s \in [\mathbb{N}]^{<\omega} : |s| = \min s\},$$

where $|s|$ denotes the cardinality or s.

It is easy to see that $\mathcal{L}$ and $\mathcal{R}$ are ω–uniform on $\mathbb{N}$.

The family $\mathcal{F}_1 = \mathcal{R}_*$ is hereditary, closed (Lemma 1.17) and $s_{\mathbb{N}}(\mathcal{F}) = \omega + 1$, according to Theorem 1.18. Since $\mathcal{L} \cap \mathcal{F}_1 = \emptyset$ and $\mathcal{L} \cap [L]^{<\omega} \neq \emptyset$ for every $L \in [\mathbb{N}]$ the first alternative of Theorem 2.16 does not occur. Hence there exists $L \in [\mathbb{N}]$ such that

$$\mathcal{F}_1 \cap [L]^{<\omega} \subseteq \mathcal{L}^* \setminus \mathcal{L}.$$

Example 2. On the other hand (refering the notation of Example 1) for the hereditary and closed family $\mathcal{F}_2 = \mathcal{L}_*$ with $s_{\mathbb{N}}(\mathcal{F}_2) = \omega + 1$ (Theorem 1.18) and the ω–uniform family $\mathcal{R}$ on $\mathbb{N}$ we have that

$$\mathcal{R}^* \subseteq \mathcal{F}_2$$

hence the first alternative of Theorem 2.16 occurs and the second does not occur since for every $L \in [\mathbb{N}]$ there exists $s \in \mathcal{F}_2 \cap [L]^{<\omega}$ such that $s \notin \mathcal{R}^*$ (take $s \in [L]^{<\omega}$ with $\min s + 1 \leq |s| \leq 2\min s + 1$).

Recapitulation of the main results

Let $\mathcal{F}$ a hereditary family of finite subsets of $\mathbb{N}$. We have the following two cases:

1st case. The family $\mathcal{F}$ is not closed. Then according to Proposition 2.14 there exists $L \in [\mathbb{N}]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$.

2nd case. The family $\mathcal{F}$ is closed. Then there exists $L \in [\mathbb{N}]$ such that $[L]^{<\omega} \subseteq ([\mathbb{N}]^{<\omega} \setminus \mathcal{F})_*$, (Corollary 2.15). Moreover, for a given infinite subset M of $\mathbb{N}$ and a system of uniform families $(\mathcal{A}_\zeta)_{\zeta<\omega_1}$ on M, setting

$$\xi = \sup\{s_L(\mathcal{F}) : L \in [M]\}$$

the following obtain:

(i) For every ordinal ζ with $\zeta + 1 < \xi$ there exists $L \in [M]$ such that:

$$(\mathcal{A}_\zeta)_* \cap [L]^{<\omega} \subseteq \mathcal{F};$$

(Theorem 2.16).

(ii) For every ordinal ζ with $\xi < \zeta + 1$ and for every $M_1 \in [M]$ there exists $L \in [M_1]$ such that:

$$\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\zeta)^* \setminus \mathcal{A}_\zeta$$

which gives that

$$\mathcal{A}_\zeta \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}$$

(Theorem 2.16).

(iii) If $\xi = \zeta + 1$ then there exists $M_1 \in [M]$ such that $s_{M_1}(\mathcal{F}) = \xi$. According to Theorem 2.12 there exists $L \in [M_1]$ such that

either $(\mathcal{A}_\zeta)_* \cap [L]^{<\omega} \subseteq \mathcal{F}$, or $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\zeta)^* \setminus \mathcal{A}_\zeta$.

According to Remark 2.17 both alternatives may materialize.

3 Some remarks and applications

The results of the previous section constitute a far reaching and powerful generalization of the classical Ramsey theorem, a generalization that is stated in terms of a countable ordinal index ξ (in place of a natural number as in the classical case); these ordinal index dichotomies are in turn analogous to the Galvin–Prikry ([G–P]) infinitary form of the Ramsey dichotomy (stated for all infinite subsets of $\mathbb{N}$, partitioned by an analytic partition). Our results, then, on the one hand generalize the classical Ramsey theorem and on the other hand they have the Galvin–Prikry infinitary theorem as the limiting (ω_1-) case.

It is to be expected that such general combinatorial principles will have wide applications in many instances, where either classical Ramsey theory, or the infinitary Galvin–Prikry theorem has been successfully applied. Some applications of the dichotomy results established in this paper, and their relation to existing applications of similar combinatorial techniques involving mostly generalized Schreier families $F_\xi, \xi < \omega_1$ (as in [A–M–T], [F1], [F2], [M–N]), will appear in a separate publication.

Here we will limit ourselves to exhibit the way in which our techniques can be applied to provide simple derivations of the combinatorial basis in the theory of Banach spaces of two recent results, one by Argyros–Mercourakis–Tsarparlias and the other by Judd.

Thus in Proposition 3.1 we indicate the close connection that exists between the Schreier family F_ξ and an ω^ξ–uniform family, particularly the family $\mathcal{B}_\xi$ (Definition 1.5). Then using the results of Section 2 we reprove a dichotomy result of Judd [J], obtaining in fact a more general expression; and additionally a combinatorial result of Argyros–Mercourakis–Tsarpalias [A–M–T] which was the basis for establishing a general form of an ℓ^1–dichotomy, initially proved in a special form by Rosenthal [RO].

Proposition 3.1 Let ξ be a countable ordinal, M an infinite subset of $\mathbb{N}$ and $\mathcal{L}$ an ω^ξ–uniform family on M. Then there exists $L \in [M]$ such that $\mathcal{F}_\xi(L) \subseteq \mathcal{L}^*$.

Proof. Let $\mathcal{L}^1 = \{\{m\} \cup s : m \in M, s \in \mathcal{L} \text{ and } \{m\} < s\}$. According to Remark 1.2 (iii) $\mathcal{L}^1$ is $\omega^\xi + 1$–uniform on M. Using Theorem 2.12, either there exists $N \in [M]$ such that $(\mathcal{L}^1)_* \cap [N]^{<\omega} \subseteq \mathcal{F}_\xi$, which is impossible, since $s_N[(\mathcal{L}^1)_*] = \omega^\xi + 2$ (according to Theorem 1.18) and $s_N(\mathcal{F}_\xi) = \omega^\xi + 1$ (according to [A–M–T]); or there exists $N \in [M]$ such that $\mathcal{F}_\xi \cap [N]^{<\omega} \subseteq (\mathcal{L}^1)^* \setminus \mathcal{L}^1$. In this case, set $L = (n_i)_{i=3}^\infty$ if $N = (n_i)_{i=1}^\infty$. Then it is easy to see that $\mathcal{F}_\xi(L) \subseteq \mathcal{L}^*$, using the fact that if $(k_1, \ldots, k_p) \in \mathcal{F}_\xi$ then $(k_1 + 1, k_1 + 2, k_2 + 2, \ldots, k_p + 2) \in \mathcal{F}_\xi$ for every $\xi < \omega_1$.

Corollary 3.2 For every $\xi < \omega_1$, $M \in [\mathbb{N}]$, there exists $L \in [M]$ such that

$$\mathcal{F}_\xi(L) \subseteq (\mathcal{B}_\xi)^* \subseteq \mathcal{F}_\xi$$

Proof. It is immediate after Theorem 1.6 and Proposition 3.1

After Proposition 3.1 we will give a Corollary of the general Ramsey Theorem 2.2 which can be used for the families $\mathcal{F}_\xi, \xi < \omega_1$.

Proposition 3.3 Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}$, M an infinite subset of $\mathbb{N}$ and ξ a countable ordinal number. If $\mathcal{F} \cap \mathcal{B}_\xi \cap [L]^{<\omega} \neq \emptyset$ for every $L \in [M]$ then there exists $L \in [M]$ such that $\mathcal{F}_\xi(L) \subseteq \mathcal{F}$.

Proof. According to Corollary 2.4 and since $\mathcal{B}_\xi \cap [M]^{<\omega}$ is ω^ξ–uniform family on M (Corollary 1.8) there exists $N \in [M]$ such that $\mathcal{B}_\xi \cap [N]^{<\omega} \subseteq \mathcal{F}$. From the previous proposition there exist $L \in [N]$ such that $\mathcal{F}_\xi(L) \subseteq (\mathcal{B}_\xi \cap [N]^{<\omega})_*$. Since $\mathcal{F}$ is hereditary and $\mathcal{B}_\xi \cap [N]^{<\omega} \subseteq \mathcal{F}$ we have that $(\mathcal{B}_\xi \cap [N]^{<\omega})_* \subseteq \mathcal{F}$. Hence, $\mathcal{F}_\xi(L) \subseteq \mathcal{F}$, as required.

R. Judd in [J] had provided, using Schreier games, that for every hereditary family $\mathcal{F}$ of finite subsets of $\mathbb{N}$, $\xi < \omega_1$ and $M \in [\mathbb{N}]$, either there exists $L \in [M]$ such that $\mathcal{F}_\xi(L) \subseteq \mathcal{F}$ or there exists $L \in [M]$ and $N \in [\mathbb{N}]$ such that $\mathcal{F} \cap [N]^{<\omega}(L) \subseteq \mathcal{F}_\xi$.

We will prove a stronger version of this result using our results of Section 2.

Theorem 3.4 *For every hereditary family $\mathcal{F}$ of finite subsets of $\mathbb{N}$, every countable ordinal ξ and $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that either $\mathcal{F}_\xi(L) \subseteq \mathcal{F}$ or $\mathcal{F} \cap [L]^{<\omega} \subseteq \mathcal{F}_\xi$.*

Proof. According to Theorem 2.12 there exist $N \in [M]$ such that:

either $(\mathcal{B}_\xi)_* \cap [N]^{<\omega} \subseteq \mathcal{F}$, or $\mathcal{F} \cap [N]^{<\omega} \subseteq (\mathcal{B}_\xi)^*$.

Using Proposition 3.1 there exists $L \in [N]$ such that

$$\mathcal{F}_\xi(L) \subseteq (\mathcal{B}_\xi)^* \cap [L]^{<\omega} \subseteq (\mathcal{B}_\xi)_* \cap [N]^{<\omega}.$$

Hence, there exists $L \in [M]$ such that either $\mathcal{F}_\xi(L) \subseteq \mathcal{F}$, or $\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{B}_\xi)^* \subseteq \mathcal{F}_\xi$.

As a corollary of Theorem 3.3 we have the following result of Argyros, Mercourakis and Tsarpalias in [A–M–T]. An analogous proof for this result was given in [J].

Theorem 3.5 *Let $\mathcal{F}$ be a hereditary and closed family of finite subsets of $\mathbb{N}$. If there exists $M \in [\mathbb{N}]$ such that $s_M(\mathcal{F}) \geq \omega^\xi$, then there exists $L \in [M]$ such that $\mathcal{F}_\xi(L) \subseteq \mathcal{F}$.*

Proof. If $s_M[\mathcal{F}] > \omega^\xi + 1$, then according to Theorem 2.16 (i), there exists $N \in [M]$ such that $(\mathcal{B}_\xi)_* \cap [N]^{<\omega} \subseteq \mathcal{F}$. Also, according to Proposition 3.1 there exists $L \in [N]$ such that

$$\mathcal{F}_\xi(L) \subseteq (\mathcal{B}_\xi)^* \cap [L]^{<\omega}$$

Hence, $\mathcal{F}_\xi(L) \subseteq \mathcal{F}$.

Now, if $s_M[\mathcal{F}] = \omega^\xi + 1$ then set $\overline{\mathcal{F}} = \{\{m\} \cup s : s \in \mathcal{F}, m \in M \text{ and } \{m\} < s\}$. It is easy to see that $s_M[\overline{\mathcal{F}}] > \omega^\xi + 1$. If we apply the previous case to $\overline{\mathcal{F}}$ we can find $(n_i)_{i=1}^\infty = N \in [M]$ such that $\mathcal{F}_\xi(N) \subseteq \overline{\mathcal{F}}$ and setting $L = (n_i)_{i=3}^\infty$ we have that $\mathcal{F}_\xi(L) \subseteq \mathcal{F}$ as required.

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