On a classification of functions

(preliminary version)

by

V. Farmaki
Department of Mathematics
University of Athens
Panepistemiopolis, 15784
15784, Athens,
GREECE

A. Louveau
Equipe d'Analyse
Universite Paris VI
4, Place Jussieu
75252-Paris Cedex 05,
FRANCE

Introduction and preliminaries. Let K be a compact, metrizable space. The class of continuous functions on K is denoted by C(K) and the class of Baire-1 functions on K (i.e. the pointwise limits of uniformly bounded sequences of continuous functions on K) by $B_1(K)$. DBSC(K) denotes the subclass of $B_1(K)$ consisting of differences of bounded semi-continuous functions. It is easy to see that

$DBSC(K)=\{f\in B_1(K)$:there exists $(f_n)\subseteq C(K)$ such that $f=\sum f_n$ and $\sum|f_n|$ is bounded$\}$

The class DBSC(K) is a Banach space with respect to the norm

$$\|f\|_D = \inf\{ \| \sum|f_n| \|_\infty : (f_n)\subseteq C(K) \text{ and } \sum f_n=f \}.$$

It is not hard to check that $\|f\|_\infty\leq\|f\|_D$, but the two norms are not equivalent in general. The norm-closure of DBSC(K) is denoted by $B_{1/2}(K)$ in [H-O-R]. In the same paper the authors defined the subclass $B_{1/4}(K)$, by

$B_{1/4}(K)=\{f\in B_1(K)$:there exists $(f_n)\subseteq DBSC(K)$ such that $\|f_n-f\|_\infty\to 0$ and $\sup_n\|f_n\|_D<\infty\}$

The space $B_{1/4}(K)$ is complete with respect to the norm

$$\|f\|_{1/4} = \inf\{ \sup_n \|f_n\|_D : (f_n)\subseteq DBSC(K) \text{ and } \|f_n - f\|_\infty \to 0 \}.$$

In [F] this definition was extended in the transfinite as follows:

$$V_1(K)=B_{1/4}(K) \text{ and } \|\ \|_1 = \|\ \|_{1/4}.$$

If the normed space $(V_\xi(K), \|\ \|_\xi)$ has been defined, then

$$V_{\xi+1}(K)=\{ f\in B_1(K): \text{there exists } (f_n)\subseteq DBSC(K) \text{ with } \|f_n - f\|_\xi \to 0 \text{ and } \sup_n \|f_n\|_D<\infty \}$$

and

$$\|f\|_{\xi+1}=\inf\{ \sup_n \|f_n\|_D : (f_n)\subseteq DBSC(K) \text{ and } \|f_n - f\|_\xi \to 0 \}.$$

For a limit ordinal ξ

$$\|f\|_\xi=\sup\{ \|f\|_\beta : 1\leq\beta<\xi \} \text{ for every } f\in \cap_{\beta<\xi} V_\beta(K) \quad \text{and}$$

$$V_\xi(K)=\{ f\in B_1(K): \|f\|_\xi<\infty \}.$$

The spaces $(V_\xi(K), \|\ \|_\xi)$, $1\leq\xi<\omega_1$, are complete and their intersection coincides with DBSC(K) ([F]). It is easy to see, that $V_\xi(K)\subseteq V_\beta(K)$ and $\|f\|_\infty\leq\|f\|_\beta\leq\|f\|_\xi$ for every $f\in V_\xi(K)$ and $\beta<\xi<\omega_1$. Hence we have a classification of $B_1(K)$ into a transfinite, decreasing hierarchy of Banach spaces. According to [H-O-R] and [F], the functions in $B_{1/4}(K)\setminus C(K)$ are characterized in terms of c_0-spreading models and the functions in $V_\xi(K)\setminus C(K)$ have an analogous stronger property ([F]).

In [K-L] the authors define a natural rank on functions not in DBSC(K) which is unbounded below ω_1 on $B_1(K)\setminus DBSC(K)$. We will define this rank.

Let f be a bounded function on K. One defines the upper regularization of f, ur(f) (usually it is denoted by $\hat{f}$), by

$$ur(f)=\inf\{g: g\in C(K) \text{ and } g\geq f\}.$$

The function ur(f) is upper semmi-continuous and one has

$$ur(f)(x)=\inf\{ \sup_V f : V \text{ is a neighbourhood of } x \}.$$

In [K-L] the authors associate with each bounded function f on K an increasing sequence $(f_\xi)_{1\leq\xi<\omega_1}$ of upper semmi-continuous functions as follows:

$$f_1=ur(f).$$

If f_ξ is defined,

$$f_{\xi+1}=ur(\ f+ur(f_\xi-f)\ ).$$

For limit ξ, f_ξ is defined if and only if for all $\beta<\xi$, f_β is defined and $\sup_{\beta<\xi} f_\beta$ is bounded and then

$$f_\xi=ur(\ \sup_{\beta<\xi} f_\beta\ ).$$

According to [K-L], f is in DBSC(K) if and only if f_ξ is defined for all $\xi<\omega_1$ or equivalently if there exists a $\xi<\omega_1$ such that f_ξ is defined and $f_{\xi+1}=f_\xi$. In this case, if ξ is the least ordinal with this property, the pair $(f_\xi,f_\xi-f)$ is the least pair of upper semmi-continuous functions (u,v) with $u\geq f$ and $f=u-v$.

Hence, to every bounded function f on K there corresponds a rank

$$r_{ND}(f)=\inf\{\ 1\leq\xi<\omega_1 : f_\xi \text{ is undefined }\}, \quad \text{if such a } \xi \text{ exists}$$

and $\quad r_{ND}(f)=\omega_1 \quad$ otherwise.

Note that the values of this rank is always limit ordinals. We will prove that the values ω^ξ are the only possible values of $r_{ND}(f)$ if $f\notin DBSC(K)$ and by [K-L], all such ordinals are obtained.

Main Theorem. Let K be a compact, metrizable space. For every $1\leq\xi<\omega_1$ we have:

$$V_\xi(K)=\{\ f:K\to\mathbb{R} : f \text{ is bounded and } r_{ND}(f)>\omega^\xi\ \}.$$

For f in $V_\xi(K)$ with $f\geq0$

$$\|f_{\omega^\xi}\|_\infty \leq \|f\|_\xi \leq 4\|f_{\omega^\xi}\|_\infty,$$

and in general,

$$(1/3)\ \|f_{\omega^\xi}\|_\infty \leq \|f\|_\xi \leq 4\|f_{\omega^\xi}\|_\infty + 5\|f\|_\infty\ .$$

Before the proof of the main Theorem we will give some other connections of the previous notions. In the following, K is a compact, metrizable space.

Proposition 1. Let $f\in DBSC(K)$ and $\bar{f}=f_\xi$, where ξ is the least ordinal such that $f_{\xi+1}=f_\xi$. Then, for $f\geq 0$ we have

$$\|\bar{f}\|_\infty \leq \|f\|_D \leq 4\|\bar{f}\|_\infty$$

and in general,

$$(1/3)\|\bar{f}\|_\infty \leq \|f\|_D \leq 4\|\bar{f}\|_\infty+5\|f\|_\infty.$$

Proof. For a bounded, upper semmi-continuous function g on K with $g\geq 0$ there exists a sequence $(f_n)\subseteq C(K)$ such that $f_n\geq 0$ for every $n\in\mathbb{N}$ and $g=\sup g-\sum f_n$, hence

$$\|g\|_D \leq \sup g + \| \sum f_n\|_\infty \leq 2\sup g = 2\|g\|_\infty.$$

So for $f\geq 0$ one gets

$$\|f\|_D \leq \|\bar{f}-(\bar{f}-f)\|_D \leq \|\bar{f}\|_D + \|\bar{f}-f\|_D \leq 2\|\bar{f}\|_\infty + 2\|\bar{f}-f\|_\infty \leq 4\|\bar{f}\|_\infty,$$

since $\bar{f}-f$ is an upper semmi-continuous function ([K-L]) and $\bar{f}\geq\bar{f}-f\geq 0$.

In general

$$\|f\|_D \leq \|f-\inf f\|_D + |\inf f| \leq 4\|\bar{f}-\inf f\|_\infty + |\inf f| \leq 4\|\bar{f}\|_\infty + 5\|f\|_\infty.$$

In the other direction, if $f=\sum h_n$, where $(h_n)\subseteq C(K)$ and $\sum|h_n|$ is bounded, we set $g_1=\sum h_n^+$ and $g_2=\sum h_n^-$. Then $f=(\sup g_1-g_2)-(\sup g_1-g_1)$, where $\sup g_1-g_1$ is a non-negative, upper semmi-continuous function. Hence $\bar{f} \leq \sup g_1-g_2 \leq \sup g_1$ ([K-L]).

For $f\geq 0$ we have $\bar{f}\geq 0$ and then

$$\|\bar{f}\|_\infty \leq \|g_1\|_\infty \leq \|g_1+g_2\|_\infty = \| \sum|h_n| \|_\infty.$$

As this is true for all such (h_n),

$$\|\bar{f}\|_\infty \leq \|f\|_D.$$

In general

$$\|\bar{f}\|_\infty \leq \|\bar{f}-\inf f\|_\infty + |\inf f| \leq \|f-\inf f\|_D + |\inf f| \leq \|f\|_D + 2\|f\|_\infty \leq 3\|f\|_D.$$

Remark 2. For $f\in DBSC(K)$ it is easy to prove that

$\|f\|_D=\inf\{ \|g_1+g_2\|_\infty : g_1,g_2$ are lower semmi-continuous, $g_1,g_2\geq 0$ and $f=g_1-g_2 \}$

If $\|f\|_U=\inf\{ \|h_1+h_2\|_\infty : h_1,h_2$ are upper semmi-continuous, $h_2\geq 0$ and $f=h_1-h_2 \}$

then $\|f\|_U = \| 2\tilde{f} - f\|_\infty$, $(1/3)\|f\|_U \le \|f\|_D \le 3\|f\|_U$ for $f \ge 0$ and in general $(1/3)\|f\|_U \le \|f\|_D \le 3\|f\|_U + 4\|f\|_\infty$.

In the next Lemma we give some elementary relations which are used in the following.

Lemma 3. Let f,g be bounded functions on K, M a real number and ξ,β ordinal numbers with $1 \le \xi < \omega_1$ and $\beta < \xi$.

(1) If $f \le g$ then $ur(f) \le ur(g)$.

(2) $(f+M)_\xi = f_\xi + M$.

(3) $f_\beta \le f_\xi$, if f_ξ is defined.

(4) $ur(f+g) \le ur(f) + ur(g)$.

(5) $ur(f - ur(g)) = ur(ur(f) - ur(g)) \le ur(f-g)$.

(6) $f_\xi - g_\xi \le (f-g)_\xi$.

The following Lemma is the main point for proving that the values ω^ξ are the only possible values of the index r_{ND}.

Lemma 4. Let f be bounded function on K with $f \ge 0$ and suppose that f_ξ ($1 < \xi < \omega_1$ is a limit ordinal) is defined. Then for all $n \in \omega$ the function $f_{\xi . n}$ is defined, and $\|f_{\xi . n}\|_\infty \le n\|f_\xi\|_\infty$.

Proof. Let $M = \|f_\xi\|_\infty$. Then $M \ge ur(f_\xi - f)$, hence by induction $f_{\xi+\beta}$ is defined and $M + f_\beta \ge f_{\xi+\beta}$ for every $\beta \le \xi$. Finally, $f_{\xi.2}$ is defined and $\|f_{\xi . 2}\|_\infty \le 2\|f_\xi\|_\infty$. The result then follows by induction on n.

Corrolary 5. For a bounded function f on K not in DBSC(K) the $r_{ND}(f)$ is an ordinal of the form ω^ξ and by [K-L], all such ordinals are obtained.

Before the proof of the main Theorem we will give a Lemma.

Lemma 6. Let $(g_n)_{n=0}^{\infty}$ be a sequence of bounded, upper semmi-continuous functions on K with $g_0=0$. If the sequence $(\ ur(g_{n+1}-g_n)\ )_{n=0}^{\infty}$ is decreasing then $ur(g_{n+1}-g_n) \le \frac{1}{n+1} g_{n+1}$ for every $n\in\mathbb{N}$.

Proof. For n=0, it reduces to $\tilde{g}_1 \le g_1$, which is trivial. Suppose we know it for n, and assume it is not true for n+1. Then there exists $x\in K$ with

$$ur(g_{n+2}-g_{n+1})(x) > \frac{1}{n+2} g_{n+2}(x).$$

Pick a real number M such that

$$ur(g_{n+2}-g_{n+1})(x) > \frac{M}{n+2} > \frac{1}{n+2} g_{n+2}(x).$$

Let $V=\{\ y\in K:\ g_{n+2}(y)<M\ \}$ an open neighbourhood of x. By the definition of the upper regularization, there must be $y\in V$ with

$$(g_{n+2}-g_{n+1})(y) > \frac{M}{n+2}\ .$$

Now $ur(g_{n+1}-g_n)(y) \ge ur(g_{n+2}-g_{n+1})(y) \ge (g_{n+2}-g_{n+1})(y) > \frac{M}{n+2}$.

Hence, by the induction hypothesis

$$g_{n+1}(y) > (n+1).\frac{M}{n+2}\ .$$

But then

$$g_{n+2}(y) > \frac{M}{n+2} + (n+1).\frac{M}{n+2} = M.$$

A contradiction, which finishes the proof.

Proof of the main Theorem. First assume that $f\ge 0$. If $f\in V_\xi$ then $r_{ND}(f)>\omega^\xi$ and $\|f_{\omega^\xi}\|_\infty \le \|f\|_\xi$. We will prove this claim by induction on ξ.

For $\xi=1$, one has a sequence (g_n) in DBSC(K) with $\|g_n-f\|_\infty\to 0$ and $\sup_n\|g_n\|_D$ bounded. If $\varepsilon_n=\|g_n-f\|_\infty$, then $-\varepsilon_n\le f-g_n \le\varepsilon_n$ and by induction on k we have

$$f_k \le (g_n)_k+2k\varepsilon_n \quad \text{for every } k,n\in\mathbb{N}.$$

Hence $f_k \le \tilde{g}_n+2k\varepsilon_n$ for every $k,n\in\mathbb{N}$. Since $g_n+\varepsilon_n\ge 0$, we have according to Proposition 1

$$f_k \le \|\tilde{g}_n+\varepsilon_n\|_\infty +(2k-1)\varepsilon_n \le \sup_n\|g_n\|_D +2k\varepsilon_n \quad \text{for every } k,n\in\mathbb{N}.$$

Letting first $n\to\infty$ and then $k\to\infty$ we get

$$f_\omega \le \|f_\omega\|_\infty \le \sup_n\|g_n\|_D.$$

Finally, $$\|f_\omega\|_\infty \le \|f\|_1.$$

Assume the result is known for ξ, and that $f\in V_{\xi+1}$. Let $(g_n)\subseteq DBSC(K)$ be such that $\varepsilon_n=\|f-g_n\|_\xi \to 0$ and $M=\sup_n\|g_n\|_D<\infty$. Since $f-g_n+\varepsilon_n\ge 0$ for every $n\in\mathbb{N}$, by the induction hypothesis

$$\|(f-g_n+\varepsilon_n)_\omega\xi\|_\infty \le \|f-g_n+\varepsilon_n\|_\xi \le 2\varepsilon_n \text{ for every } n\in\mathbb{N}.$$

By Lemma 4 $(f-g_n)_{\omega}\xi_{.k}$ is defined and

$$(f-g_n+\varepsilon_n)_\omega\xi_{.k} \le 2k.\varepsilon_n \text{ for all } n,k\in\mathbb{N}.$$

Hence, according to Lemma 3 and Proposition 1, for all $n,k\in\mathbb{N}$ we have

$$f_\omega\xi_{.k}\le (g_n)_\omega\xi_{.k}+(2k-1)\varepsilon_n\le \|\tilde{g}_n+\varepsilon_n\|_\infty+(2k-2)\varepsilon_n\le \|g_n\|_D+(2k-1)\varepsilon_n \le M+(2k-1)\varepsilon_n.$$

Letting $n\to\infty$, $k\to\infty$ in this order we get that $f_\omega\xi+1$ is defined and

$$\|f_\omega\xi+1\|_\infty \le \sup_n\|g_n\|_D.$$

So finally, $$\|f_\omega\xi+1\|_\infty \le \|f\|_{\xi+1}.$$

For limit ξ one has $f\in\bigcap_{\beta<\xi}V_\beta(K)$ and $\|f\|_\xi=\sup_{\beta<\xi}\|f\|_\beta<\infty$. By induction,

$$f_\omega\beta \le \|f_\omega\beta\|_\infty \le \|f\|_\beta \le \|f\|_\xi \quad \text{for every } \beta<\xi.$$

Hence, $f_\omega\xi$ is defined and $\|f_\omega\xi\|_\infty \le \|f\|_\xi$, which finishes the proof of the claim.

In general, if $f\in V_\xi(K)$ then $f-\inf f\in V_\xi$ and $f-\inf f\ge 0$. Hence, from the previous result $f_\omega\xi$ is defined and

$$\|f_\omega\xi\|_\infty \le \|f_\omega\xi-\inf f\|_\infty + |\inf f| \le \|f-\inf f\|_\xi + |\inf f| \le \|f\|_\xi+2\|f\|_\infty \le 3\|f\|_\xi.$$

On the other direction, let $f\ge 0$ with $r_{ND}(f)>\omega^\xi$. We will prove, by induction on ξ, that $f\in V_\xi(K)$ and $\|f\|_\xi \le 4\|f_\omega\xi\|_\infty$.

For $\xi=1$, suppose f_ω is defined and bounded by M. Define a sequence (g_n) in DBSC(K) by $g_n=f_n-ur(f_n-f)$ for every $n\in\mathbb{N}$. Obviously, $0\le g_n\le f$ and $\tilde{g}_n\le f_n$ ([K-L]) for every $n\in\mathbb{N}$. According to Proposition 1

$$\|g_n\|_D \le 4\|\tilde{g}_n\|_\infty \le 4\|f_n\|_\infty \le 4\|f_\omega\|_\infty \text{ for every } n\in\mathbb{N}.$$

If we can prove that $\|f-g_n\|_\infty \to 0$, then one has that $f\in V_1(K)$ and $\|f\|_1 \le 4\|f_\omega\|_\infty$.

Now $$ur(f-g_n)=ur(\ f+ur(f_n-f)-f_n\ )=ur(f_{n+1}-f_n).$$

Moreover for n=1,2,...

$$ur(f_{n+2}-f_{n+1}) = ur(\ f+ur(f_{n+1}-f)-f_{n+1}\ ) \le ur(\ f+ur(f_{n+1}-f)-ur(f_n-f)-f\ ) =$$
$$= ur(\ ur(f_{n+1}-f)-ur(f_n-f)\ ) \le ur(f_{n+1}-f_n).$$

Hence, the sequence $(\ ur(f_{n+1}-f_n)\ )_{n=0}^{\infty}$ is decreasing (setting $f_0=0$). Also, according to Lemma 6

$$ur(f_{n+1}-f_n) \le \frac{1}{n+1} f_{n+1} \le \frac{1}{n+1} f_\omega \le \frac{M}{n+1} \quad \text{for every } n\in\mathbb{N}.$$

Thus $\|ur(f_{n+1}-f_n)\|_\infty \to 0$ and finally $\|f-g_n\|_\infty \to 0$.

Assume the result is known for ξ and let $r_{ND}(f) > \omega^{\xi+1}$. Then the function $f_{\omega^{\xi+1}}$ is defined and let M an upper bound for it. Set $g_n=f_{\omega^\xi.n}-ur(f_{\omega^\xi.n}-f)$ for every $n\in\mathbb{N}$. Then $(g_n)\subseteq DBSC(K)$ and $0\le g_n\le f$, $0\le\tilde{g}_n\le f_{\omega^\xi.n}$ for all $n\in\mathbb{N}$. From Proposition 1

$$\|g_n\|_D \le 4\|\tilde{g}_n\|_\infty \le 4\|f_{\omega^\xi.n}\|_\infty \le 4\|f_{\omega^{\xi+1}}\|_\infty \text{ for every } n\in\mathbb{N}.$$

Hence, if $\|f-g_n\|_\xi \to 0$ then one has that $f\in V_\xi(K)$ and $\|f\|_{\xi+1} \le 4\|f_{\omega^{\xi+1}}\|_\infty$.

By the induction hypothesis

$$\|(f-g_n)_{\omega^\xi}\|_\infty \le \|f-g_n\|_\xi \le 4\|(f-g_n)_{\omega^\xi}\|_\infty ,$$

thus, if $\|(f-g_n)_{\omega^\xi}\|_\infty \to 0$, then $\|f-g\|_\xi \to 0$.

Claim $(f-g_n)_\beta \le ur(f_{\omega^\xi.n+\beta}-f_{\omega^\xi.n})$ for every $1\le\beta\le\omega^\xi$ and $n\in\mathbb{N}$.

Let $n\in\mathbb{N}$. We will prove the claim by induction on β.

For $\beta=1$ the claim is true because

$$(f-g_n)_1 = ur(f-g_n) = ur(\ ur(f_{\omega^\xi.n}-f)-f_{\omega^\xi.n}+f\ ) = ur(f_{\omega^\xi.n+1}-f_{\omega^\xi.n})$$

Let the claim is true for β with $1\le\beta<\omega_1$. We will prove it for $\beta+1$. Using the induction hypothesis it is enough to prove that

$$A = (f-g_n) + ur(\ ur(f_{\omega^\xi.n+\beta}-f_{\omega^\xi.n}) -f+g_n\ ) - ur(\ ur(f_{\omega^\xi.n+\beta}-f)+f-f_{\omega^\xi.n}) \le 0$$

But $A \le (f-g_n) + ur(\ ur(f-f_{\omega^\xi.n})+f_{\omega^\xi.n}-f-(f-g_n)\ ) \le$

$$\le ur(f_{\omega^\xi.n}-f) - (f_{\omega^\xi.n}-f) + ur(\ ur(f-f_{\omega^\xi.n})-ur(f_{\omega^\xi.n}-f)\ ) \le 0.$$

The last inequality is true, since for every $x\in K$ and $\varepsilon>0$ there exists a continous function G on K such that $G\ge F$ and $ur(F)(x)\le G(x)\le ur(F)(x)+\varepsilon$ where $F=f_{\omega^\xi.n}-f$ and then it is easy to see that $ur(F) - F + ur(\ -F-ur(F)\ )(x) \le \varepsilon$.

For a limit ordinal β it is easy to prove the claim if it is true for every ordinal $\zeta<\beta$.

Hence using the claim we have that

$$(f-g_n)_{\omega^\xi} \leq ur(f_{\omega^\xi(n+1)}-f_{\omega^\xi.n}) \text{ for every } n\in\mathbb{N}.$$

We will prove that $\| ur(f_{\omega^\xi(n+1)}-f_{\omega^\xi.n}) \|_\infty \to 0$ and immediately one has that $\|(f-g_n)_{\omega^\xi}\|_\infty \to 0$.

The sequence $(ur(f_{\omega^\xi(n+1)}-f_{\omega^\xi.n}))_{n=0}^{\infty}$ is decreasing for every $0\leq\xi<\omega_1$. Indeed, by induction on β one has

$$ur(f_{\omega^\xi(n+1)+\beta}-f_{\omega^\xi.n+\beta}) \leq ur(f_{\omega^\xi(n+1)}-f_{\omega^\xi.n}) \quad \text{for every } 0\leq\beta<\omega^\xi.$$

Hence, for every n=1,2,... we have

$$ur(f_{\omega^\xi(n+2)}-f_{\omega^\xi(n+1)}) \leq ur(\sup_{\beta<\omega^\xi}(f_{\omega^\xi(n+1)+\beta}-f_{\omega^\xi.n+\beta})) \leq$$
$$ur(f_{\omega^\xi(n+1)}-f_{\omega^\xi.n}).$$

For n=0, one has by induction on β that $f_{\omega^\xi+\beta} \leq f_{\omega^\xi}+f_\beta$ for every $0\leq\beta\leq\omega^\xi$, hence $f_{2\omega^\xi} -f_{\omega^\xi}\leq f_{\omega^\xi}$.

According to Lemma 6,

$$ur(f_{\omega^\xi(n+1)}-f_{\omega^\xi.n}) \leq \frac{1}{n+1} f_{\omega^\xi(n+1)} \leq \frac{1}{n+1} f_{\omega^{\xi+1}} \leq \frac{1}{n+1}\| f_{\omega^{\xi+1}} \|_\infty$$

for every $n\in\mathbb{N}$.

Hence $\| ur(f_{\omega^\xi(n+1)}-f_{\omega^\xi.n}) \|_\infty \to 0$ and finally, $\| f-g_n\|_\xi \to 0$. Thus, $f\in V_{\xi+1}(K)$ and $\|f\|_{\xi+1} \leq 4\|f_{\omega^{\xi+1}}\|_\infty$.

For a limit ordinal ξ, let $r_{ND}(f) > \omega^\xi$. Then, by the induction hypothesis, we have that $f\in V_\beta(K)$ and $\|f\|_\beta \leq 4\|f_{\omega^\beta}\|_\infty$ for every $\beta<\xi$. Hence,

$$\|f\|_\xi =\sup_{\beta<\xi}\|f\|_\beta \leq 4\sup_{\beta<\xi}\|f_{\omega^\beta}\|_\infty \leq 4\|f_{\omega^\xi}\|_\infty \qquad \text{and } f\in V_\xi.$$

Thus, we proved that, for every $1\leq\xi<\omega_1$, $f\in V_\xi$ and $\|f\|_\xi\leq4\|f_{\omega^\xi}\|_\infty$ if $f\geq0$ and $r_{ND}(f)>\omega^\xi$.

In general, if $r_{ND}(f)>\omega^\xi$ then $f-\inf f\geq0$ and $r_{ND}(f-\inf f)>\omega^\xi$. Thus from the previous result $f\in V_\xi(K)$ and

$$\|f\|_\xi \leq \|f-\inf f\|_\xi + |\inf f| \leq 4\|(f-\inf f)_{\omega^\xi}\|_\infty + |\inf f| \leq 4\|f_{\omega^\xi}\|_\infty +5\|f\|_\infty,$$

which finishes the proof of the main Theorem.

Corrolary 7. For every uncountable, compact, metrizable space K and every countable ordinal ξ there exists a function f in $V_\xi(K)$ and not in $V_{\xi+1}(K)$.

Proof. According to a Theorem in [K-L] we can construct a function f on K with $r_{ND}(f)=\omega^{\xi+1}$. From the main Theorem this function belongs to $V_\xi(K)$ and certainly it does not belong to $V_{\xi+1}(K)$.

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