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GENERALIZED FRACTION RULES FOR MONOTONICITY WITH HIGHER ANTIDERIVATIVES AND DERIVATIVES

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Abstract
We first introduce the generic versions of the fraction rules for monotonicity, i.e., the one that involves integrals known as Gromov's theorem and the other that involves derivatives known as L'Hôpital's rule for monotonicity, which we then extend to high-order antiderivatives and derivatives, respectively.

Keywords Fraction rules for monotonicity · Gromov's theorem · L'Hôpital's rule for monotonicity · High-order antiderivative · High-order mean · Cauchy formula of repeated integration · High-order derivative · Taylor polynomial · Taylor remainder

Introduction

Roughly speaking, the application of either the integral or the differential operation to both the numerator and the denominator of a fraction, preserves the monotonicity of the fraction. The integral case of such fact is known as *Gromov's theorem* (see, e.g., [6, 12]), while the differential case is called *L'Hôpital's rule for monotonicity* (see, e.g., [2, 12, 14, 16, 20]). Gromov's theorem first appeared in [7], i.e., about a decade *before* the introduction of L'Hôpital's rule for monotonicity in [1].

These results have been proven to be quite useful analytical tools with many applications to a plethora of mathematical areas, such as differential geometry (see, e.g., [6, 7]), quasiconformal theory (see, e.g., [1]), information theory

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(see, e.g., [16]), probability theory (see, e.g., [15]), approximation theory (see, e.g., [17]), theory of special functions (see, e.g., [2, 8, 20]) and theory of analytic functions (see, e.g., [12]).
Below follow the most generic versions of these fraction rules for monotonicity, for the statement of which we remind that a real function, defined in an interval of the extended real line, $[-\infty, \infty]$, is locally characterized by a property when it is characterized by that property in every compact subinterval of its domain (we remind that an unbounded interval of the form $[-\infty, \infty]$, $[-\infty, a]$ or $[a, \infty]$, for some $a \in \mathbb{R}$, is compact).

Theorem 1.1 (Gromov's theorem). *Consider*

1. *an interval* $I \subseteq [-\infty, \infty]$,
2. *a point* $c \in I$ *and*
3. *two functions* $f, g : I \to \mathbb{R}$, *such that*

   i. f *and* g *are both locally Lebesgue integrable and*
   ii. g *preserves Lebesgue integrability almost everywhere a non-zero sign*.

If

$$\frac{f}{g} : I \to [-\infty, \infty]$$

is Lebesgue integrable almost everywhere (strictly) monotonic, then

$$\frac{\int_c^{\cdot} f(t)\mathrm{d}t}{\int_c^{\cdot} g(t)\mathrm{d}t} : I \setminus \{c\} \to \mathbb{R}$$

is (strictly) monotonic of the same (strict) monotonicity.

Theorem 1.2 (L'Hôpital's rule for monotonicity). *Consider*

1. *an interval* $I \subseteq [-\infty, \infty]$,
2. *a point* $c \in I$ *and*
3. *two functions* $f, g : I \to \mathbb{R}$, *such that*

   i. $f|_{I\cap\mathbb{R}}$ *and* $g|_{I\cap\mathbb{R}}$ *are both differentiable and*
   ii. $g'(x) \neq 0$, *for all* $x \in I \cap \mathbb{R}$.

If

$$\frac{f'}{g'} : I \cap \mathbb{R} \to \mathbb{R}$$

is (strictly) monotonic, then

$$\frac{f - f(c)}{g - g(c)} : I \setminus \{c\} \to \mathbb{R}$$

is (strictly) monotonic of the same (strict) monotonicity.

It can be shown (see the "Equivalence?" section) that Theorem 1.1 is stronger than Theorem 1.2, a fact that has already been observed in [12]. However, the latter is an independent result of differential calculus, for the proof of which no tools of the integration theory are needed (see Appendix A).

The goal of the present manuscript is not only the proofs of Theorems 1.1 and 1.2, but also the introduction of their generalizations to higher antiderivatives and derivatives, respectively. Our analysis is organized as follows. In the "Basic notions" section, we review some necessary notions used for the compact statement of the aforementioned generalizations. In the "Generalized fraction rules for monotonicity" section, after the statement of the main results, we examine the relation between them and we proceed to their proof. In the "Corollaries and examples" section, we employ our findings in some novel applications. In Appendix A, we provide an alternative proof of the generalized L'Hôpital's rule for monotonicity with the exclusive utilization of the differential calculus toolbox.

Basic notions

For the statement of our results, we make a short, necessary note on the notation used.

1. For every

   i. $n \in \mathbb{N}$,
   ii. interval $I \subseteq [-\infty, \infty]$ when $n = 1$ or $I \subseteq \mathbb{R}$ when $n \neq 1$,
   iii. $c \in I$ and
   iv. locally Lebesgue integrable $f : I \to \mathbb{R}$,

   $A_{n,f,c}$ stands for *the antiderivative* of order n for f at c, i.e.,

   $$A_{n,f,c} : I \to \mathbb{R}$$
   $$x \mapsto A_{n,f,c}(x) := \frac{1}{(n-1)!} \int_c^x f(t)(x-t)^{n-1} \mathrm{d}t.$$

   The name of this function is nothing but random. It comes from *the Cauchy formula of repeated integration*,

   $$A_{n,f,c} = \int_c^{\cdot} \int_c^{t_1} \cdots \int_c^{t_{n-1}} f\left(t_n\right) \mathrm{d}t_n \ldots \mathrm{d}t_2 \mathrm{d}t_1, \text{ when } n \neq 1.$$

   This formula is introduced in [5, Trente-Cinquième Leçon in page 137] with the additional assumption of f being continuous. The result is then derived from the density of continuous functions in the space of integrable ones (see, e.g., [9, Theorem 11.5.8 in page 391]). With this equality at hand, we can directly verify that

   $$A_{n,f,c} = A_{k,A_{n-k,f,c},c}, \forall k \in \{1, \ldots, n-1\}, \text{ when } n \neq 1. \tag{2.1}$$

   Moreover, using the fundamental theorem of calculus (see, e.g., [9, Theorem B.4.3 in page 497]), we obtain that, when $n \neq 1$, the function $A_{n,f,c}\big|_{I\cap\mathbb{R}}$ is $(n-1)$-times differentiable and n-times Lebesgue integrable almost everywhere differentiable, with

   $$A_{n,f,c}{}^{(k)} = A_{n-k,f,c}, \forall k \in \{1, \ldots, n-1\}$$

   and

   $$A_{n,f,c}{}^{(n)} = f, \text{ Lebesgue integrable almost everywhere.}$$

   If, in addition, f is continuous, then $A_{n,f,c}\big|_{I\cap\mathbb{R}}$ is n-times differentiable, with

$$A_{n,f,c}{}^{(n)} = f. \tag{2.2}$$

2. For every

   i. $n \in \mathbb{N}$,
   ii. interval $I \subseteq \mathbb{R}$,
   iii. $c \in I$ and
   iv. locally Lebesgue integrable $f : I \to \mathbb{R}$,

   $M_{n,f,c}$ stands for *the mean* of order n for f at c, i.e.,

$$M_{n,f,c} : I \setminus \{c\} \to \mathbb{R}$$
$$x \mapsto M_{n,f,c}(x) := \frac{n}{(x-c)^n} \int_c^x f(t)(x-t)^{n-1}\mathrm{d}t.$$

   The concept behind the above definition lies in the observation that

$$A_{n,1,c}(x) = \frac{(x-c)^n}{n!}, \forall x \in \mathbb{R},$$

   which confirms the expected equality

$$M_{n,f,c} = \frac{A_{n,f,c}}{A_{n,1,c}}.$$

3. For every

   i. $n \in \mathbb{N}_0$,
   ii. interval $I \subseteq [-\infty, \infty]$ when $n = 0$ or $I \subseteq \mathbb{R}$ when $n \neq 0$,
   iii. $c \in I$ and
   iv. n-times differentiable in $I \cap \mathbb{R}$ $f : I \to \mathbb{R}$,

   $T_{n,f,c}$ and $R_{n,f,c}$ stand for *the Taylor polynomial and remainder*, respectively, of order n for f at c, i.e.,

$$T_{n,f,c} : I \to \mathbb{R}$$
$$x \mapsto T_{n,f,c}(x) := \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k,$$

   and

$$R_{n,f,c} : I \to \mathbb{R}$$
$$x \mapsto R_{n,f,c}(x) := f(x) - T_{n,f,c}(x).$$

   If, in addition, $n \in \mathbb{N}$ and $f^{(n)} : I \cap \mathbb{R} \to \mathbb{R}$ is locally Lebesgue integrable, then *the integral form of the remainder* (see, e.g., [4, §1.6 in page 62]) implies that

$$R_{n-1,f,c} = A_{n,f^{(n)},c}. \tag{2.3}$$

Generalized fraction rules for monotonicity

Statement

For the proper statement of the main results, we need the following result.

Proposition 3.1 *Consider*

i. *a natural number* $n \in \mathbb{N}$,
ii. *an interval* $I \subseteq [-\infty, \infty]$ *when* $n = 1$ *or* $I \subseteq \mathbb{R}$ *when* $n \neq 1$,
iii. *a point* $c \in I$ *and*
iv. *a function* $f : I \to \mathbb{R}$.

1. *If f*

   a. *is locally Lebesgue integrable and*
   b. *preserves Lebesgue integrability almost everywhere a non-zero sign*,

   then ${A_{n,f,c}}^{-1}(\{0\}) = \{c\}$.
2. *If*

   a. $f|_{I\cap\mathbb{R}}$ *is n-times differentiable and*
   b. $f^{(n)}(x) \neq 0$, *for all* $x \in I \cap \mathbb{R}$,

   then ${R_{n-1,f,c}}^{-1}(\{0\}) = \{c\}$.

Proof

1. To begin with, we have that $A_{n,f,c}(c) = 0$. Since f preserves Lebesgue integrability almost everywhere a non-zero sign, we deduce that for every $x \in I \setminus \{c\}$ the function

   $$(x - \mathrm{id})^{n-1} f : (\min\{c, x\}, \max\{c, x\}) \to \mathbb{R}$$

   also preserves Lebesgue integrability almost everywhere a non-zero sign, where id stands for the identity function. Thus, $A_{n,f,c}(x) \neq 0$ and the result then follows.
2. We have that $R_{n-1,f,c}(c) = 0$. Since $f^{(n)}(x) \neq 0$, for all $x \in I \cap \mathbb{R}$, from Darboux's theorem (see, e.g., [9, Theorem 8.3.2 in page 228]), we have that $f^{(n)}$ preserves a non-zero sign, that is $f^{(n-1)} : I \cap \mathbb{R} \to \mathbb{R}$ is strictly monotonic, hence $f^{(n)}$ is locally Lebesgue integrable (see, e.g., [9, Theorem B.2.5 in 490]). Now, we first apply point 1. for the function $f^{(n)}$ and we then employ Eq. 2.3, in order to get the desired result. □

With Proposition 3.1 at hand, we can now state the generalizations of Theorems 1.1 and 1.2 to higher antiderivatives and derivatives, respectively.

Theorem 3.1 (generalization to higher antiderivatives). *Consider*

1. *a natural number* $n \in \mathbb{N}$,
2. *an interval* $I \subseteq [-\infty, \infty]$ *when* $n = 1$ *or* $I \subseteq \mathbb{R}$ *when* $n \neq 1$,
3. *a point* $c \in I$ *and*
4. *two functions* $f, g : I \to \mathbb{R}$, *such that*

   i. *f and g are both locally Lebesgue integrable and* *ii. g preserves Lebesgue integrability almost everywhere a non-zero sign.*

If

$$\frac{f}{g} \; : \; I \to [-\infty, \infty]$$

is Lebesgue integrable almost everywhere (strictly) monotonic, then

$$\frac{A_{n,f,c}}{A_{n,g,c}} \; : \; I \setminus \{c\} \to \mathbb{R}$$

is (strictly) monotonic of the same (strict) monotonicity.

Theorem 3.2 (generalization to higher derivatives). *Consider*

1. *a natural number $n \in \mathbb{N}$,*
2. *an interval $I \subseteq [-\infty, \infty]$ when $n = 1$ or $I \subseteq \mathbb{R}$ when $n \neq 1$,*
3. *a point $c \in I$ and*
4. *two functions $f, g \; : \; I \to \mathbb{R}$, such that*

   i. $f|_{I\cap\mathbb{R}}$ and $g|_{I\cap\mathbb{R}}$ are both n-times differentiable and
   ii. $g^{(n)}(x) \neq 0$, for all $x \in I \cap \mathbb{R}$.

If

$$\frac{f^{(n)}}{g^{(n)}} \; : \; I \cap \mathbb{R} \to \mathbb{R}$$

is (strictly) monotonic, then

$$\frac{R_{n-1,f,c}}{R_{n-1,g,c}} \; : \; I \setminus \{c\} \to \mathbb{R}$$

is (strictly) monotonic of the same (strict) monotonicity.

Equivalence?

In general, Theorem 3.1 is stronger than Theorem 3.2.

Proposition 3.2 *Theorem 3.1 implies Theorem 3.2.*

Proof Under the hypothesis of Theorem 3.2, we first deduce that both

$$f^{(n)}, g^{(n)} \; : \; I \cap \mathbb{R} \to \mathbb{R}$$

are locally Lebesgue integrable. Indeed, we can argue as in the proof of point 2. of Proposition 3.1, in order to show that $g^{(n)}$ is locally Lebesgue integrable. Moreover, $\frac{f^{(n)}}{g^{(n)}}$ is locally bounded since it is (strictly) monotonic, hence we write

$$f^{(n)} = \frac{f^{(n)}}{g^{(n)}} g^{(n)}$$

and we conclude that $f^{(n)}$ is also locally Lebesgue integrable as a product of a locally bounded function and a locally Lebesgue integrable one.

Now, we first apply Theorem 3.1 for the functions $f^{(n)}$ and $g^{(n)}$ and we then employ Eq. 2.3. □

We can weaken Theorem 3.1 in a specific manner, in order to get the reverse implication of Proposition 3.2.

Proposition 3.3 *Theorem 3.2 implies Theorem 3.1, when the latter is equipped with the hypothesis that f and g are both continuous instead of being just locally Lebesgue integrable.*

Proof Under the hypothesis of the weakened Theorem 3.1, Eq. 2.2 implies that $A_{n,f,c}\Big|_{I\cap\mathbb{R}}$ and $A_{n,g,c}\Big|_{I\cap\mathbb{R}}$ are both n-times differentiable.

Now, all we have to do is to apply Theorem 3.2 for the functions $A_{n,f,c}$ and $A_{n,g,c}$. □

Proof

In view of Proposition 3.2, we only need to prove the stronger of the main results, namely, Theorem 3.1.

Proof of Theorem 3.1 It suffices to show the result only for the case where g preserves Lebesgue integrability almost everywhere the positive sign. Indeed, we can employ such a result for -f and -g instead of f and g, respectively, in order to get the corresponding one for g that preserves Lebesgue integrability almost everywhere the negative sign.

Moreover, it suffices to show Theorem 3.1 only for the case where $\frac{f}{g}$ is Lebesgue integrable almost everywhere (strictly) increasing. Indeed, we can employ such a result for -f instead of f, in order to get the corresponding one for $\frac{f}{g}$ that is Lebesgue integrable almost everywhere (strictly) decreasing.

Hence, we assume, without loss of generality, that g preserves Lebesgue integrability almost everywhere the positive sign and that f is Lebesgue integrable almost everywhere (strictly) increasing.

We will show the desired result by induction on n.

1. *The base case*. The case where $n = 1$ is nothing but Theorem 1.1 itself. Since g preserves Lebesgue integrability almost everywhere the positive sign, the function $A_{1,g,c}$ is strictly increasing, which implies that its inverse

$$A_{1,g,c}{}^{-1} : A_{1,g,c}(I) \to I$$

is not only well defined but also strictly increasing. In addition, the continuity of $A_{1,g,c}$ guarantees that $A_{1,g,c}(I)$ is an interval. We then consider the function

$$h := A_{1,f,c} \circ A_{1,g,c}{}^{-1} : A_{1,g,c}(I) \to \mathbb{R}$$

and we claim that

$$h = A_{1,\frac{f}{g}\circ A_{1,g,c}{}^{-1},0},$$

that is

$$\int_{c}^{A_{1,g,c}{}^{-1}(\cdot)} f(t)\mathrm{d}t = \int_{0}^{\cdot} \frac{f\left(A_{1,g,c}{}^{-1}(t)\right)}{g\left(A_{1,g,c}{}^{-1}(t)\right)}\mathrm{d}t.$$

Indeed, observing that

$$\int_{c}^{{A_{1,g,c}}^{-1}(\cdot)} f(t)\mathrm{d}t = \int_{c}^{{A_{1,g,c}}^{-1}(\cdot)} \frac{f(t)}{g(t)} g(t)\mathrm{d}t = \int_{c}^{{A_{1,g,c}}^{-1}(\cdot)} \frac{f(t)}{g(t)} {A_{1,g,c}}'(t)\mathrm{d}t,$$

we get the desired equality using the change of variable formula (see, e.g., [19, point (i) of Corollary 6.97 in page 326]). Moreover,

$$\frac{f}{g} \circ {A_{1,g,c}}^{-1} : A_{1,g,c}(I) \to [-\infty, \infty]$$

is Lebesgue integrable almost everywhere (strictly) increasing as a composition of a strictly increasing function and a Lebesgue integrable almost everywhere (strictly) increasing function. The combination of the above two facts implies that h is (strictly) convex (see, e.g., [18, Theorem A in page 9 and Remark B in page 13] or [21, Theorem 14.14 in page 334]). Hence, from the equality $h(0) = 0$ along with the three chords lemma (a.k.a. Galvani's lemma) (see, e.g., [13, Theorem 1.3.1 in page 20]), we deduce that the function

$$\frac{h}{\mathrm{id}} : A_{1,g,c}(I) \setminus \{0\} \to \mathbb{R}$$

is (strictly) increasing and so is

$$\frac{h \circ A_{1,g,c}}{A_{1,g,c}} : I \setminus \{c\} \to \mathbb{R},$$

since $A_{1,g,c}$ is strictly increasing. The result then follows from the fact that $h \circ A_{1,g,c} = A_{1,f,c}$.

2. *The induction step*. For $n \neq 1$, we fix a natural number $k \in \{1, \ldots, n-1\}$. In view of point 1. of Proposition 3.1, both

$$\frac{A_{k,f,c}}{A_{k,g,c}}, \frac{A_{k+1,f,c}}{A_{k+1,g,c}} : I \setminus \{c\} \to \mathbb{R}$$

are well defined. We assume that $\frac{A_{k,f,c}}{A_{k,g,c}}$ is (strictly) increasing and we will show that $\frac{A_{k+1,f,c}}{A_{k+1,g,c}}$ is (strictly) increasing. We consider the functions

$$\tilde{f} := (\mathrm{sgn} \circ (\mathrm{id} - c))^k A_{k,f,c} : I \to \mathbb{R}$$

and

$$\tilde{g} := (\mathrm{sgn} \circ (\mathrm{id} - c))^k A_{k,g,c} : I \to \mathbb{R},$$

which are both locally Lebesgue integrable. We claim that $\tilde{g}$ preserves Lebesgue integrability almost everywhere the positive sign. Indeed, we have that

$$\tilde{g}(x) = \frac{\mathrm{sgn}(x-c)}{(k-1)!} \int_c^x g(t)|x-t|^{k-1}\mathrm{d}t, \forall x \in I,$$

since

$$\mathrm{sgn}(x-c) = \mathrm{sgn}(x-t), \forall t \in (\min\{c,x\}, \max\{c,x\}), \forall x \in I \setminus \{c\},$$

therefore, $\tilde{g}|_{I \setminus \{c\}}$ preserves the positive sign. In addition, the function

$$\frac{\tilde{f}}{\tilde{g}} : I \setminus \{c\} \to \mathbb{R}$$

is (strictly) increasing, since

$$\frac{\tilde{f}}{\tilde{g}} = \frac{A_{k,f,c}}{A_{k,g,c}}.$$

With the above facts at hand, all we have to do is first to apply Theorem 1.1 for the functions $\tilde{f}$ and $\tilde{g}$ and second to employ Eq. 2.1, in order to obtain the desired result. □

Corollaries and examples

Below are some applications of the generalized fraction rules for monotonicity.

1. *Monotonicity of high-order mean*: We consider

   i. a natural number $n \in \mathbb{N}$,
   ii. an interval $I \subseteq \mathbb{R}$,
   iii. a point $c \in I$ and
   iv. a locally Lebesgue integrable function $f : I \to \mathbb{R}$.

   If f is Lebesgue almost everywhere (strictly) monotonic, then from Theorem 3.1 for $g \equiv 1$, we deduce that $M_{n,f,c}$ is (strictly) monotonic of the same (strict) monotonicity.
2. *Convexity of high-order mean*: We consider

   i. a natural number $n \in \mathbb{N}$,
   ii. an interval $I \subseteq \mathbb{R}$,
   iii. a point $c \in I$ and
   iv. a convex function $f : I \to \mathbb{R}$.

   From the three chords lemma, we have that the function

   $$\frac{f - f(c)}{\mathrm{id} - c} : I \setminus \{c\} \to \mathbb{R}$$

   is (strictly) increasing. Extending the above function as

   $$\frac{(f - f(c))\mathrm{sgn} \circ (\mathrm{id} - c)}{|\mathrm{id} - c|} : I \to [-\infty, \infty]$$

   and remembering that every convex function is locally Lebesgue integrable, we employ Theorem 3.1, in order to obtain that the function

   $$\frac{A_{n,(f-f(c))\mathrm{sgn}\circ(\mathrm{id}-c),c}}{(n+1)A_{n,|\mathrm{id}-c|,c}} = \frac{A_{n,f-f(c),c}}{(n+1)A_{n,\mathrm{id}-c,c}} = \frac{M_{n,f-f(c),c}}{(n+1)M_{n,\mathrm{id}-c,c}} = \\ = \frac{M_{n,f,c} - f(c)}{\mathrm{id} - c} : I \setminus \{c\} \to \mathbb{R}$$

   is also (strictly) increasing. Hence, again from the three chords lemma, we deduce that $M_{n,f,c}$ is (strictly) convex.
3. *An application in ordinary differential equations*: We consider the classic nondimensionalized epidemiological model of the single epidemic outbreak for non-negative times $t \in [0, \infty)$,

$$S'(t) = -\mathcal{R}_0 S(t)I(t)$$
$$I'(t) = -I(t) + \mathcal{R}_0 S(t)I(t)$$
$$R'(t) = I(t),$$

where S are the susceptible, I are the infected/infectious and R are the recovered individuals of a total constant population. Moreover, *the basic reproductive number (ratio, rate)*, $\mathcal{R}_0$, represents the average number of secondary cases arising by a single infected/infectious individual in a completely susceptible population (see, e.g., [3, 11]). The value of $\mathcal{R}_0$ plays a key role in the epidemic as it determines whether an infectious disease will spread in a population or not. $\mathcal{R}_0 > 1$ reflects an outbreak of the epidemic and we search for $S, I, R : [0,\infty] \to [0,1]$, when the initial values $S(0)$, $I(0)$ and $R(0)$ are given. In the non trivial epidemiological situation of $S(0), I(0) \in (0,1)$ and $R(0) \in [0,1)$, there exists such functions satisfying the following properties,

i. $S(t), I(t) \in (0,1)$, for every $t \in [0,\infty)$, with

$$I(t) + S(t) - \frac{1}{\mathcal{R}_0}\ln S(t) = I(0) + S(0) - \frac{1}{\mathcal{R}_0}\ln S(0), \forall t \in [0,\infty)$$

and

ii. $I(\infty) = 0$ and

$$S(\infty) = -\frac{1}{\mathcal{R}_0} W\left(-\mathcal{R}_0 S(0)\mathrm{e}^{-\mathcal{R}_0(S(0)+I(0))}\right) \in \left(0, \frac{1}{\mathcal{R}_0}\right),$$

where W stands for the Lambert function (see, e.g., [10]).

Hence, $S'(t) < 0$, for all $t \in [0,\infty)$, which implies that S is strictly decreasing. Making use of Theorem 1.2, we deduce that

$$\frac{I - I(c)}{S - S(c)} : [0,\infty] \setminus \{c\} \to \mathbb{R}$$

is strictly increasing for every $c \in [0,\infty]$, since

$$\frac{I'}{S'} = \frac{1}{\mathcal{R}_0 S} - 1$$

is strictly increasing. Thus,

$$\lim_{t\to c} \frac{I(t) - I(c)}{S(t) - S(c)} \overset{\frac{0}{0}}{=} \lim_{t\to c} \frac{I'(t)}{S'(t)} = \frac{1}{\mathcal{R}_0 S(c)} - 1.$$

These facts imply that

$$I(t) < I(c) + \left(\frac{1}{\mathcal{R}_0 S(c)} - 1\right)(S(t) - S(c)), \forall t \in [0,\infty] \setminus \{c\},$$

i.e., a useful *a priori* estimate when $c = 0$. Moreover, using Theorem 3.1, we deduce that

$$\frac{M_{n,I,c} - I(c)}{M_{n,S,c} - S(c)} : [0,\infty) \setminus \{c\} \to \mathbb{R}$$

is strictly increasing for every $n \in \mathbb{N}$ and $c \in [0,\infty)$. The corresponding inequality is

$$M_{n,I,c}(t) < I(c) + \left(\frac{1}{\mathcal{R}_0 S(c)} - 1\right)\left(M_{n,S,c}(t) - S(c)\right), \forall t \in [0,\infty) \setminus \{c\},$$

which can also be deduced directly from the previous one.

4. *Multidimensional analog for specific radial functions*: We consider

   i. a natural number $n \in \mathbb{N}$ and
   ii. two functions $f, g : [0, \infty) \to \mathbb{R}$, such that

      a. f and g are both locally Lebesgue integrable and
      b. g preserves Lebesgue integrability almost everywhere a non-zero sign.

   We then set

$$\phi : \coprod_{r \in (0,\infty)} B\left(0_n, r\right) \to \mathbb{R}$$
$$(r, x) \mapsto \phi(r, x) := f(r - |x|)$$

and

$$\psi : \coprod_{r \in (0,\infty)} B\left(0_n, r\right) \to \mathbb{R}$$
$$(r, x) \mapsto \psi(r, x) := g(r - |x|),$$

where $B\left(0_n, r\right)$ stands for the n-dimensional ball of radius $r > 0$ centered at the origin $0_n \in \mathbb{R}^n$, $\coprod_{r \in (0,\infty)} B\left(0_n, r\right)$ stands for the dependent type $\left\{(r, x) \in (0, \infty) \times \mathbb{R}^n \,\middle|\, x \in B\left(0_n, r\right)\right\}$ and $|\cdot|$ stands for the standard Euclidean norm in $\mathbb{R}^n$. Employing the change of variables formula, we can deduce that, for every fixed $r > 0$, the functions $\phi(r, \cdot), \psi(r, \cdot) : B\left(0_n, r\right) \to \mathbb{R}$ are both Lebesgue integrable. Indeed, we have

$$\int_{B\left(0_n, r\right)} \phi(r, x)\mathrm{d}x = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^r f(r - t)t^{n-1}\mathrm{d}t = \frac{2\pi^{\frac{n}{2}}(n-1)!}{\Gamma\left(\frac{n}{2}\right)} A_{n,f,0}(r),$$

where for the first equality, we employed the polar coordinates change of variables formula for the radial functions (see, e.g., [21, Theorem 26.20 in page 695]). Similarly, there follows the result for the other function, ψ, for which we also note that, in view of Proposition 3.1, we have

$$\int_{B\left(0_n, r\right)} \psi(r, x)\mathrm{d}x \neq 0, \forall r > 0.$$

We now claim that if

$$\frac{f}{g} : [0, \infty) \to [-\infty, \infty]$$

is Lebesgue almost everywhere (strictly) monotonic, then the well-defined function

$$\frac{\int_{B\left(0_n, \cdot\right)} \phi(\cdot, x)\mathrm{d}x}{\int_{B\left(0_n, \cdot\right)} \psi(\cdot, x)\mathrm{d}x} : (0, \infty) \to \mathbb{R}$$

is (strictly) monotonic of the same (strict) monotonicity. Indeed, from Theorem 3.1, we have that the function

$$\frac{A_{n,f,0}}{A_{n,g,0}} : (0, \infty) \to \mathbb{R}^n$$

is (strictly) monotonic of the same (strict) monotonicity as of $\frac{f}{g}$ and the result then follows since

$$\frac{A_{n,f,0}}{A_{n,g,0}} = \frac{\int\limits_{B(0_n,\cdot)} \phi(\cdot, x)\mathrm{d}x}{\int\limits_{B(0_n,\cdot)} \psi(\cdot, x)\mathrm{d}x}.$$

Appendix A L'Hôpital's rule for monotonicity via differential calculus

We need the following straightforward extension to unbounded intervals of a well-known result (see, e.g., [9, Theorem 8.3.3 in page 229]), the proof of which is omitted.

Theorem A.1 (Rolle's theorem). *Consider*

1. *a compact interval* $I \subseteq [-\infty, \infty]$ *and*
2. *a function* $f : I \to \mathbb{R}$, *such that*

   i. f *is continuous and*
   ii. $f|_{I^\circ}$ *is differentiable.*

If $f(\partial I)$ is a singleton, then there exists a point $\xi \in I^\circ$, such that $f'(\xi) = 0$.
We also need the following extension of point 2. of Proposition 3.1.

Proposition A.1 *Consider*

1. *a natural number* $n \in \mathbb{N}$,
2. *an interval* $I \subseteq [-\infty, \infty]$ *when* $n = 1$ *or* $I \subseteq \mathbb{R}$ *when* $n \neq 1$,
3. *a point* $c \in I$ *and*
4. *a function* $f : I \to \mathbb{R}$, *such that*

   i. $f|_{I\cap\mathbb{R}}$ *is n-times differentiable and*
   ii. $f^{(n)}(x) \neq 0$, *for all* $x \in I \cap \mathbb{R}$.

Then,

$$\left(R_{n-1,f,c}{}^{(k)}\right)^{-1}(\{0\}) = \{c\}, \forall k \in \{0, \dots, n-1\}.$$

Proof To begin with, we have the equalities

$$R_{n-1,f,c}{}^{(k)}(c) = 0, \forall k \in \{0, \dots, n-1\}.$$

We assume that there exists a natural number $k \in \{0, \dots, n-1\}$ and a point $x \in I \setminus \{c\}$, such that $R_{n-1,f,c}{}^{(k)}(x) = 0$. In view of the above sequence of equalities, we inductively apply Theorem A.1 $n - k$ times, in order to deduce that there exists a point $\xi \in (\min\{c, x\}, \max\{c, x\})$, such that $f^{(n)}(\xi) = 0$, which contradicts the assumption of the non vanishing $f^{(n)}$. □

With Proposition A.1 at hand, Theorem 3.2 is properly stated in the context of differential calculus. We now proceed to its proof.

Proof of Theorem 1.2 Since $g'(x) \neq 0$, for all $x \in I \cap \mathbb{R}$, from Darboux's theorem, we have that g' preserves a non-zero sign, hence g is strictly monotonous. Hence, the inverse function of g, $g^{-1} : g(I) \to I$, is well defined. Additionally, g^{-1} is differentiable with

$$\left(g^{-1}\right)' = \frac{1}{g' \circ g^{-1}}.$$

Arguing as in the proof of Theorem 3.1, it suffices to show the result for g being strictly increasing and f being (strictly) increasing. Therefore, we make such assumptions. From the strict monotonicity of g, the function g^{-1} is also strictly increasing.

Now, we consider the function $h := f \circ g^{-1} : g(I) \to \mathbb{R}$, which is differentiable, due to the chain rule, with

$$h' = f' \circ g^{-1} \left(g^{-1}\right)' = \frac{f'}{g'} \circ g^{-1},$$

thus h' is (strictly) increasing as a composition of a strictly increasing function and a (strictly) increasing function. Hence, h is (strictly) convex.

We then consider two arbitrary $x_1, x_2 \in I \setminus \{c\}$, such that $x_1 < x_2$. Since $g\left(x_1\right) < g\left(x_2\right)$, from the three chords lemma, we deduce that

$$\frac{h\left(g\left(x_1\right)\right) - h(g(c))}{g\left(x_1\right) - g(c)} \underset{(<)}{\leq} \frac{h\left(g\left(x_2\right)\right) - h(g(c))}{g\left(x_2\right) - g(c)},$$

or else

$$\frac{f\left(x_1\right) - f(c)}{g\left(x_1\right) - g(c)} \underset{(<)}{\leq} \frac{f\left(x_2\right) - f(c)}{g\left(x_2\right) - g(c)}.$$

□

Proof of Theorem 3.2 It is only left to show the result for $n > 1$ (with $I \subseteq \mathbb{R}$), thus we make such an assumption.

To begin with, in view of Proposition A.1, we have the following sequence of equalities

$$\frac{{R_{n-1,f,c}}^{(k)}}{{R_{n-1,g,c}}^{(k)}} = \frac{{R_{n-1,f,c}}^{(k)} - {R_{n-1,f,c}}^{(k)}(c)}{{R_{n-1,g,c}}^{(k)} - {R_{n-1,g,c}}^{(k)}(c)}, \forall x \in I, \forall k \in \{0, 1, \dots, n-1\}.$$

Additionally, the following

$$\frac{{R_{n-1,f,c}}^{(n)}}{{R_{n-1,g,c}}^{(n)}} = \frac{f^{(n)}}{g^{(n)}}$$

is true.

Now, we inductively apply Theorem 1.2 n times, in order to get that both

$$\left.\frac{R_{n-1,f,c}}{R_{n-1,g,c}}\right|_{I \cap [-\infty, c)} \text{ and } \left.\frac{R_{n-1,f,c}}{R_{n-1,g,c}}\right|_{I \cap (c, \infty]}$$

are (strict) monotonic of the same (strict) monotonicity as of $\frac{f^{(n)}}{g^{(n)}}$. If $c \in \partial I$, then the proof is complete.

Next, we deal with the case where $c \in I^{\circ}$. From the above (strict) monotonicity, we deduce that the one-sided limits to c of these functions exist in $[-\infty, \infty]$, i.e.,

$$\lim_{x\to c^-}\frac{R_{n-1,f,c}(x)}{R_{n-1,g,c}(x)}\in[-\infty,\infty]\ni\lim_{x\to c^+}\frac{R_{n-1,f,c}(x)}{R_{n-1,g,c}(x)}.$$

Moreover, making use of *the Lagrange form of the remainder* (see, e.g., [9, Theorem 8.4.1 in page 235]), we have that

$$R_{n-1,f,c}(x)=\frac{f^{(n)}\left(\xi_{f,x}\right)}{n!}(x-c)^n,\ \text{for some } \xi_{f,x}\in(\min\{x,c\},\max\{x,c\}),\forall x\in I$$

and

$$R_{n-1,g,c}(x)=\frac{g^{(n)}\left(\xi_{g,x}\right)}{n!}(x-c)^n,\ \text{for some } \xi_{g,x}\in(\min\{x,c\},\max\{x,c\}),\forall x\in I.$$

Therefore,

$$\lim_{x\to c^-}\frac{R_{n-1,f,c}(x)}{R_{n-1,g,c}(x)}=\lim_{x\to c^-}\frac{f^{(n)}\left(\xi_{f,x}\right)}{g^{(n)}\left(\xi_{g,x}\right)}\in\mathbb{R}\ni\lim_{x\to c^+}\frac{f^{(n)}\left(\xi_{f,x}\right)}{g^{(n)}\left(\xi_{g,x}\right)}=\lim_{x\to c^+}\frac{R_{n-1,f,c}(x)}{R_{n-1,g,c}(x)},$$

since the function $\frac{f^{(n)}}{g^{(n)}}$ is (strictly) monotonous. By the same reason, we deduce that

$$\lim_{x\to c^-}\frac{R_{n-1,f,c}(x)}{R_{n-1,g,c}(x)}\left\{\begin{matrix}\underset{(<)}{\leq}\\ \underset{(>)}{\geq}\end{matrix}\right\}\lim_{x\to c^+}\frac{R_{n-1,f,c}(x)}{R_{n-1,g,c}(x)}\ \text{if}\ \frac{f^{(n)}}{g^{(n)}}\ \text{is}\ \left\{\begin{matrix}\text{(strictly) increasing}\\ \text{(strictly) decreasing}\end{matrix}\right\},$$

thus

$$\frac{R_{n-1,f,c}\left(x_1\right)}{R_{n-1,g,c}\left(x_1\right)}\left\{\begin{matrix}\underset{(<)}{\leq}\\ \underset{(>)}{\geq}\end{matrix}\right\}\frac{R_{n-1,f,c}\left(x_2\right)}{R_{n-1,g,c}\left(x_2\right)}\ \text{if}\ \frac{f^{(n)}}{g^{(n)}}\ \text{is}\ \left\{\begin{matrix}\text{(strictly) increasing}\\ \text{(strictly) decreasing}\end{matrix}\right\},$$

for every $x_1, x_2 \in I$, such that $x_1 < c < x_2$, which completes the proof. □

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