Enumeration of Rosenberg–type Hypercompositional structures defined by binary relations

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Abstract

Every binary relation ρ on a set H, $(card(H) > 1)$ can define a hypercomposition and thus endow H with a hypercompositional structure. In this paper the binary relations are represented by Boolean matrices. With their help, the hypercompositional structures (hypergroupoids, hypergroups, join hypergroups) that derive with the use of the Rosenberg's hyperoperation, are characterized, constructed and enumerated using symbolic manipulation packages. Moreover, the hyperoperation $x \circ x = \{z \in H \mid (z, x) \in \rho\}$ and $x \circ y = x \circ x \cup y \circ y$, is introduced and connected to Rosenberg's hyperoperation, which assigns to every (x, y) the set of all z such that either $(x, z) \in \rho$ or $(y, z) \in \rho$.

Keywords: HyperGroups, Binary relations, Mathematica®
2000 MSC: 20N20, 68W30

1. Introduction

A hypercomposition in a non empty set H, is a function from $H \times H$ to the power set $P(H)$ of H. This notion was introduced in Mathematics together with the notion of the hypergroup, by F. Marty in 1934 during the 8th congress of the Scandinavian Mathematicians, held in Stockholm, [5].

The axioms that endow the pair $(H,)$, where H is a nonempty set and " $\cdot$ " is a hypercomposition on H, with the hypergroup structure are:

(i) $a(bc) = (ab)c$ for all $a, b, c \in H$ (associativity)

(ii) $aH = Ha = H$ for all $a \in H$ (reproductivity)

If only (i) is valid then $(H, \cdot)$ is called semi–hypergroup, while it is called quasi–hypergroup if only (ii) holds. In a hypergroup, the result of the hypercomposition is always a nonempty set. Indeed, suppose that for two elements a, b in

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H it hold that $ab = \emptyset$. Then $H = aH = a(bH) = (ab)H = \emptyset H = \emptyset$, which is absurd. Thus if $(H, \cdot)$ is not associative and reproductive, then the empty set can be among the results of the hypercomposition. $(H, \cdot)$ is called hypergroupoid if $xy \neq \emptyset$ for all x, y in H, otherwise it is called partial hypergroupoid.

Furthermore F. Marty [5] defined the two **induced hypercompositions** (the left and the right division) that derive from the hypercomposition, i.e.

$a/b = \{x \in H \mid a \in xb\}$ and $b\backslash a = \{y \in H \mid a \in by\}$

When "$\cdot$" is commutative, $a/b = b\backslash a$ is valid. In a hypergroup a/b and a\b are nonempty for all a, b in H and this is equivalent to the reproductive axiom [8]. A **transposition hypergroup** [4] is a hypergroup (H,$\cdot$) that satisfies a postulated property of transposition i.e. $(b\backslash a) \cap (c/d) \neq \emptyset \Rightarrow (ad) \cap (bc) \neq \emptyset$. A **join space** or **join hypergroup**, is a commutative transposition hypergroup. Here it is worth mentioning that the hypergroup, which is a very general structure, was progressively enriched with additional (either less or more powerful) axioms, creating, thus, a significant number of specific hypergroups as the above mentioned transposition and join ones, their fortifications [4, 14], the canonicals and quasicanonicals ones [7] etc, with many widespread applications e.g. [6, 14, 12].

Several papers were written on the construction of hypergroups, since hypergroups are much more varied than groups, e.g. for each prime number p, there is only one group, up to isomorphism, with cardinality p, while there is a very large number of non isomorphic hypergroups. For example there are 3999 non isomorphic hypergroups with 3 elements [16].

Among others, Rosenberg [15], Corsini [1], De Salvo and Lo Faro [3] studied hypercompositional structures defined in terms of binary relations. Corsini constructed partial hypegroupoids by introducing in a non empty set H the hypercomposition

$$x \cdot y = \{z \in H \mid \ (x, z) \in \rho \ and \ (z, y) \in \rho\}$$

where ρ is a binary relation on H. Obviously such a partial hypergroupoid is hypergroupoid if for each pair of elements x, y in H, there exists z in H such that $(x, z) \in \rho$ and $(z, y) \in \rho$. In [10] it is proved that this hypercomposition generates only one semihypergroup and only one quasihypergroup which are coincide with the total hypergroup. Also in [6] it is computed that this hypercomposition generates 2, 17, 304, 20660 non isomorphic hypergroupoids of order 2, 3, 4, 5 respectively.

De Salvo and Lo Faro introduced in a non empty set H the hypercomposition

$$x \cdot y = \{z \in H \mid \ (x, z) \in \rho \ \ or \ \ (z, y) \in \rho\}$$

where ρ is a binary relation on H and they characterized in [3] the relations ρ which give quasihypergroups, semihypergroups and hypergroups.

On the other hand Rosenberg introduced in a non empty set H the hypercomposition

$$x \bullet x = \{z \in H \mid \ (x, z) \in \rho\} \ \text{ and } \ x \bullet y = x \bullet x \cup y \bullet y$$

where ρ is a binary relation on H.

This paper deals with the Rosemberg-type hypercompositional structures, the properties of their generative binary relations and their representations by Boolean matrices. Some conclusions from [15] are restated with the use of Boolean Matrices, in order to develop Mathematica programs, which enumerate the hypergroupoids, the hypergroups and the join hypergroups with 2, 3, 4 and 5 elements. During the preparation of this paper, the authors became familiar with [2], where an extensive program written in C#, computes the Rosemberg hypergroups with 2, 3 and 4 elements. Regarding the hypergroups with 2, 3 and 4 elements (which both papers enumerate), the results are the same, even though they are reached through completely different (in software and size) computational methods.

2. The Rosemberg–type hypercompositional structures

Let H be a non empty set and ρ a binary relation on H. As usual:

$$\rho^2 := \{(x,y) \mid (x,z),(z,y) \in \rho, \text{ for some } z \in H\}$$

An element x of H is called outer element of ρ if $(z,x) \notin \rho^2$ for some $z \in H$, otherwise x is called inner element. The domain of ρ is the set

$$D = \{x \in H \mid \ (x,z) \in \rho, \ \text{ for some } \ z \in H\}$$

and the range of ρ is the set

$$R = \{x \in H \mid \ (z,x) \in \rho, \ \text{ for some } \ z \in H\}.$$

Rosenberg introduced in H the hypercomposition

$$x \bullet x = \{z \in H \mid \ (x,z) \in \rho\} \ \text{ and } \ x \bullet y = x \bullet x \cup y \bullet y$$

and he observed that $H_\rho = (H, \bullet)$ is a hypergroupoid if and only if H is the domain of ρ and that H_ρ is a quasihypergroup if and only if H is the domain and the range of ρ. Also he proved that:

Proposition 1. *[15] H_ρ is a semihypergroup if and only if:*

(i) H is the domain of ρ

(ii) $\rho \subseteq \rho^2$

(iii) $(a,x) \in \rho^2 \Rightarrow (a,x) \in \rho$ whenever x is an outer element of ρ

From the last two elements of this proposition it derives that whenever x is an outer element of ρ for some $a \in H_\rho$, then (a,y) is in ρ, if and only if (a,y) is in ρ^2. Thus one can easily observe that Proposition 1 is equivalent to

Proposition 2. *H_ρ is a semihypergroup if and only if:*

(i) H is the domain of ρ

(ii) $(y,x) \in \rho^2 \Leftrightarrow (y,x) \in \rho$ for all $y \in H$, whenever x is an outer element of ρ.

Thus:

Proposition 3. *H_ρ is a hypergroup if and only if:*

(i) H is the domain and the range of ρ

(ii) $(y,x) \in \rho^2 \Leftrightarrow (y,x) \in \rho$ for all $y \in H$, whenever x is an outer element of ρ.

On the other hand the binary relation ρ can define in H another hypercomposition, which is the following one:

$$x \circ x = \{z \in H \mid (z,x) \in \rho\} \text{ and } x \circ y = x \circ x \cup y \circ y$$

Proposition 4. *If ρ is symmetric, then the hypercompositional structures $(H, \bullet)$ and $(H, \circ)$ are coincide.*

One can easily observe that $(H, \circ)$ is a hypegroupoid if and only if H is the range of ρ. For $(a,b) \in \rho$, a is called a predecessor of b and b a successor of a [15]. Following Rosenberg's terminology an element x will be called predecessor outer element of ρ if $(x,z) \notin \rho^2$ for some $z \in H$.

The following two Propositions are proved in a similar way as Propositions 1 and 2.

Proposition 5. *$(H, \circ)$ is a semihypergroup if and only if:*

(i) H is the range of ρ

(ii) $(x,y) \in \rho^2 \Leftrightarrow (x,y) \in \rho$ for all $y \in H$, whenever x is a predecessor outer element of ρ.

Proposition 6. *$(H, \circ)$ is a hypergroup if and only if:*

(i) H is the domain and the range of ρ

(ii) $(x,y) \in \rho^2 \Leftrightarrow (x,y) \in \rho$ for all $y \in H$, whenever x is a predecessor outer element of ρ

From the definitions of the two above hypercompositions it derives that the hypercompositional structures constructed through them are always commutative. Since "$\bullet$" is commutative, the two induced hypercompositions "/" and "\" are coincide. The same holds for the hypercomposition "$\circ$".

Proposition 7. *If $H_\rho = (H, \bullet)$ is a hypergroup, then it holds:*

$$x/y = \{ \begin{array}{ll} H, & \text{if } (y,x) \in \rho \\ x \circ x, & \text{if } (y,x) \notin \rho \end{array}$$

for all $x,\ y$ in H

Proof: $x/y = \{v \in H \mid x \in v \bullet y\} = \{v \in H \mid x \in v \bullet v \cup y \bullet y\} =$
$= \{v \in H \mid\ (v,x) \in \rho \text{ or } (y,x) \in \rho\}$
which is equal to H, if $(y,x) \in \rho$ or equal to $x \circ x$, if $(y,x) \notin \rho$.

Corrolary 1. *If ρ is reflexive, then $x/x = H$, for each $x \in H$*

From Proposition 7 directly derives

Proposition 8. *Let $x,\ y,\ z,\ w$ be in H. If $x/y \cap w/z \neq \emptyset$, then there are three cases:*

(i) $x/y \cap w/z = H$, when $(y,x) \in \rho$ and $(z,w) \in \rho$

(ii) $x/y \cap w/z = x \circ x$, when $(y,x) \notin \rho$ and $(z,w) \in \rho$ or
$x/y \cap w/z = w \circ w$, when $(y,x) \in \rho$ and $(z,w) \notin \rho$

(iii) $x/y \cap w/z = x \circ x \cap w \circ w$, when $(y,x) \notin \rho$ and $(z,w) \notin \rho$

Lemma 1. *If ρ is reflexive, then the transposition axiom is fulfilled in the cases (i) and (ii) of Proposition 8.*

Proof: (i) Consider the intersection $x{\bullet}z{\cap}w{\bullet}y$. It is $x{\bullet}z{\cap}w{\bullet}y = (x \bullet x \cup z \bullet z) \cap (w \bullet w \cup y \bullet y)$. Since $(y,x) \in \rho$, it derives that $x \in y{\bullet}y$. Also $(x,x) \in \rho$, because ρ is reflexive. Thus $x \in x \bullet x$. Consequently $x \bullet z \cap w \bullet y \neq \emptyset$. Similar is the proof of (ii).

Lemma 2. *If $\rho^2 = \rho$, then the transposition axiom is fulfilled in the cases (i) and (ii) of Proposition 8.*

Proof: (i) Consider the intersection $xz \cap wy$ which is equal to $(x \bullet x \cup z \bullet z) \cap (w \bullet w \cup y \bullet y)$. Suppose that $(z,w) \in \rho$. Since H is the domain and the range of ρ, there exists $t \in H$ such that $(w,t) \in \rho$. Thus $t \in w \bullet w$. Next $(z,t) \in \rho^2$, because $(z,w) \in \rho$ and $(w,t) \in \rho$. But $\rho^2 = \rho$, hence $(z,t) \in \rho$ and therefore $t \in z \bullet z$. Consequently $t \in x \bullet z \cap w \bullet y$, so the intersection is non void. Similar is the proof of (ii).

Corrolary 2. *If ρ is transitive, then the transposition axiom is fulfilled in the cases (i) and (ii) of Proposition 8.*

Proof: If ρ is transitive, $\rho^2 \subseteq \rho$. Since H_ρ is a hypergroup it holds $\rho \subseteq \rho^2$. Thus $\rho^2 = \rho$.

Proposition 9. *If ρ is compatible (i.e reflexive and symmetric), then the transposition axiom is valid in H_ρ.*

Proof: Since ρ is reflexive, according to Lemma 1 the transposition axiom is valid in the cases (i) and (ii) of Proposition 8. Now for the case (iii) suppose that $x/y \cap w/z \neq \emptyset$. Since $x/y \cap w/z = x \circ x \cap w \circ w$, it derives that the intersection $x \circ x \cap w \circ w$ is non empty. But $x \circ x \cap w \circ w = x \bullet x \cap w \bullet w$, because ρ is symmetric. Thus $x \bullet x \cap w \bullet w \neq \emptyset$. Next the inclusion $x \bullet x \cap w \bullet w \subseteq x \bullet z \cap w \bullet y$ holds. Hence $x \bullet z \cap w \bullet y \neq \emptyset$ and so the transposition axiom is valid.

Also from the above Lemmas it derives that:

Proposition 10. *If ρ is reflexive or transitive and the implication:*

$$x \circ x \cap w \circ w \neq \emptyset \Rightarrow x \bullet x \cap w \bullet w \neq \emptyset$$

holds, for all x, w in H, then the transposition axiom is valid in H_ρ.

The implication $x \circ x \cap w \circ w \neq \emptyset \Rightarrow x \cdot x \cap w \cdot w \neq \emptyset$ means that a pair of elements with common predecessor has a common successor.

Proposition 11. *If $(y, x) \in \rho$ and $x \bullet x$ contains an outer element, then*

$$x/y \cap w/z \neq \emptyset \Rightarrow x \bullet z \cap w \bullet y \neq \emptyset$$

Proof: Let $(y, x) \in \rho$ and let t be an outer element in $x \bullet x$. Then $(x, t) \in \rho$. Therefore $(y, t) \in \rho^2$. But t is an outer element, so $(y, t) \in \rho$. Thus $t \in y \bullet y$.

The two hypercompositions "$\bullet$" and "$\circ$" can be seen in the case of graphs. A directed graph consists of a finite set V, whose members are called vertices and a subset A of $V \times V$ whose members are called arcs. Thus A is a binary relation in V and so through A the two hypercompositions "$\bullet$"and $\circ$ can be defined. Then $x \bullet x$ consists of all vertices z for which an arrow exists pointing from x to z, while $x \circ x$ consists of all vertices z for which an arrow exists pointing from z to x (see also [9]).

3. Boolean matrices and finite hypergroupoids

The Boolean domain $B = \{0, 1\}$ becomes a semiring under the addition

$$0 + 1 = 1 + 0 = 1 + 1 = 1, \; 0 + 0 = 0$$

and the multiplication:

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \; 1 \cdot 1 = 1.$$

This semiring is called *binary Boolean semiring.* A *Boolean matrix* is a matrix with entries from the binary Boolean semiring. Every binary relation ρ in a finite set H with $cardH = n \neq 0$, can be represented by a Boolean matrix M_ρ and conversely every $n \times n$ square Boolean matrix defines on H a binary relation. Indeed, let H be the set $\{a_1, \ \ldots, \ a_n\}$. Then a $n \times n$ Boolean matrix is constructed as follows: the element $(i, \ j)$ of the matrix is 1, if $(a_i, \ a_j) \in \rho$

and it is 0 if $(a_i,\ a_j) \notin \rho$ and vice versa. Hence in every set with n elements, 2^{n^2} partial hypergroupoids can be defined. The element a_k of H is an outer element of ρ if the k column of M_{ρ^2} has a zero entry. If all the entries of the k column are 1, then a_k is an inner element of ρ. Moreover $M_{\rho^2} = (M_\rho)^2$. A square Boolean matrix is called *total* if all its entries are equal to 1. A Boolean matrix is called *good* if its square is the total matrix [1], i.e. the good matrices are the square roots of the total matrix [11]. *Basic* Boolean matrix is a good matrix which is converted to a not good one, through the replacement of any unit entry to 0 [11]. It is proved that all the good matrices are generated from the basic ones [11]. A *nxn* Boolean matrix which has all the entries of its i row and its i column equal to 1, $i = 1, \ldots, n$, is called minimum basic matrix [11].

Let H_ρ denotes the above mentioned partial hypergroupoid, which is defined by a binary relation ρ through the hypercomposition "$\bullet$". Then the Propositions of the previous paragraph can be restated using Boolean matrices. Thus

Theorem 1. *H_ρ is a hypergroupoid if and only if M_ρ has no zero rows.*

Theorem 2. *H_ρ is a quasihypergroup if and only if M_ρ has no zero rows and no zero columns.*

From Proposition 2 it derives that

Theorem 3. *H_ρ is a semihypergroup if and only if*

(i) M_ρ consists only of non zero rows

(ii) if a column of the matrix M_{ρ^2} has a zero entry, then it coincides with the same column of M_ρ.

Also from Proposition 3 it derives that

Theorem 4. *H_ρ is a hypergroup if and only if*

(i) M_ρ consists only of non zero rows and non zero columns

(ii) whenever a column of the matrix M_{ρ^2} has a zero entry, it coincides with the same column of M_ρ.

Since the square roots of the total Boolean matrices consists only of non zero rows and non zero columns [11], it derives that

Theorem 5. *The square roots of the total Boolean matrices give Rosenberg hypergroups.*

Moreover from Proposition 10 it derives that

Theorem 6. *A hypergroup H_ρ is a join one, if*

(i) all the elements on the main diagonal of M_ρ are equal to 1 or $M_\rho = M_{\rho^2}$

(ii) the entrywise product of two rows a_{i*} and a_{j*} of M_ρ contains a non zero entry whenever the entrywise product of the corresponding columns $a_{*\,i}$ and a_{*j} contains a non zero entry.

More generally if $(a_{i*})\,(a_{j*})$ is the entrywise product of the two rows a_{i*} and a_{j*}, then from Proposition 8 it derives that:

Theorem 7. *A hypergroup H_ρ is a join one, if and only if*

(i) whenever an entry (j,i) is 1, then the row vector $(a_{i*}+a_{l*})\,(a_{j*}+a_{k*})$ is not the zero one, for all the row vectors a_{l*}, a_{k*} of M_ρ.

(ii) the entrywise product of two rows a_{i*} and a_{j*} of M_ρ contains a non zero entry whenever the entrywise product of the corresponding columns $a_{*\,i}$ and a_{*j} contains a non zero entry.

Corrolary 3. *The Rosenberg hypergroup which derives from the minimum basic matrix is a join one.*

Relevant Propositions hold for the hypercompositional structures which are defined by a binary relation ρ through the hypercomposition "$\circ$" e. g. from Proposition 6 it derives that:

Theorem 8. $(H,\circ)$ *is a hypergroup if and only if*

(i) M_ρ consists only of non zero rows and non zero columns

(ii) whenever a row of the matrix M_{ρ^2} has a zero entry, it coincides with the same row of M_ρ.

Thus a principle of duality folds between the two hypercompositions "$\bullet$" and "$\circ$":

Given a theorem, the dual statement, which results from the interchanging of one hypercomposition with the other, is also a theorem.

Hence:

Theorem 9. *The hypergroup $(H,\circ)$ is a join one, if and only if*

(i) whenever an entry (i,j) is 1, then the column vector $(a_{*i}+a_{*l})\,(a_{*j}+a_{*k})$ is not the zero one, for all the column vectors a_{*l}, a_{*k} of M_ρ.

(ii) the entrywise product of two columns $a_{*\,i}$ and a_{*j} of M_ρ contains a non zero entry whenever the entrywise product of the corresponding rows a_{i*} and a_{j*} contains a non zero entry.

Next, as to when two hypergroupoids generated by binary relations, are isomorphic, the answer has been given in [10] by the following Proposition and Theorem:

Proposition 12. *If in the Boolean matrix M_ρ the i and j rows are interchanged and, at the same time, the corresponding i and j columns are interchanged as well, then the deriving new matrix and the initial one, give isomorphic hypergroupoids.*

Theorem 10. *If the Boolean matrix M_σ derives from M_ρ by interchanging rows and the corresponding columns, then the hypergroupoids H_σ and H_ρ are isomorphic.*

4. Mathematica packages

The Mathematica [17] packages follow

4.1. Counting all HyperGroups

The function `Good[di]` returns the Boolean matrices that form a hypergroupoid

```
Good[di_] :=
Module[{c, i1, z},
        c = Tuples[Tuples[{0, 1}, di], di];
        z = Table[Min[Flatten[
                  - c[[i1]] + Sign[c[[i1]].c[[i1]]]]]*2^(di*di)
                  + Length[Position[c[[i1]], Table[0, {i2, 1, di}]]],
            {i1, 1, 2^(di*di)}];
        Return[c[[Flatten[Position[z, 0]]]]]
];
```

For example the 8 Boolean matrices of 2nd order that give hypergroupoids are the following ones

```
In[1]:=Good[2]
Out[1]:={{{0, 1}, {0, 1}}, {{0, 1}, {1, 1}},
         {{1, 0}, {0, 1}}, {{1, 0}, {1, 0}},
         {{1, 0}, {1, 1}}, {{1, 1}, {0, 1}},
         {{1, 1}, {1, 0}}, {{1, 1}, {1, 1}}}
```

The results of the enumeration of hypergroupoids of order 2, 3, 4, 5 are as follows

```
In[2]:= Length[Good[2]]
Out[2]= 8
In[3]:= Length[Good[3]]
Out[3]= 236
In[4]:= Length[Good[4]]
Out[4]= 28023
In[5]:= Length[Good[5]]
Out[5]= 13419636
```

The code that follows constructs a hypergroupoid from a Boolean Matrix

```
HyperGroupoid[a_List, order_] :=
   Table[Table[Complement[
             Union[Sign[a[[i1]] + a[[i2]]]*Table[j3,
                                     {j3, 1, order}]
                   ], {0}],
             {i2, 1, order}],
        {i1, 1, order}];
```

Example: The 99th Boolean matrix of 3rd order that results to a hypergroupoid is the following one

```
In[6]:=Good[3][[99]] // MatrixForm
```

$$\texttt{Out}[6] = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

The hypergroupoid which derives from the above matrix is the next one

```
In[7]:=HyperGroupoid[Good[3][[99]], 3] // MatrixForm
```

$$\texttt{Out}[7] = \begin{pmatrix} \{1\} & \{1,2\} & \{1,3\} \\ \{1,2\} & \{1,2\} & \{1,2,3\} \\ \{1,3\} & \{1,2,3\} & \{1,3\} \end{pmatrix}$$

The function `GoodH[di]` returns the Boolean matrices that form a hypergroup.

```
GoodH[di_] :=
Module[{c, i1, z, h2, outer, indexes},
       c = Tuples[Tuples[{0, 1}, di], di];
       z = Table[ Min[Flatten[-c[[i1]]
            + Sign[c[[i1]].c[[i1]]]]]*2^(di*di)
            + Length[Position[c[[i1]], Table[0, {i2, 1, di}]]]
            + Length[Position[Transpose[c[[i1]]],
                   Table[0, {i2, 1, di}]]], {i1, 1, 2^(di*di)}];
       h2 = c[[Flatten[Position[z, 0]]]];
       outer = Table[Complement[
               Sign[di - Total[Sign[h2[[j1]].h2[[j1]]]]]*
               Table[j3, {j3, 1, di}], {0}], {j1, 1, Length[h2]}];
       indexes = Complement[Range[1, Length[h2]],
                        Flatten[Position[
                          Table[Max[
                            Sign[h2[[j1]].h2[[j1]]][[All,outer[[j1]]]]
                          - h2[[j1]][[All, outer[[j1]]]]],
                           {j1, 1, Length[h2]}], 1]]
                        ];
Return[h2[[indexes]]]
];
```

For example we get the 6 Boolean matrices of 2nd order that form a hypergroup

```
In[8]:=GoodH[2]
Out[8]:={{{0, 1}, {1, 1}}, {{1, 0}, {0, 1}}, {{1, 0}, {1, 1}},
        {{1, 1}, {0, 1}}, {{1, 1}, {1, 0}}, {{1, 1}, {1, 1}}}
```

Enumeration of hypergroups

```
In[9]:= Length[GoodH[2]]
Out[9]= 6
In[10]:= Length[GoodH[3]]
Out[10]= 149
In[11]:= Length[GoodH[4]]
Out[11]= 9729
In[12]:= Length[GoodH[5]]
Out[12]= 2921442
```

These are the only hypergroups that derive from the hypercompositions which are defined from binary relations.

With a small modification of the above codes we found the join hypergroups of orders $2, 3, 4$, and 5 to be $5, 106, 6979$ and 2122681 respectively.

4.2. Counting NonIsomorphic Hypergroups

The packages that enumerate the NonIsomorphic classes follow. `IsomorphTest1` returns all isomorphisms of a Matrix.

```
IsomorphTest1[a_List] :=
Module[{p, a1},
        p = Permutations[Range[1, Length[a]]];
        Return[Table[a1 = a;
                     a1 = ReplaceAll[a1, a1[[All, Table[j2,
                                     {j2, 1, Length[a1]}]]] ->
                                         a1[[All, p[[j1]]]]];
                          ReplaceAll[a1, a1[[Table[j2,
                                     {j2, 1, Length[a]}]]] ->
                                         a1[[p[[j1]]]]],
                     {j1, 1, Length[p]}]
              ]]
```

Let's see the six permutations of the matrix

$$M_\rho = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

which are defined by corresponding binary relations, that give isomorphic hypergroupoids:

```
In[13]:= IsomorphTest1[{{1, 0, 1}, {1, 1, 0}, {0, 1, 1}}]
Out[13]:= {{{1,0,1}, {1,1,0}, {0,1,1}}, {{1,1,0}, {0,1,1}, {1,0,1}},
          {{1,1,0}, {0,1,1}, {1,0,1}}, {{1,0,1}, {1,1,0}, {0,1,1}},
          {{1,0,1}, {1,1,0}, {0,1,1}}, {{1,1,0}, {0,1,1}, {1,0,1}}}
```

Now we count the isomorphic classes of the hypergroupoids

```
Cardin[d_] :=
Module[{h2, cardinalities, len, temp1, temp},
  h2 = Good[d];
  cardinalities = Table[0, {j1, 1, Factorial[d]}];
  While[Length[h2] > 0,
        temp = Union[IsomorphTest1[h2[[1]]]];
        len = Length[Union[temp]];
        cardinalities[[len]] = cardinalities[[len]] + 1;
        h2 = Complement[h2, temp]
       ];
Return[cardinalities]]
```

Then we get

```
In[14]:= Cardin[2]
Out[14]:= {2, 3}
In[15]:= Total[%]
Out[15]:= 5
In[16]:= Cardin[3]
Out[16]:= {3, 1, 13, 0, 0, 32}
In[17]:= Total[%]
Out[17]:= 49
In[18]:= Cardin[4]
Out[18]:= {3, 0, 2, 17, 0, 15, 0, 8, 0, 0, 0, 238,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1039}
In[19]:= Total[%]
Out[19]:= 1322
In[20]:= Cardin[5]
Out[20]= ...
In[21]:= Total[%]
Out[21]= 117534
```

By changing the line `h2=Good[di]` in the above function `Cardin[]` with the line `h2=GoodH[di]` we get the isomorphic classes of the Hypergroups.

```
In[22]:= Cardin[2]
Out[22]= {2, 2}
In[23]:= Total[%]
Out[23]= 4
In[24]:= Cardin[3]
Out[24]= {3, 1, 10, 0, 0, 19}
In[25]:= Total[%]
Out[25]= 33
In[26]:= Cardin[4]
Out[26]= {3, 0, 2, 11, 0, 12, 0, 5, 0, 0, 0, 139,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 329}
In[27]:= Total[%]
Out[27]= 501
In[28]:= Cardin[5]
Out[28]= ...
In[29]:= Total[%]
Out[29]= 26409
```

5. Conclusion

This paper shows that there exist lots of Rosenberg-type hypercompositional structures, the number of which is calculated with the use of Mathematica packages that are constructed for this purpose. The results of these calculations are given in the cumulative Table–1 below for the orders 2, 3, 4 and 5. Because of the principle of Duality enunciated above, the same number of hypercompositional structures exists for the dual hypercomposition

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Table 1: Cumulative results

order→	2	3	4	5
Boolean Matrices (BM)	16	512	65536	33554432
BM forming Hypergroupoids	8	236	28023	13419636
Nonisomorphic BM forming Hypergroupoids	5	49	1322	117534
BM forming Hypergroups	6	149	9729	2921442
Nonisomorphic BM forming Hypergroups	4	33	501	26409
BM forming join Hypergroups	5	106	6979	2122681

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