Optimized Runge-Kutta pairs for problems with oscillating solutions

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Abstract

Three type of methods for integrating periodic initial value problems are presented. These methods are i) phase-fitted, ii) zero dissipation iii) both zero dissipative and phase-fitted. Some particular modifications of well known explicit Runge-Kutta pairs of orders five and four are constructed. Numerical experiments show the efficiency of the new pairs in a wide range of oscillatory problems.

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1 Introduction

Explicit Runge-Kutta (RK) pairs are widely used for the numerical solution of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \in \Re^m, \quad x \in [x_0, x_e] \tag{1}$$

where $f : \Re \times \Re^m \mapsto \Re^m$. These pairs are characterized by the extended Butcher tableau [1, 6]:

$$\begin{array}{c|c} c & A \\ \hline & b \\ & \hat{b} \end{array}$$

with b^T, $\hat{b}^T$, $c \in \Re^s$ and $A \in \Re^{s\times s}$ is strictly lower triangular. The procedure that advances the solution from (x_n, y_n) to $x_{n+1} = x_n + h_n$ computes at each step two approximations y_{n+1}, $\hat{y}_{n+1}$ to $y(x_{n+1})$ of orders p and $p-1$ respectively, given by

$$y_{n+1} = y_n + h_n \sum_{i=1}^{s} b_i f_{ni}$$

and

$$\hat{y}_{n+1} = y_n + h_n \sum_{i=1}^{s} \hat{b}_i f_{ni},$$

with

$$f_{ni} = f(x_n + c_i h_n, y_n + h_n \sum_{j=1}^{i-1} a_{ij} f_{nj})$$

for $i = 1, 2, .., s$. From this embedded form we can obtain an estimate $E_{n+1} = \|y_{n+1} - \hat{y}_{n+1}\|$ of the local truncation error of the $p-1$ order formula. So the step-size control algorithm

$$h_{n+1} = 0.9 \cdot h_n \cdot (\frac{\mathrm{TOL}}{E_{n+1}})^{1/p},$$

is in common use, with TOL being the requested tolerance. The above formula is used even if TOL is exceeded by E_{n+1}, but then h_{n+1} is simply the recomputed current step. See [16] for more details on the implementation of these type of step size policies.

2 Basic theory

The application of a RK method to the test problem

$$y' = i\omega y, \quad \omega \in \Re, i = \sqrt{-1}, \tag{2}$$

leads to the numerical scheme

$$y_{n+1} = P\left(i\omega h_n\right) y_n,$$

$h_n = x_{n+1} - x_n$, where the function $P\left(iv\right) = P\left(i\omega h\right)$ satisfies the relation

$$P\left(iv\right) = 1 + ivb\left(I - ivA\right)^{-1} e = \sum_{j=0}^{\infty} t_j \left(iv\right)^j, \tag{3}$$

and for $j \geq 1$, $t_j = bA^{j-1}e$, $t_0 = 1$, [9]. The numbers t_j depend only on the coefficients of the method. It must be observed that for explicit methods (that is for A lower triangular), the summation in the determination of $P\left(iv\right)$ above is finite (specifically, j runs from 0 through s).

The phase-lag (or dispersion) order of a RK method is defined as the order of approximation of the argument of the exponential function by the argument of P along the imaginary axis. Symbolically, the phase-lag order of a method is q, whenever $\delta\left(v\right) = O\left(v^{q+1}\right)$, for $\delta\left(v\right) = v - \arg\left(P\left(iv\right)\right)$. For RK methods this notion has been introduced in [19]. The imaginary stability interval of a RK method $I_I = \left(0, v_0\right)$ is defined by the relations $\left|P\left(iv\right)\right| < 1$ and $\left|P\left(v_0 + \theta\right)\right| > 1$, for every $v \in I_I$ and every suitably small positive θ. A method characterized by a non-vanishing imaginary stability interval is called dissipative.

Although for a RK method the phase-lag property is defined for the special problem (2), as it was shown by the numerical tests presented in [7, 9], RK pairs of high phase-lag order exhibit a remarkable numerical performance on a much wider class of test problems. It seems that for a certain class of initial value problems (as those whose solutions are described by free oscillations or free oscillations of low frequency with forced oscillations of high frequency superimposed, over long integration intervals), it might be advantageous to use pairs of methods of high phase-lag order with minimized their leading truncation error coefficients instead of pairs of the same algebraic order as the latter, but with a phase-lag order equal to the minimal allowed by the number of stages and their algebraic order.

The other interesting property is that of zero dissipation, which can not be applied to explicit RK methods. On the other hand it is straightforward to derive zero-dissipative explicit Runge-Kutta-Nyström methods [14], for problems of the form

$$y'' = f(x, y).$$

For problems of the type (1) we demand $|P(iv)| - 1 = O\left(v^{r+1}\right)$ to be as small as possible. In this case r is the dissipation order of the method, [15].

In both cases it is impossible to achieve $q = \infty$ or $r = \infty$. Thinks are better if a good estimate of the frequency ω is known in advance.

3 Methods with known frequency

The first who tried to fit a method to a set of linearly independent trigonometric functions was Gautschi [5]. Since then a lot of methods trying to do something similar have been constructed. Here we will exploit the knowledge of $v = \omega h$ in the direction we discussed in the previous section.

Observe that

$$\begin{aligned} P(iv) &= Q(v) + iR(v) = \\ &= \left(1 - t_2 v^2 + t_4 v^4 - t_6 v^6 \pm \cdots\right) + i\left(v - t_3 v^3 + t_5 v^5 - t_7 v^7 \pm \cdots\right). \end{aligned}$$

This series is finite for explicit methods.

If we require $\delta(v) = v - \arg(P(iv)) = 0$ then $\tan v = \frac{R(v)}{Q(v)}$ and finally

$$Q(v)\tan v = R(v)$$

holds, restricting just one t_{ji} to some expression of v. A new method can be derived solving all the order conditions and the equation for the restricted t_{ji}. Since the number of the stages of a method is greater than its order ($s > p$) when $p > 4$, then there is always some free t_j, $j > p$ to solve for. This method is a *phase-fitted* method. We note here that the term *phase-fitted* was first introduced by Raptis and Simos [22].

Then we may ask for $|P(iv)| \equiv 1$, i.e. $Q(v)^2 + R(v)^2 = 1$, leading to a nonlinear equation which can be solved for some t_j. Such a method is called *zero-dissipative*.

Combining both cases above we demand

$$Q(v) = \cos v, \;\; R(v) = \sin v.$$

In this case both $v - \arg(P(iv)) = 0$ and $|P(iv)| \equiv 1$ hold. The method is zero dissipative and phase-fitted, but we have two separated linear equations to solve. Simple expressions for some t_{ji} and t_{j+1} can be derived. Again the order conditions are solved along with the two new equations.

In the next section we shall seek explicit RK formulas of orders 5(4) and we will derive coefficients for the three cases we studied here.

4 Runge-Kutta pairs of orders 5(4)

We are interested for 5(4) pairs, that are the most popular ones [4, 3, 8, 2].

Then (3) takes the form

$$\begin{aligned} P &= \left(1 - \frac{1}{2}v^2 + \frac{1}{24}v^4 - t_6 v^6\right) + \sqrt{-1}\left(v - \frac{1}{6}v^3 + t_5 v^5\right) = \\ &\quad Q(v) + iR(v) \end{aligned}$$

If we have an estimation of v, then there are three choices.

4.1 $(\;) = \cos$ and $(\;) = \sin$

This leads to the selection

$$t_5 = \frac{\sin v - v + v^3/6}{v^5}$$

$$t_6 = \frac{1 - v^2/2 + v^4/24 - \cos v}{v^6}.$$

Of course $t_5 = \frac{1}{120}$ for a fifth order method but observe that the series expansion of t_5 and t_6 are

$$\begin{aligned} t_5 &= \frac{1}{120} - \frac{1}{5040}v^2 + \frac{1}{362880}v^4 - \frac{1}{3991680}v^6 + O\left(v^8\right), \\ t_6 &= \frac{1}{720} - \frac{1}{40320}v^2 + \frac{1}{3628800}v^4 - \frac{1}{479001600}v^6 + O\left(v^8\right). \end{aligned}$$

So from the series expansion and since $\lim_{v\to 0} t_5 = \frac{1}{120}$, the method satisfying the above expression is of fifth order.

An FSAL-type pair from the family of Dormand and Prince [3] uses seven stages per step. Since the first is reused in the next step, it effectively uses only six stages per step. We may express all the coefficients of such a pair with respect to the free parameters c_3, c_4, c_5 and $\widehat{b}_7$, with the assistance of a symbolic manipulation package [21]. Requiring $t_5 = bA^3c$ and $t_6 = bA^4c$ we concluded to a method with the following coefficients:

$$\begin{aligned} c_2 &= \frac{16}{75}, c_3 = \frac{8}{25}, c_5 = \frac{49}{50}, \\ c_4 &= \frac{(15 \cdot (2 - 540t_5 + 36000t_5^2 + 491t_6 - 55080t_5t_6))}{(16 \cdot (-1 + 144t_5)(-1 + 150t_5))} \end{aligned}$$

$$\begin{aligned} b_1 &= \frac{91 + 352c_4}{4704c_4}, b_3 = \frac{15625 \cdot (-19 + 48c_4)}{53856 \cdot (-8 + 25c_4)} \\ b_4 &= \frac{91}{12 \cdot (-1 + c_4) \cdot c_4 \cdot (-8 + 25c_4) \cdot (-49 + 50c_4)} \\ b_5 &= \frac{62500 \cdot (-7 + 9c_4)}{4851 \cdot (-49 + 50c_4)}, b_6 = -\frac{-307 + 398c_4}{204(-1 + c_4)} \end{aligned}$$

$$a_{32} = \frac{6}{25}, a_{42} = \frac{75c_4 \cdot (-75 + 213c_4 - 125c_4^2 + 9000t_5 - 27000c_4t_5 + 18000c_4^2t_5)}{4 \cdot (-491 + 55080t_5)}$$

$$a_{43} = \frac{-125c_4 \cdot (-8 + 25c_4) \cdot (15 - 8c_4 - 1800t_5 + 1152c_4t_5)}{16 \cdot (-491 + 55080t_5)}$$

$$a_{52} = \frac{-147 \cdot (28987 - 32121c_4 - 3031560t_5 + 3125520c_4t_5)}{800 \cdot (-7 + 9c_4) \cdot (-491 + 55080t_5)}$$

$$a_{53} = \frac{\begin{array}{c} 4851 \cdot (1820 + 13391c_4 - 17425c_4^2 - 1180760t_5 - 444824c_4t_5 \\ +1858200c_4^2t_5 + 107956800t_5^2 - 110160000c_4t_5^2) \end{array}}{320 \cdot (-7 + 9c_4) \cdot (-8 + 25c_4) \cdot (-491 + 55080t_5)}$$

$$a_{54} = \frac{1617 \cdot (-49 + 50c_4) * (-1 + 150t_5)}{1250 \cdot c_4(-7 + 9c_4)(-8 + 25c_4)}$$

$$a_{62} = \frac{-75 \cdot (14650 - 15833c_4 - 1530000t_5 + 1530000c_4t_5)}{4 \cdot (-307 + 398c_4)(-491 + 55080t_5)}$$

$$a_{63} = \frac{\begin{array}{c} 2125 \cdot (453650 + 2403463c_4 - 3214470c_4^2 - 248144400t_5 - 60259752c_4t_5 \\ +341485200c_4^2t_5 + 21811680000t_5^2 - 21811680000c_4t_5^2) \end{array}}{528 \cdot (-8 + 25c_4)(-307 + 398c_4)(-491 + 55080t_5)}$$

$$a_{64} = \frac{17 \cdot (-1 + c_4) \cdot (9891 - 10000c_4 - 1470000t_5 + 1500000c_4t_5)}{c_4(-8 + 25c_4)(-49 + 50c_4)(-307 + 398c_4)}$$

$$a_{65} = \frac{-85000 \cdot (-1 + c_4)(-7 + 9c_4)}{1617 \cdot (-49 + 50c_4)(-307 + 398c_4)}$$

$$\widehat{b}_3 = \frac{\begin{array}{c} 125 \cdot (-1218800 + 4435431c_4 - 3610497c_4^2 + 133260000t_5 \\ -482280000c_4t_5 + 388170000c_4^2t_5) \end{array}}{107712 \cdot (-8 + 25c_4)(235 - 289c_4 - 25800t_5 + 31200c_4t_5)}$$

$$\widehat{b}_4 = -\frac{-316400 + 505671c_4 - 142497c_4^2 + 34188000t_5 - 52872000c_4t_5 + 13770000c_4^2t_5}{120 \cdot (-1 + c_4)c_4(-8 + 25c_4)(-49 + 50c_4)(235 - 289c_4 - 25800t_5 + 31200c_4t_5)}$$

$$\widehat{b}_5 = \frac{125 \cdot (-7 + 9c_4)(102850 - 128667c_4 - 11370000t_5 + 14070000c_4t_5)}{4851 \cdot (-49 + 50c_4)(235 - 289c_4 - 25800t_5 + 31200c_4t_5)}$$

$$\widehat{b}_5 = -\frac{(-307 + 398c_4)(2055 - 2569c_4 - 227400t_5 + 281400c_4t_5)}{2040 \cdot (-1 + c_4)(235 - 289c_4 - 25800t_5 + 31200c_4t_5)}$$

$$\widehat{b}_7 = \frac{1}{40}, \widehat{b}_1 = \frac{39}{40} - \widehat{b}_3 - \widehat{b}_4 - \widehat{b}_5 - \widehat{b}_6$$

$$\begin{aligned} a_{21} &= c_2, a_{31} = c_3 - a_{32}, a_{41} = c_4 - a_{42} - a_{43}, a_{51} = c_5 - a_{52} - a_{53} - a_{54}, \\ a_{61} &= 1 - a_{62} - a_{63} - a_{64} - a_{65}, a_{7i} = b_i, i = 1, 2, ..., 6. \end{aligned}$$

4.2 $= \arg(\;(\;))$ Phase-fitted pair

Now we fix $t_5 = \frac{1}{120}$, getting

$$t_6 = \frac{120 - 60v^2 + 5v^4 + \cot v \cdot (-120v + 20v^3 - v^5)}{120v^6}$$

The corresponding series expansion (useful when say $v < .05$) is

$$t_6 = \frac{1}{840} + \frac{1}{22680}v^2 + \frac{1}{267300}v^4 + \frac{373}{1021620600}v^6 + O\left(v^8\right).$$

The coefficients of the pair are now even simpler expressions. The new coefficients are:

$$\begin{aligned} c_4 &= 600t_6 \\ a_{42} &= -2531250t_6^2(-1 + 1250t_6) \\ a_{43} &= 1125000t_6^2(-1 + 1875t_6) \end{aligned}$$

$$\begin{aligned} a_{52} &= \frac{-147 \cdot (-931 + 911250t_6)}{6400 \cdot (-7 + 5400t_6)} \\ a_{53} &= \frac{1617 \cdot (49 - 114420t_6 + 65475000t_6^2)}{2560 \cdot (-1 + 1875t_6) \cdot (-7 + 5400t_6)} \\ a_{54} &= \frac{539 \cdot (-49 + 30000t_6)}{8000000t_6 \cdot (-1 + 1875t_6) \cdot (-7 + 5400t_6)} \end{aligned}$$

$$a_{62} = \frac{-1875 \cdot (-19 + 18498 t_6)}{32 \cdot (-307 + 238800 t_6)}$$
$$a_{63} = \frac{10625 \cdot (311 - 724872 t_6 + 414855000 t_6^2)}{2112 \cdot (-1 + 1875 t_6) \cdot (-307 + 238800 t_6)}$$
$$a_{64} = \frac{17 \cdot (-1 + 600 t_6) \cdot (-2359 + 1500000 t_6)}{4800 t_6 \cdot (-1 + 1875 t_6) \cdot (-49 + 30000 t_6) \cdot (-307 + 238800 t_6)}$$
$$a_{65} = \frac{-85000 \cdot (-1 + 600 t_6) \cdot (-7 + 5400 t_6)}{1617 \cdot (-49 + 30000 t_6) \cdot (-307 + 238800 t_6)}$$

$$\widehat{b}_3 = \frac{625 \cdot (361 - 832862 t_6 + 450896400 t_6^2)}{287232 \cdot (1 - 2745 t_6 + 1631250 t_6^2)}$$
$$\widehat{b}_4 = \frac{-105 + 130142 t_6 - 33296400 t_6^2}{38400 t_6 \cdot (-1 + 600 t_6)(-1 + 870 t_6)(-1 + 1875 t_6)(-49 + 30000 t_6)}$$
$$\widehat{b}_5 = \frac{625 \cdot (-7 + 5400 t_6)(-27 + 22834 t_6)}{1617 \cdot (-1 + 870 t_6)(-49 + 30000 t_6)}$$
$$\widehat{b}_6 = -\frac{(-1 + 840 t_6)(-307 + 238800 t_6)}{255(-1 + 600 t_6)(-1 + 870 t_6)}$$

All the other coefficients do not depend on t_5 and remain the same.

4.3 ( ) 1 zero dissipative method

Fixing again $t_5 = \frac{1}{120}$, we get

$$t_6 = \frac{1}{120 v^6} \left(120 - 60 v^2 + 5 v^4 - \sqrt{14400 - 14400 v^2 + 4800 v^4 - 640 v^6 + 40 v^8 - v^{10}} \right)$$

with corresponding series expansion

$$t_6 = \frac{1}{720} + \frac{1}{5760} v^2 + \frac{11}{172800} v^4 + \frac{53}{2073600} v^6 + \frac{43}{4147200} v^8 + O\left(v^{10}\right).$$

The coefficients have exactly the same expressions as in the previous case.

The implementation described in this section may extended to sixth order pairs since we have explicit algorithms for the derivation of the coefficients [10, 13].

5 Numerical Results

5.1 The methods

The methods chosen to be tested are:

1. The Runge-Kutta pair of orders 5(4) due to Dormand and Prince [3]. This pair is one of the most used RK pairs in the literature and can be found as the ode45 file with matlab [2].

2. The trigonometric fitted RK5(4) pair given here.

3. The phase-fitted RK5(4) pair given here.

4. The zero dissipative RK5(4) pair given here.

5. The fourth order exponentially fitted RK method of [18].

The pairs 1-4 were run for tolerances $10^{-3}, 10^{-4}, \cdots, 10^{-9}$ in variable step-size mode while the fifth method were run with constant step through the interval of integration.

5.2 The problems

Five well known problems from the literature were chosen for our tests.

5.2.1 Bessel equation

$$y'' = -\left(100 + \frac{1}{4x^2}\right) y,$$

with initial conditions $y(1) = -0.2459357644513483$, $y'(1) = -0.5576953439142885$, for $x \in [1, 32.59406213134967]$.

The theoretical solution of this problem is $y(x) = \sqrt{x} J_0(10x)$. The 100th zero of this problem was observed for $x = 32.59406213134967$, [17]. We used $\omega = 10$ for derivation of the coefficients of fitted methods.

5.2.2 Inhomogeneous equation

$$y'' = -100y + 99 \sin x,$$

with $y(0) = 1,\ y'(0) = 11$ for $x \in [0, 20\pi]$.

Its theoretical solution is $y(x) = \cos 10x + \cos x + \sin x$, and the estimation $\omega = 10$ was used for this problem.

5.2.3 Duffing equation

$$y'' = -y - y^3 + .002 \cos 1.01x,$$

with $y(0) = 0.200426728067,\ y'(0) = 0$, for $x \in [0, 24.5\pi/1.01]$.

Theoretical solution, [20]:

$$\begin{aligned} y(x) &= 0.200179477536 \cos 1.01x + 2.46946143 \cdot 10^{-4} \cos 3.03x \\ &\quad +3.04014 \cdot 10^{-7} \cos 5.05x + 3.74 \cdot 10^{-10} \cos 7.07x + \cdots \end{aligned}$$

where the rest coefficients are smaller than 10^{-12}. Here we used $\omega = 1$, while for the exponentially fitted method we used $\omega = 1.01$ [18].

5.2.4 Hyperbolic problem

The hyperbolic PDE,

$$\begin{aligned} \frac{\vartheta u}{\vartheta x} &= \frac{\vartheta u}{\vartheta r},\ u(x,0) = 0,\ u(0,r) = \sin \pi^2 r^2, \\ 0 &\le r \le 1,\ x \ge 0 \end{aligned}$$

is discretisized by symmetric differences (with $\Delta r = 1/50$) to the system of ODEs

$$\begin{bmatrix} y_1' \\ y_2' \\ \\ \\ y_{50}' \end{bmatrix} = \frac{1}{2 \cdot 50} \begin{bmatrix} 0 & -1 & & & \\ 1 & 0 & -1 & & \\ & & & & \\ & & 1 & 0 & -1 \\ & & -1 & 4 & -3 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \\ \\ y_{50} \end{bmatrix}.$$

In [19] it was found that the 500th zero of the 20th component in the above equation was reached for $x = 33.509996948$. So we integrated the methods to that point. There is not some dominant frequency for this problem. So we use a rough estimation of $\omega = 50$ which not fat from the greater eigenvalue of the problem.

5.2.5 Non Linear

$$y'' = -100y + \sin y,$$

with $y(0) = 0$, $y(0) = 1$ for $x \in [0, 20\pi]$. The analytic solution is not known but we have found that $y(20\pi) = 3.92823991 \cdot 10^{-4}$, [12]. We used $\omega = 10$ for this problem.

Notice that most of the problems we choose have oscillatory solutions, that are not described by trivial trigonometric solutions as the majority of problems in [11, 18]. The behavior of the methods with the frequency in some region around the correct value was

5.3 The tables

We notify the steps used and the value $-\log(\text{end-point error})$ at y. Especially for the hyperbolic problem we recorded end point error of y_{20}.

The results over the 5 problems were summarized in tables 1-5.

The 13 columns in these tables have the following meaning.

1st column, Tolerance

2nd column, DP5(4) steps

3nd column, DP5(4), error

4th-5th column, NEW5(4) steps and error, (1st case)

6th-7th column, NEW5(4) steps and error, (2nd case)

8th-9th column, NEW5(4) steps and error, (3nd case)

10th-11th column, NEW5(4) steps and error, (1st case and v is selected randomly in the range $[.9v_0, 1.1v_0]$, v_0 the true frequency. e.g. for the Bessel equation $v_0 = 10$.)

12th column, error of the exponentially fitted method using step-size that produces the same cost as 1st case.

Table 1: Bessel equation

	DP54		case1		case2		case3		case1-rand		[18]	
TOL	steps	error	steps	error	steps	error	steps	error	steps	error	error	error
10^{-3}	494	2.0	434	6.5	538	7.5	502	2.1	434	4.4	5.3	3.9
10^{-4}	761	3.1	705	7.5	827	9.2	728	3.3	705	5.5	6.2	4.8
10^{-5}	1075	4.1	1044	8.6	1222	10.5	1067	4.4	1040	7.5	6.9	5.4
10^{-6}	1602	5.3	1559	9.7	1912	11.2	1555	5.5	1558	8.7	7.6	6.1
10^{-7}	2408	6.4	2357	10.8	2808	11.9	2374	6.7	2357	9.0	8.3	6.8
10^{-8}	3714	7.6	3639	12	4347	13.6	3645	7.9	3639	10.4	9.0	7.6
10^{-9}	5734	8.7	5633	12.6	6751	>14	5644	9.1	5633	11.4	9.7	8.4

Table 2: Inhomogeneous equation

	DP54		case1		case2		case3		case1-rand		[18]	
TOL	steps	error	steps	error	steps	error	steps	error	steps	error	error	error
10^{-3}	1315	2.3	1234	6.8	1600	2.2	1282	2.1	1234	3.9	11.6	3.6
10^{-4}	1949	3.6	1889	7.7	2218	3.2	1851	3.2	1883	5.4	11.1	4.3
10^{-5}	2947	4.1	2799	9.3	3239	4.2	2823	4.4	2799	7.2	10.6	5.0
10^{-6}	4314	4.9	4244	9.9	5012	5.2	4331	5.6	4240	7.5	11.4	5.8
10^{-7}	6601	5.8	6473	11.9	7756	6.2	6481	6.7	6470	9.1	10.0	6.5
10^{-8}	10190	6.7	10018	12.4	11982	7.1	10031	7.9	10012	10.0	10.0	7.3
10^{-9}	16021	7.7	15718	13.1	18956	8.1	15723	9.1	15718	11.4	9.4	8.1

13th column, error of the exponentially fitted method as 12th column but with $v = 0.999v_0$ (the same results were observed for $v = 1.001v_0$.). For the Duffing equation this column was produced using $\omega = 1.00$. In case that there is not 13th column the results are identical with that of 12th column.

The results justify our effort. In all cases the trigonometric fitted pair performed very well. The phase-fitted pair performed better than the other methods in Bessel, Hyperbolic and Nonlinear equation. Zero dissipative pair alone did not presented as good results as the other pairs but it was better than DP5(4). Generally the best fitted pair gained about $1.5 - 5.5$ digits of accuracy at the same cost.

Table 3: Duffing equation

	DP54		case1		case2		case3		case1-rand		[18]	
TOL	steps	error	steps	error	steps	error	steps	error	steps	error	$\omega = 1.01$	$\omega = 1$
10^{-3}	76	2.5	76	3.4	78	2.4	76	2.2	76	3.5	4.3	2.9
10^{-4}	94	3.1	81	3.1	105	3.1	94	2.8	81	3.1	4.4	3.0
10^{-5}	149	4.7	138	5.0	173	4.2	146	4.1	138	5.0	5.4	4.0
10^{-6}	236	5.0	227	7.2	278	5.2	232	5.4	227	7.0	6.3	4.9
10^{-7}	374	5.8	364	7.3	443	6.2	368	6.7	364	7.3	7.2	5.8
10^{-8}	592	6.8	581	8.3	704	7.2	583	8.1	581	8.3	8.1	6.6
10^{-9}	938	7.7	922	9.3	1116	8.2	924	9.4	922	9.3	9.0	7.4

Table 4: Hyperbolic equation

	DP54		case1		case2		case3		case1-rand		[18]	
TOL	steps	error	steps	error	steps	error	steps	error	steps	error	error	error
10^{-3}	1113	1.2	447	2.1	1188	2.1	2094	1.8	508	2.5	-125	-132
10^{-4}	1129	1.9	457	3.1	1154	3.4	2305	3.1	519	3.4	-110	-114
10^{-5}	1152	3.0	527	4.8	1179	4.9	2562	4.0	570	4.4	-44	-46
10^{-6}	1363	3.8	1002	7.2	1459	6.7	1361	4.2	1000	6.1	4.9	4.6
10^{-7}	2042	4.7	1782	6.7	2283	8.0	2017	4.8	1782	6.5	5.2	5.2
10^{-8}	3235	5.8	3007	7.9	3722	8.8	3170	6.1	3007	7.7	5.2	5.2
10^{-9}	5125	7.0	4915	9.1	5993	9.5	5023	7.3	4915	9.0	5.2	5.2

Table 5: Non linear problem

	DP54		case1		case2		case3		case1-rand		[18]
TOL	steps	error	steps	error	steps	error	steps	error	steps	error	error
10^{-3}	840	1.6	731	3.8	898	4.9	858	1.7	725	4.0	3.1
10^{-4}	1393	2.7	1191	4.9	1344	6.5	1209	2.9	1191	4.6	3.9
10^{-5}	1818	3.8	1713	6.0	2123	7.8	1747	4.0	1729	5.9	4.5
10^{-6}	2687	4.8	2638	7.2	2990	8.9	2699	5.2	2639	7.0	5.3
10^{-7}	4177	6.1	3969	8.3	4690	9.9	3955	6.4	3968	8.1	6.0
10^{-8}	6154	7.2	6052	9.5	7215	11.0	6088	7.5	6053	9.3	6.7
10^{-9}	9510	8.4	9325	10.7	11168	12.3	9350	8.7	9332	10.5	7.5

The exponential fitted method was competitive only in Duffing equation using the exact value of ω. It showed some better results from DP5(4) in Bessel and Inhomogeneous equation but it was worse than DP5(4) in Hyperbolic and Nonlinear equation. Its results in Inhomogeneous equation were very peculiar. In Hyperbolic problem gave very bad results and it seems that it is not appropriate for problems without dominant frequency or for problems with solution not described by simple trigonometric functions. In case of a little error in estimation of ω, the results were disappointing.

The comparison of function evaluations used by RK pairs in comparison is common in relevant literature, but a little overhead is expected here from the evaluation of the coefficients every step. This overhead is meaningless for Hyperbolic equation and it is about 15% in smaller problems. e.g. for Inhomogeneous equation 16021 steps are required for DP5(4) to achieve accuracy $10^{-7.7}$, while trigonometric fitted pair needs only 1889 steps. Traditionally this means DP5(4) is about $\frac{16021-1889}{1889}-1 \approx 650\%$ more expensive.. Now including overhead we may admit an $\frac{16021-1889\cdot 1.15}{1889\cdot 1.15}-1 \approx 540\%$ difference in efficiency.

Anyway, $10-20\%$ difference in efficiency (or gaining half digit of accuracy) is very important for pairs of the same order [8, 10, 13, 9, 16].

6 Conclusion

Zero dissipative, trigonometric and phase fitted methods are presented in this article. We give the equations that ought to hold for each case. Then particular pairs of orders 5(4) are derived for each case. Exhaustive numerical results on problems with oscillatory solutions indicate that the new trigonometric and phase fitted pairs are appropriate for initial value problems with oscillatory solutions.

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