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FOUR-STEP, TWO-STAGE, SIXTH-ORDER, P-STABLE METHODS

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An implicit four-step, sixth-order, P-stable method for initial value problem of the form $y'' = f(x, y)$ is suggested. It is recommended for systems with stiff oscillatory solutions. Only two stages required per step, instead of three stage methods found until now in the literature.

1. The problem and the methods

Consider the special second order initial value problem

$$y'' = f(x, y), y(x_0) = y_0, y'(x_0) = y_0' \tag{1}$$

which is of continuous interest in many fields of sciences and engineering.

Numerical methods for solving (1), produce approximations over a set of points $y_i \approx y(x_0+ih)$, $i = 1,2,\ldots$ There are various types of popular methods that integrate numerically (1), such as Runge-Kutta-Nystrom (RKN) [3,8] or Stormer-Cowell (SC) [4,9]. Numerov (NU) [4, pg 464] formula is the classical two-step representative of implicit SC methods. Their coefficients can be derived using common interpolatory techniques. Many authors use modifications of NU [1,2,12] methods in order to achieve special characteristics for their suggestions. The classical way is using off-step nodes. This procedure can be applied to four step methods as well [5,10].

The new four step formula we propose for approximating $y_4 \approx y(x_0+4h)$ while y_0, y_1, y_2 and y_3 are given, is

$$\begin{aligned} & y_0 - \frac{9688}{2425} y_1 + \frac{14526}{2425} y_2 - \frac{9688}{2425} y_3 + y_4 = \\ & h^2\left(\frac{6061}{72750} y_0'' + \frac{24268}{36375} y_1'' - \frac{3}{2} y_2'' + \frac{24268}{36375} y_3'' + \frac{6061}{72750} y_4'' + \frac{1}{250} y_a''\right) \end{aligned} \tag{2}$$

with off step node

$$\begin{aligned} y_a = {} & \frac{3}{5} y_0 - \frac{7}{3} y_1 + \frac{67}{15} y_2 - \frac{7}{3} y_3 + \frac{3}{5} y_4 + \\ & h^2\left(-\frac{9}{40} y_0'' + \frac{1511}{180} y_1'' - \frac{556}{45} y_2'' + \frac{1511}{180} y_3'' - \frac{9}{40} y_4''\right) \end{aligned} \tag{3}$$

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It is obvious that two functions evaluations are needed every step, namely y''_a and y''_4.

2. Zero Stability and Accuracy

The method (2-3) is zero stable since the polynomial

$$p(x) = 1 - \frac{9688}{2425}x + \frac{14526}{2425}x^2 - \frac{9688}{2425}x^3 + x^4$$

has four roots on the unit circle and only two of them are equal to 1. There are no stability requirements for y_a.

The method is of sixth order of accuracy since $y_4 = y(x_0+4h) + O(h^8)$. To verify this observe first that $y_a = y(x+2h) + O(h^6)$. Taking this in account we may expand (2) in Taylor series to get the desired result. We may consider the new method as a symmetric one because y''_a can grouped with y''_2 in (2).

3. P-stability

Lambert and Watson [6], used the test equation

$$y'' = -\lambda^2 y, \qquad (4)$$

for introducing the concept of the so-called interval of periodicity.

Especially when applying a symmetric four-step numerical method, like the one introduced here by formulas (2-3), to the problem (4) we obtain a difference equation, with characteristic equation of the form

$$p_1(v^2) + p_2(v^2)\cdot x + p_3(v^2)\cdot x^2 p_2(v^2)\cdot x^3 + p_1(v^2)\cdot x^4 = 0 \qquad (5)$$

where $v = \lambda h$ and $p_1(v^2)$, $p_2(v^2)$, $p_3(v^2)$ polynomials in v^2.

The method considered here is said to have an interval of periodicity $(0, v_0^2)$ if for all $0 < v < v_0$ the roots $r_1(v)$, $r_2(v)$, $r_3(v)$, $r_4(v)$ of (5) satisfy:

$$|r_1| = |r_2| = 1,\ |r_3| \le 1,\ |r_4| \le 1$$

A method is said P-stable if its interval of periodicity is $(0,\infty)$. It is obligatory for such methods to be implicit.

Many authors have constructed P-stable methods. Papageorgiou et. al. gave a fifth order RKN method [7]. Cash [1], Chawla and Rao [2] derived sixth order hybrid Numerov methods while Simos and Tsitouras were the first who produced eigth order methods of this type. Jain et. al. [5], constructed an implicit, P-stable, four-step method sharing three stages per step. Its local truncation error is

$$\text{LTE} \approx 0.00018\cdot h^8 + O(h^9)$$

while the truncation error of the new method is

$$\text{LTE} \approx 0.00464 \cdot h^8 + O(h^9)$$

Considering the efficiency of those methods as [11],

$$\text{Efficiency} = \#\text{stages} \cdot \text{PTE}^{1/8},$$

where PTE the principal term in local truncation error, we may conclude that both methods are of comparable efficiency.

The method derived in [5] was a two-parameter modification of a sixth order implicit method by Lambert and Watson [6]. Our new method has five free parameters even if it requires only two stages. This is due to full exploitation of all possible coefficients appearing in a two-stage method. So it is believed that a better method could be derived with a comprehensive search.

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