Quadratic Störmer–type methods for the solution of the Boussinesq equation by the methods of lines.

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We study the numerical treatment of Boussinesq PDE equation using the methods of lines. For the space discretization we choose either Classical Finite differences or Fourier pseudospectral methods. Both cases result in a system of second order ordinary differential equations (ODEs) that is quadratic. In order to take advantage of this special feature, we choose to solve the ODE system using a new type of hybrid Numerov method specially constructed for such problems. Other efficient ODE solvers taken from the literature are used to solve the system of ODEs as well. By taking all the combinations of space discretization methods and ODE solvers we discuss the stability and accuracy features revealed from the numerical tests.

Keywords: Differentiation matrices, Finite differences, Quadratic ODEs, two-step methods, order conditions.

I. INTRODUCTION

The Boussinesq nonlinear equation that describes motions of long waves in shallow water under gravity and in one dimensional nonlinear lattice, is given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + q\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 \left(u^2\right)}{\partial x^2} \tag{1.1}$$

$$L_0 < x < L_1, \quad t > t_0$$

where $u = u\,(x, t)$ and $|q| = 1$ is a real parameter. Taking $q = -1$ gives the good Boussinesq equation (GB) while taking $q = 1$ we get the bad Boussinesq equation (BB).

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It is assumed that the initial displacement

$$u(x,t_0) = g_1(x);\ L_0 \le x \le L_1,$$

and the initial velocity

$$\frac{\partial u(x,t_0)}{\partial t} = g_2(x),\ L_0 \le x \le L_1,$$

are known.

According to Manoranjan et. al. [8], the function

$$u(x,t) = q_1\{Q\text{sech}^2(\sqrt{\frac{Q}{6}}(x-ct+x^o)) + b - \frac{q_1}{2}\} \tag{1.2}$$

satisfies (1.1). In the above equation Q is the amplitude of the pulse, b is an arbitrary parameter, x^o is the initial position of the pulse and

$$c = \pm\sqrt{2q_1 \cdot (b + Q/3)}$$

is the velocity. If $q_1 = 1$ then (1.2) satisfies the single soliton BB, while for $q_1 = -1$ satisfies the GB equation.

There is a constant interest in numerical methods for solving (1.1). Abrosi [1], Bratsos [2], Feng [5], Wazwaz [12] have dealt with this subject producing mainly finite difference methods. In [11] an earlier version of the current was presented.

II. THE METHOD OF LINES

A. Finite differences

If we semi-disrcetize (1.1) along space using central differences we obtain a system of ordinary differential equations of the form

$$U'' = (R + q \cdot T) \cdot U + R \cdot U^* \tag{2.1}$$

where

$$\begin{aligned} U &= [U_0\, U_1\ \cdots U_N]^T \\ &\cong [u(L_0,t)\ u(L_0+h,t) \cdots u(L_1,t)]^T, \end{aligned}$$

is a vector approximation of $u(x,t)$ at the lines $L_0, L_0+h, \cdots, L_1-h, L_1$ with step length $h = (L_1 - L_0)/N$.

Whereas we define the vectors

$$U^* = [U_0^2\ U_1^2\ \cdots U_N^2]^T,$$

and

$$\begin{aligned} U'' &= [U_0''\, U_1'' \cdots U_N''] \\ &\cong [\frac{\partial^2 u(L_0,t)}{\partial t^2}\ \frac{\partial^2 u(L_0+h,t)}{\partial t^2}\ \cdots \frac{\partial^2 u(L_1,t)}{\partial t^2}] \end{aligned}$$

as well. The matrices R and T are $O\left(h^4\right)$ finite difference approximations for second and fourth derivatives respectively having the form

$$R = \frac{1}{h^2} \cdot \begin{bmatrix} \frac{35}{12} & -\frac{26}{3} & \frac{19}{2} & -\frac{14}{3} & \frac{11}{12} & & \cdots\ 0 \\ \frac{11}{12} & -\frac{5}{3} & \frac{1}{2} & \frac{1}{3} & -\frac{1}{12} & & \\ -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & & \\ & \ddots & & \ddots & & \ddots & \\ & & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} \\ & & -\frac{1}{12} & \frac{1}{3} & \frac{1}{2} & -\frac{5}{3} & \frac{11}{12} \\ 0 & \cdots & \frac{11}{12} & -\frac{14}{3} & \frac{19}{2} & -\frac{26}{3} & \frac{35}{12} \end{bmatrix},$$

and

$$T = \frac{1}{h^4} \cdot \begin{bmatrix} \frac{35}{6} & -31 & \frac{137}{2} & -\frac{242}{3} & \frac{107}{2} & -19 & \frac{17}{6} & \cdots & 0 \\ \frac{17}{6} & -14 & \frac{57}{2} & -\frac{92}{3} & \frac{37}{2} & -6 & \frac{5}{6} & & \\ \frac{5}{6} & -3 & \frac{7}{2} & -\frac{2}{3} & -\frac{3}{2} & 1 & -\frac{1}{6} & & \\ -\frac{1}{6} & 2 & -\frac{13}{2} & \frac{28}{3} & -\frac{13}{2} & 2 & -\frac{1}{6} & & \\ & & \ddots & & \ddots & & & \ddots & \\ & & -\frac{1}{6} & 2 & -\frac{13}{2} & \frac{28}{3} & -\frac{13}{2} & 2 & -\frac{1}{6} \\ & & -\frac{1}{6} & 1 & -\frac{3}{2} & -\frac{2}{3} & \frac{7}{2} & -3 & \frac{5}{6} \\ & & \frac{5}{6} & -6 & \frac{37}{2} & -\frac{92}{3} & \frac{57}{2} & -14 & \frac{17}{6} \\ 0 & \cdots & \frac{17}{6} & -19 & \frac{107}{2} & -\frac{242}{3} & \frac{137}{2} & -31 & \frac{35}{6} \end{bmatrix}$$

B. Pseudospectral methods

We may also solve equation (2.1) using Fourier pseudospectral differentiation matrices R and T [6]. First we discretesize the interval $[0, 2\pi]$ in N equistand points and compute in case that N is odd [13]:

$$R_{kj} = \begin{cases} -\frac{N^2+2}{12}, & k = j \\ -\frac{1}{2}\left(-1\right)^{k-j} \csc^2 \frac{(k-j)\pi}{N}, & k \neq j \end{cases}$$

For N even matrix R elements can be found in [13]. Now the matrix $T = R^2$.

Since (1.1) is posed on $[L_0, L_1]$ we convert it to $[0, 2\pi]$ through the linear transformation

$$x \longleftrightarrow L_0 + \frac{1}{2\pi}(L_1 - L_0)x.$$

So, matrix R becomes $(\frac{2\pi}{L_1 - L_0})^2 R$, and matrix T converts to $(\frac{2\pi}{L_1 - L_0})^4 T$.

III. HYBRID STÖRMER–TYPE METHODS

We consider the initial value problem of second order

$$U'' = f(U),\ U(t_0) = U^{[0]},\ U'(t_0) = -U'^{[0]}, \tag{3.1}$$

where $f : \Re^{N+1} \longrightarrow \Re^{N+1}$ and $U^{[0]},\ U'^{[0]} \in \Re^{N+1}$. Observe that U' is not involved in (3.1). The independent variable t can be considered as an extra component of U, setting

$$U''_{N+1} = 0,\ U^{[0]}_{N+1} = t_0,\ U'^{[0]}_{N+1} = 0.$$

For the last 20-25 years hybrid Numerov type methods have been widely used to solve this problem [3]. Recently a new methodology for deriving such methods has been proposed [9, 10] and studied more theoretically in [4].

In vector notation an s-stage Numerov type method takes the form

$$\begin{array}{c} U^{[k+1]} = 2U^{[k]} - U^{[k-1]} + \\ \lambda^2 \cdot \left(b^T \otimes I_s\right) \cdot f\left(Y\right) \\ Y = (e + c) \otimes U^{[k]} - c \otimes U^{[k-1]} + \\ \lambda^2 \cdot \left(A \otimes I_s\right) \cdot f\left(Y\right) \end{array} \tag{3.2}$$

where λ is the step forward in time, $e = [1\ 1\ 1 \cdots 1]^T \in \Re^s$ and I_s is the identity matrix in $\Re^{s\times s}$. The coefficients of the method are the two s-dimensional vectors b, c and the matrix $A \in \Re^{s\times s}$. In case that $a_{ij} = 0$ when $i \geq j$ the method is explicit.

The two step nature of the method is obvious from (3.2) since

$$\begin{aligned} U^{[k+1]} &= \left[U_0^{[k+1]}\ U_1^{[k+1]} \cdots U_N^{[k+1]}\right] \approx U\left(t_0 + k\lambda\right), \\ U^{[k]} &\approx U\left(t_0 + (k-1)\lambda\right), \\ U^{[k-1]} &\approx U\left(t_0 + (k-2)\lambda\right), \end{aligned}$$

etc.

A. Order conditions

The coefficients in b, c, A are derived after matching the corresponding Taylor expansions of the numerical scheme and the theoretical solution. This leads to an expression of the form

$$\begin{array}{c} q_{21}F_{21}\lambda^2 + q_{31}F_{31}\lambda^3 + \cdots \\ +(q_{71}F_{71} + \cdots + q_{7,10}F_{7,10})\lambda^7 \\ +(q_{81}F_{81} + \cdots + q_{8,21}F_{8,21})\lambda^8 + O(\lambda^9) \end{array} \tag{3.3}$$

where q's are expressions of the coefficients while F's are elementary differentials with respect to U', f and $\frac{\partial^m f}{\partial t^m}$ $m = 1, 2, \cdots$. In order to satisfy a seventh order of accuracy we have to nullify all coefficients of λ in (3.3) up to λ^8. So, various equations $q_{ij} = 0$ have to be solved such as

$$\begin{array}{lll} q_{11} = be - 1, & q_{21} = bc, & q_{31} = bc^2 - \frac{1}{6}, \\ q_{41} = bc^3, & q_{42} = bAc, & q_{51} = bc^4 - \frac{1}{12}, \text{ etc.} \end{array}$$

There are 43 in total equations to be solved for achieving 7th order [4].

B. A Particular quadratic method

Here we intend to solve (2.1) which is a problem in the form of (3.1). This problem is quadratic and we observe that many of the elementary differentials from (3.3) vanish (e.g. $F_{41} = 0,\ F_{51} = 0$). So, we may derive a special method that needs to satisfy a subset of the equations of the general case. Since q_{41}, q_{51} and many other equations of condition are discarded, only 14 equations remain to be solved, are given in Table I. In this table powers of vectors and the operation "*", may be understood as component-wise ones.

TABLE I. Order conditions for 7th order quadratic Numerov-type methods

$be = 1$
$bc = 0$
$bc^2 - \frac{1}{6}$
$bAc^2 = \frac{1}{180}$
$b\,(c * Ae) = -\frac{1}{12}$
$\frac{1}{2}b(Ae)^2 + \frac{1}{2}b(c * Ae) + \frac{1}{8}bc^2 = \frac{1}{120}$
$b(c * Ac^2) = \frac{1}{72}$
$bA^2c = 0$
$b.(Ae * Ac) + \frac{1}{6}b(c * Ae) + \frac{1}{2}b(c * Ac) = -\frac{1}{72}$
$\frac{1}{2}bA^2c^2 = \frac{2}{40320}$
$bA(c * Ac) + \frac{1}{6}bAc^2 + \frac{1}{180}bc = \frac{8}{40320}$
$b(c * A^2c) + \frac{1}{6}b(c * Ac) + \frac{1}{120}bc^2 = \frac{12}{40320}$
$\frac{1}{2}b(Ac)^2 + \frac{1}{6}b(c * Ac) + \frac{1}{72}bc^2 = \frac{20}{40320}$
$\frac{1}{2}b(Ae * Ac^2) + \frac{1}{24}b(c * Ae) + \frac{1}{4}b(c * Ac^2) + \frac{1}{48}bc^2 = \frac{1}{1344}$

A particular choice is given in Table II. The coefficients not reported in this table are zero.

Notice that $s = 6$ but the method uses only five new stages in every step because the $f(Y_1) = f(U^{[k-1]})$ has already been evaluated in the previous step.

IV. NUMERICAL RESULTS

In our numerical testing we integrate both Bad and Good Boussinesq equations for $x \in [-100, 100]$ and $t \in [0, 60]$.

TABLE II. Coefficients of the new 7th order method

$c_1 = -1, c_2 = 0,$	$c_3 = 0.3893598766411142$
$c_4 = -0.4099755503817072,$	$c_5 = 0.7848105925840123$
$c_6 = -0.8320502943378437$	
$b_1 = -0.00416400431581421,$	$b_2 = 0.3516135225291974$
$b_3 = 0.2541320627325857,$	$b_4 = 0.2632121979604164$
$b_5 = 0.07254557879412268,$	$b_6 = 0.06266064229949197$
$a_{31} = 0.05505541429588489,$	$a_{32} = 0.215425080793664$
$a_{41} = -0.05259964171766104,$	$a_{42} = -.07925025213127237$
$a_{43} = 0.01090209461347168,$	$a_{51} = -0.05941628917804282$
$a_{52} = 0.02348438379454109,$	$a_{53} = 0.2404644996321801$
$a_{54} = 0.495836535159362,$	$a_{61} = 0.06878346045410173$
$a_{62} = 0.2483265298658323,$	$a_{63} = -0.07412211033513141$
$a_{64} = -0.3229432175469921,$	$a_{65} = 0.01008403654711381$

We consider the initial displacement

$$U^{[0]} = u(x,0) = q_1\{Q\text{sech}^2(\sqrt{\frac{Q}{6}}x)\}$$

and we solve using three methods for the integration over time.

The first is the simplest Numerov Type method known as Strömer method

$$U^{[k+1]} = 2U^{[k]} - U^{[k-1]} + \lambda^2 \cdot f\left(U^{[k]}\right).$$

The second is the new Numerov Type method presented in Table II. The third method taken from [9] is of sixth algebraic order method and it was specially suited for oscillatory problems.

We combine these three methods with both the choices of Methods of Lines presented in this paper i.e. with finite differences and Fourier pseudospectral method. The mesh–size along space is taken to be $N = 64$ as for bigger values the numerical results face stability problems.

A. Bad Boussinesq

For the case of BB we consider the values of the theoretical parameters in (1.2) $Q = 0.369$, $b = 0.5$, $x^0 = 0$, $q_1 = 1$ and $c \approx 1.11624$.

We solve the BB equation for various choices of time step-size λ using the three methods and finite differences. We present in Table III the maximum absolute error to the grid of the numerical solution. We observe that for $\lambda = 3$, the Strömer method, and for $\lambda = 4$, the Numerov [9] method, become unstable. In contradiction the new Method is very stable and keeps the error considerably low as λ increases.

TABLE III. BB Maximum Error with Finite Differences

	Störmer	New Method	Numerov [9]
$\lambda = 0.1$	0.050	0.050	0.049
$\lambda = 0.5$	0.043	0.050	0.048
$\lambda = 1$	0.021	0.050	0.045
$\lambda = 2$	0.114	0.049	0.055
$\lambda = 3$	∞	0.046	0.274
$\lambda = 4$	∞	0.056	∞
$\lambda = 5$	∞	0.136	∞

When we solve the BB equation for various choices of time step-length λ using the three methods and Fourier Pseudospectral we get the best performance for all the methods, (Table IV). Though, as time step-size increases the maximum error of both Strömer method and Numerov [9] increases considerably. Moreover, we observe that the new method achieves an amount of error at a time–step almost $10 - 20$ times larger than its competitors. In addition, the new Method retains its remarkable stability as time step length increases whereas the other competitors become once again unstable.

TABLE IV. BB Maximum Error with Fourier Pseudospectral

	Störmer	New Method	Numerov [9]
$\lambda = 0.1$	0.0032	0.0032	0.0044
$\lambda = 0.5$	0.0085	0.0030	0.0128
$\lambda = 1$	0.0399	0.0036	0.0230
$\lambda = 2$	0.2310	0.0046	0.0863
$\lambda = 3$	∞	0.0065	1.5324
$\lambda = 4$	∞	0.0315	∞
$\lambda = 5$	∞	0.2535	∞

It is remarkable as time stepsize increases the numerical solution maintains its nice error characteristics.

B. Good Boussinesq

We employ the same numerical testing for the case o GB equation where the values of the theoretical parameters in (1.2) are $Q = 0.369$, $b = -0.5$, $x^0 = 0$, $q_1 = -1$, $c \approx 0.86833$. As we can see in Tables V and VI we have similar behavior even though the error is considerably bigger.

TABLE V. GB Maximum Error with Finite Differences

	Störmer	New Method	Numerov [9]
$\lambda = 0.1$	0.124	0.124	0.124
$\lambda = 0.5$	0.121	0.122	0.122
$\lambda = 1$	0.116	0.121	0.122
$\lambda = 2$	0.097	0.117	0.121
$\lambda = 3$	0.067	0.116	0.146
$\lambda = 4$	∞	0.114	∞
$\lambda = 5$	∞	0.120	∞

TABLE VI. GB Maximum Error with Fourier Pseudospectral

	Störmer	New Method	Numerov [9]
$\lambda = 0.1$	0.009	0.009	0.010
$\lambda = 0.5$	0.011	0.009	0.014
$\lambda = 1$	0.019	0.009	0.020
$\lambda = 2$	∞	0.009	∞
$\lambda = 3$	∞	0.032	∞
$\lambda = 4$	∞	∞	∞
$\lambda = 5$	∞	∞	∞

V. CONCLUSIONS

We implemented the the methods of lines for the numerical treatment of Boussinesq equations. Classical Finite Differences and Fourier pseudospectral methods were used to form a system of quadratic differential equations. We solved the system using a new type of hybrid Numerov method hoping to take advantage of its special form.

Numerical testing revealed quite interesting features. First of all, for all the methods chosen to solve over time, the choice of Fourier pseudospectral method gives a considerably smaller errors. In addition, the proposed Quadratic Numerov type method makes the overall numerical procedure remarkably stable even if time stepsize takes considerably big values.

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