Enumeration of hypercompositional structures defined by binary relations

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Abstract

This paper deals with hyperoperations that derive from binary relations and it studies the hypercompositional structures that are created by them. It is proved that if ρ is a binary relation on a non-void set H, then the hypercomposition $xy = \{z \in H : (x, z) \in \rho \text{ and } (z, y) \in \rho\}$ satisfies the associativity or the reproductivity only when it is total. There also appear routines that calculate (with the use of small computing power) the number of non isomorphic hypergroupoids, when the cardinality of H is finite.

Keywords: Binary relations, Mathematica

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1 Introduction

The theory of hypercompositional structures was born in 1934, when F. Marty introduced the notion of the hypergroup [5]. A hypergroup is a pair $(H, \cdot)$ where H is a non empty set and "$\cdot$" a hypercomposition, i.e. a mapping from $H \times H$ to the power set $P(H)$ of H, which satisfies the axioms:

i. $a(bc) = (ab)c$ for every $a, b, c \in H$ (associativity)

ii. $aH = Ha = H$ for every $a \in H$ (reproductivity)

In a hypergroup, the result of the hypercomposition is always a nonempty set. Indeed, suppose that for two elements $a, b \in H$ it holds that $ab = \varnothing$. Then $H = aH = a(bH) = (ab)H = \varnothing H = \varnothing$, which is absurd. Thus, if a non empty set H is endowed with a hypercomposition which does not satisfy the associative and the reproductive low, then the void set can possibly be among its results. A pair $(H, \cdot)$ where H is a non empty set and " $\cdot$ " a hypercomposition, is called partial hypegroupoid, while it is called hypegroupoid if $ab \neq \varnothing$, for all $a, b \in H$. A hypergroupoid in which the associativity is valid, is called semi–hypergroup, while it is called quasi–hypergroup if only the reproductivity holds.

Several papers dealing with the construction of hypergroupoids and hypergroups appear in the relevant bibliography, since hypergroups are much more varied than groups, e.g. for each prime number p, there exists only one group, up to isomorphism, with cardinality p, while the number of pairwise non isomorphic hypergroups is very large. For example there exist 3999 non isomorphic hypergroups with 3 elements

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[12]. Nieminen [8] studied hypergroups associated with graphs and G. G. Massouros studied hypergroups associated with automata [7]. Also Chvalina [1], Rosenberg [9] and Corsini [2] studied hypergroupoids and hypergroups defined in terms of binary relations. This paper deals with the hypergroupoids defined by Corsini, it proves that this family of hypergroupoids contains only one semihypergroup and only one quasihypergroup, the total hypergroup and enumerates the hypergroupoids with $2, 3, 4$ and 5 elements. The order n of a finite hypergroupoid H is defined to be the number of elements in the set H.
Let H be a non empty set and ρ a binary relation on H. Corsini introduced in H the hypercomposition.

$$x \cdot y = \{z \in H : (x, z) \in \rho \text{ and } (z, y) \in \rho\}. \tag{1}$$

With the above hypercomposition, $(H, \cdot)$ becomes a partial hypergroupoid, while it becomes a hypergroupoid if for each pair of elements $x, y \in H$, there exists $z \in H$ such that $(x, z) \in \rho$ and $(z, y) \in \rho$. Since $\rho^2 = \rho \circ \rho = \{(x, y) \in H^2 : (x, z), (z, y) \in \rho \text{ for some } z \in H\}$, it derives that $(H, \cdot)$ is a hypegroupoid if $\rho^2 = H^2$.

2 The hypercompositional structures defined by ρ

Let H_ρ denote the hypercompositional structure defined by (1) through the binary relation ρ. One can observe that the reproductivity is valid in H_ρ if and only if $(x, y) \in \rho$, for all $x, y \in H_\rho$. Indeed let x be an arbitrary element of H_ρ. For the reproductivity to be valid, it must hold: $y \in xH_\rho$, for all $y \in H_\rho$. Hence, for all $x, y \in H_\rho$, the pair (x, y) must belong to ρ. Thus:
***Proposition* 2.1:** H_ρ is a quasihypergroup, if and only if $(x, y) \in \rho$ for all $x, y \in H_\rho$.
Next suppose that H_ρ is a hypergroupoid. Then:
***Lemma* 2.1:** If H_ρ is a semihypergroup and $(z, z) \notin \rho$ for some $z \in H_\rho$, then $(s, z) \in \rho$ implies that $(z, s) \notin \rho$.
Proof: Suppose that $(s, z) \in \rho$ and $(z, s) \in \rho$. Then for zz and ss we have

$$zz = \{x \in H_\rho : (z, x) \in \rho \text{ and } (x, z) \in \rho\},$$

thus $s \in zz$ and,

$$ss = \{x \in H_\rho : (s, x) \in \rho \text{ and } (x, s) \in \rho\}$$

thus $z \in ss$. Now $z \in (zz)s$ since $ss \subseteq (zz)s$. But $z \notin z(zs)$, because:

$$z(zs) = z\{x \in H_\rho : (z, x) \in \rho \text{ and } (x, s) \in \rho\}$$

$$= \{y \in H_\rho : (z, y) \in \rho \text{ and } (y, x) \in \rho\}$$

and $(z, z) \notin \rho$. Hence the associativity is not valid, which contradicts the assumption that H_ρ is a semihypergroup.
***Corollary* 2.1:** If H_ρ is a semihypergroup and ρ is not reflexive, then ρ is not symmetric.
***Lemma* 2.2:** If H_ρ is a semihypergroup, then ρ is reflexive.
Proof: Suppose that $(x, x) \notin \rho$, for some $x \in H_\rho$. Then, according to Lemma 2.1, for every element t in H_ρ such that $(x, t) \in \rho$, it derives that $(t, x) \notin \rho$. But $xx = \{y \in H_\rho : (x, y) \in \rho \text{ and } (y, x) \in \rho\}$. Therefore $xx = \emptyset$, which is absurd, since H_ρ is a semihypergroup.

***Lemma* 2.3:** If any pair of elements of H_ρ does not belong to ρ, then H_ρ is not a semihypergroup.
Proof: According to Lemma 2.2, if $(x,x) \notin \rho$ for some $x \in H_\rho$, then H_ρ is not a semihypergroup. So, let t, z be two elements of H_ρ such that $t \neq z$ and $(t,z) \notin \rho$. Then:

$$t(tz) = t\{s \in H : (t,s) \in \rho \text{ and } (s,z) \in \rho\} = \{y \in H : (t,y) \in \rho \text{ and } (y,s) \in \rho\}$$

According to Lemma 2.2, it holds $(t,t) \in \rho$. Also $(t,s) \in \rho$. Therefore $t \in t(tz)$. On the other hand:

$$(tt)z = \{r \in H : (t,r) \in \text{ and } (r,t) \in \rho\}z = \{w \in H : (r,w) \in \text{ and } (w,z) \in \rho\}$$

Thus $(tt)z \subseteq \{w \in H : (w,z) \in \rho\}$, therefore $t \notin (tt)z$. Hence the associativity is not valid.
From the above series of lemmas, it derives that:

***Proposition* 2.2:** H_ρ is a semihypergroup if and only if $(x,y) \in \rho$, for all $x, y \in H_\rho$
Now, if $(x,y) \in \rho$ for all $x, y \in H_\rho$, then the hypercomposition which is defined through ρ is total, i.e. $xy = H_\rho$, for all $x, y \in H_\rho$. But if a hypercompositional structure is endowed with the total hypercomposition, then it is a hypergroup. Therefore, from Propositions 2.1 and 2.2, it derives that:

***Theorem* 2.1:** The only semihypergroup and the only quasihypergroup defined by the binary relation ρ is the total hypergroup.

3 Enumeration of the finite hypergroupoids

Every relation ρ in a finite set H with card$H = n$, is represented by a Boolean matrix M_ρ and conversely every $n \times n$ Boolean matrix defines in H a binary relation. Indeed, let H be the set $\{a_1, \cdots, a_n\}$. Then a $n \times n$ Boolean matrix is constructed as follows: the element (i,j) of the matrix is 1, if $(a_i, a_j) \in \rho$ and it is 0 if $(a_i, a_j) \notin \rho$ and vice versa. Hence, in every set with n elements, 2^{n^2} partial hypergroupoids can be defined.
Recall that in Boolean algebra it holds: $0+1 = 1+0 = 1+1 = 1$, while $0+0 = 0$. Also $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$ and $1 \cdot 1 = 1$. Let H_ρ be the above mentioned partial hypergroupoid, which is defined by a binary relation ρ. Then H_ρ is a hypergroupoid if and only if $M_\rho^2 = T$, where $T = (t_{ij})$ with $t_{ij} = 1$ for all i, j [2]. The matrix M_ρ is called good, if H_ρ is a hypergroupoid. Since the element c_{ij} of M_ρ^2 is equal to $\sum_{s=1}^{n} x_{is} y_{sj}$, it derives that matrices having a column or a row consisting only of 0 elements are not good.
Now from Proposition 2.1 it derives:
***Proposition* 3.1:** H_ρ is quasihypergroup if and only if $M_\rho = T$.

Also Proposition 2.2 gives:
***Proposition* 3.2:** H_ρ is semihypergroup if and only if $M_\rho = T$.

Hence the theorem holds:
***Theorem* 3.1:** The only relation ρ that gives a semihypergroup or a quasihypergroup is the one which has $M_\rho = T$, and so H_ρ is the total hypergroup.

Spartalis and Mamaloukas [11] wrote, in Visual Basic code, a 190-lines long program that enumerates the hypergroupoids associated with binary relations of orders 2, 3 and 4. Though, the following few lines of a Mathematica [13] program produces their results in a fraction of a second. It simply collects in variable `c` all the Boolean matrices of size n and computes their squares. Boolean minimum entry of these squares is recorded in table `z`. In return we count the nonzero elements of `z`.

```
Good[n_] :=
Module[{c, i1, z},
        c = Tuples[Tuples[{0, 1}, n], n];
        z = Table[Min[Flatten[c[[i1]].c[[i1]]]]
                    ,{i1, 1, 2^(n*n)}];
        Return[Count[z, _?Positive]]
      ]
```

Which gives:

```
In[1]:= Good[2]
Out[1]= 3
In[2]:= Good[3]
Out[2]= 73
In[3]:= Good[4]
Out[3]= 6003
In[4]:= Good[5]
Out[4]= 2318521
```

Thus it is confirmed that there exist $3, 73$ and 6003 binary relations that form a hypergroupoid of orders $2, 3$ and 4 respectively. It took the above program only a few minutes to count 2318521 hypergroupoids of order 5. For $n = 6$, the function `Good` fails due to memory restrictions of a small computer. One can proceed with a more slow but reliable package and form one by one the various Boolean matrices and their squares.

Remark: Notice that the above enumeration coincides with the enumeration of square roots of the total Boolean matrix, i.e. the Boolean matrix with all entries equal 1.

3.1 Isomorphisms

Naturally the question arises: When two hypergroupoids, are isomorphic?

***Proposition* 3.3:** If in the Boolean matrix M_ρ the i and j rows are interchanged and, at the same time, the corresponding i and j columns are interchanged as well, then the deriving new matrix and the initial one, give isomorphic hypergroupoids.

Proof: Suppose that $H = \{a_1, , a_n\}$ is a finite set and let $(H_{\rho_1}, \bullet_{\rho_1})$ be the hypergroupoid defined by a binary relation ρ_1. Let M_{ρ_1} be the Boolean matrix defined by ρ_1. Now suppose that the i and j rows and columns are interchanged and let M_{ρ_2} be the new Boolean matrix. Then a new binary relation ρ_2 is defined on H. Obviously for ρ_1 and ρ_2 it holds:

$$\begin{array}{lll} (a_k, a_i) \in \rho_1 \Longleftrightarrow (a_k, a_j) \in \rho_2 & (a_i, a_k) \in \rho_1 \Longleftrightarrow (a_j, a_k) \in \rho_2 & \text{if} k \neq i, j \\ (a_i, a_j) \in \rho_1 \Longleftrightarrow (a_j, a_i) \in \rho_2 & (a_j, a_i) \in \rho_1 \Longleftrightarrow (a_i, a_j) \in \rho_2 & \\ (a_i, a_i) \in \rho_1 \Longleftrightarrow (a_j, a_j) \in \rho_2 & (a_j, a_j) \in \rho_1 \Longleftrightarrow (a_i, a_i) \in \rho_2 & \end{array}$$

If $(H_{\rho_2}, \bullet_{\rho_2})$ is the hypercompositional structure defined by M_{ρ_2}, then the mapping $\phi : H_{\rho_1} \longrightarrow H_{\rho_2}$ with:

$$\phi(x) = \{ \begin{array}{l} x \text{ if } x \neq a_i, a_j \\ a_i \text{ if } x = a_j \\ a_j \text{ if } x = a_i \end{array}$$

is an isomorphism. Obviously ϕ is $1-1$ and onto. Next we distinguish the cases:

1. $\phi(a_i \bullet_{\rho_1} a_j) = \phi\{x \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\}$

   (a) If $a_i \bullet_{\rho_1} a_j \cap \{a_i, a_j\} = \emptyset$. Then
   $\phi\{x \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\} =$
   $\{\phi(x) \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\} =$
   $\{x \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\} =$
   $\{x \in H : (a_j, x) \in \rho_2 \text{ and } (x, a_i) \in \rho_2\} = a_j \bullet_{\rho_2} a_i = \phi(a_i) \bullet_{\rho_2} \phi(a_j)$

   (b) If $a_i \bullet_{\rho_1} a_j \cap \{a_i, a_j\} \neq \emptyset$. Assume e.g. that a_i belongs to $a_i \bullet_{\rho_1} a_j$, then
   $\phi\{x \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\} =$
   $\{\phi(x) \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\} =$
   $\{x \in H, x \neq a_i : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\} \bigcup \{a_j\} =$
   $\{x \in H : (a_j, x) \in \rho_2 \text{ and } (x, a_i) \in \rho_2\} = a_j \bullet_{\rho_2} a_i = \phi(a_i) \bullet_{\rho_2} \phi(a_j)$

   Similar is the proof of the rest cases, i.e. $a_j$to be in $a_i \bullet_{\rho_1} a_j$ or both a_i, a_j to be in $a_i \bullet_{\rho_1} a_j$. Also, since the principle of duality is valid [4], the dual statement holds, i.e. $\phi(a_j \bullet_{\rho_1} a_i)$=$\phi(a_j) \bullet_{\rho_2} \phi(a_i)$.

2. If $a_k, a_\lambda \notin \{a_i, a_j\}$ then $\phi(a_k) \bullet_{\rho_1} \phi(a_\lambda)$=
   $\phi\{x \in H : (a_k, x) \in \rho_1 \text{ and } (x, a_\lambda) \in \rho_1\} =$

   (a) if neither a_i nor a_j belongs to $a_k \bullet_{\rho_1} a_\lambda$ then
   $\phi\{x \in H : (a_k, x) \in \rho_1 \text{ and } (x, a_\lambda) \in \rho_1\} =$
   $\{\phi(x) \in H : (a_k, x) \in \rho_1 \text{ and } (x, a_\lambda) \in \rho_1\} =$
   $\{x \in H : (a_k, x) \in \rho_1 \text{ and } (x, a_\lambda) \in \rho_1\} =$
   $\{x \in H : (a_k, x) \in \rho_2 \text{ and } (x, a_\lambda) \in \rho_2\} = a_k \bullet_{\rho_2} a_\lambda$
   $= \phi(a_k) \bullet_{\rho_2} \phi(a_\lambda)$

   (b) if $a_k \bullet_{\rho_1} a_\lambda \cap \{a_i, a_j\} \neq \emptyset$. Assume e.g. that both a_i, a_j belongs to $a_k \bullet_{\rho_1} a_\lambda$ then
   $\phi\{x \in H : (a_k, x) \in \rho_1 \text{ and } (x, a_\lambda) \in \rho_1\} =$
   $\{\phi(x) \in H : (a_k, x) \in \rho_1 \text{ and } (x, a_\lambda) \in \rho_1\}$
   and since $\phi(a_i) = a_j$, $\phi(a_j) = a_i$ this is equal to
   $\{x \in H : (a_k, x) \in \rho_1 \text{ and } (x, a_\lambda) \in \rho_1\}$
   or $\{x \in H : (a_k, x) \in \rho_2 \text{ and } (x, a_\lambda) \in \rho_2\}$ which is
   $a_k \bullet_{\rho_2} a_\lambda$ or $\phi(a_k) \bullet_{\rho_2} \phi(a_\lambda)$

Similar is the proof for the cases $\phi(a_k) \bullet_{\rho_1} \phi(a_i)$, $\phi(a_k) \bullet_{\rho_2} \phi(a_j)$ and their duals.
From the above Proposition derives the following Theorem

***Theorem* 3.3:** If the Boolean matrix M_σ derives from M_ρ by interchanging rows and the corresponding columns, then the hypergroupoids H_σ and H_ρ are isomorphic.

The isomorphic classes of these hypergroupoids are not computed in [11]. These can be counted with a proper modification of the function `Good[]`, which will then return all the binary matrices that form a hypergroupoid. Thus, the above function changes in one of its lines and can be found in the appendix as a module of the package.
Check for example the three binary relations with matrices of size 2

```
In[4]:= h2 = Good1[2]
Out[4]= {{{0, 1}, {1, 1}}, {{1, 1}, {1, 0}}, {{1, 1}, {1, 1}}}
```

We are able now to give a function that forms all $n!$ isomorphisms of a given binary relation.

```
IsomorphTest1[a_List] :=
Module[{p, a1},
        p = Permutations[Range[1, Length[a]]];
        Return[Table[a1 = a;
                     a1 = ReplaceAll[a1, a1[[All, Table[j2,
                                     {j2, 1, Length[a1]}]]] ->
                                         a1[[All, p[[j1]]]]];
                          ReplaceAll[a1, a1[[Table[j2,
                                     {j2, 1, Length[a]}]]] ->
                                         a1[[p[[j1]]]]],
                     {j1, 1, Length[p]}]
               ]]
```

Let's see the six permutations of the matrix

$$M_\rho = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

which are defined by corresponding binary relations, that give isomorphic hypergroupoids:

```
In[5]:= IsomorphTest1[{{1, 0, 1}, {1, 1, 0}, {0, 1, 1}}]
Out[5]:= {{{1,0,1}, {1,1,0}, {0,1,1}}, {{1,1,0}, {0,1,1}, {1,0,1}},
          {{1,1,0}, {0,1,1}, {1,0,1}}, {{1,0,1}, {1,1,0}, {0,1,1}},
          {{1,0,1}, {1,1,0}, {0,1,1}}, {{1,1,0}, {0,1,1}, {1,0,1}}}
```

In order to count the number of the different non–isomorphic classes of hypergroupoids of order n, a n–digit array, called `cardinalities`, is used by the program. Each time the routine encounters an isomorphic class, it drops it from variable `h2`.

```
Cardin[d_] :=
Module[{h2, cardinalities, len, temp1, temp},
  h2 = Good1[d];
  cardinalities = Table[0, {j1, 1, Factorial[d]}];
  While[Length[h2] > 0,
        temp = Union[IsomorphTest1[h2[[1]]]];
        len = Length[Union[temp]];
        cardinalities[[len]] = cardinalities[[len]] + 1;
        h2 = Complement[h2, temp]
        ];
  Return[cardinalities]]
```

Then we get

```
In[6]:= Cardin[2]
Out[6]:= {1, 1}
In[7]:= Total[%]
Out[7]:= 2
In[8]:= Cardin[3]
```

```
Out[8]:= {2, 1, 5, 0, 0, 9}
In[9]:= Total[%]
Out[9]:= 17
In[10]:= Cardin[4]
Out[10]:= {2, 0, 1, 5, 0, 7, 0, 4, 0, 0, 0, 78,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 207}
In[11]:= Total[%]
Out[11]:= 304
In[12]:= Cardin[5]
Out[12]= {2, 0, 0, 0, 5, 0, 0, 0, 0, 13, 0, 1, 0, 0, 8,
          0, 0, 0, 0, 78, 0, 0, 0, 3, 0, 0, 0, 0, 0, 152,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 42, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2206, 0,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 18150}
In[13]:= Total[%]
Out[13]= 20660
```

So, there are $2, 17, 304$ and 20660 isomorphic classes (I.C.) of orders $2, 3, 4$ and 5 respectively. For example, for order 4 there are 2 I. C. of cardinality 1, 1 I. C. of cardinality 3, 5 I. C. of cardinality 4, 7 I. C. of cardinality 6, 4 I. C. of cardinality 8, 78 I. C. of cardinality 12 and 207 I. C. of cardinality 24. These $2 + 1 + 5 + 7 + 4 + 78 + 207 = 304$ I. C. form the

$$2 \cdot 1 + 1 \cdot 3 + 5 \cdot 4 + 7 \cdot 6 + 4 \cdot 8 + 78 \cdot 12 + 207 \cdot 24 = 6003$$

non–isomorphic hypergroupoids of order 4.
We also mention that there are $2^{n\times n}$ binary matrices of size n. We may count the non–isomorphic ones by simply changing the line
`h2 = Good[n];`
in the routine `Cardin[]` by the line
`h2=Tuples[Tuples[0,1,n],n]`.
Then we get

```
In[14]:= Cardin[1]
Out[14]= {2}
In[15]:= Cardin[2]
Out[15]= {4, 6}
In[16]:= Total[%]
Out[16]= 10
In[17]:= Cardin[3]
Out[17]= {4, 2, 28, 0, 0, 70}
In[18]:= Total[%]
Out[18]= 104
In[19]:= Cardin[4]
Out[19]= {4, 0, 4, 28, 0, 32, 0, 16, 0, 0, 0, 496,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2464}
In[20]:= Total[%]
Out[20]= 3044
```

The integer sequence $2, 10, 104, 3044$ etc. coincides with the integer sequence A000595, appeared in [10] and represents the number of non–isomorphic unlabeled binary relations on n nodes.

3.2 Weak Associativity

As proved in section– 2 above, the total hypergroup is the only hypergroupoid that fulfills the property of associativity. Thus we checked a weaker property, which is called Weak Associativity:

$$a(bc)\bigcap(ab)c \neq \emptyset \text{ for all } a, b, c \in H. \qquad (2)$$

Having, up to this point, constructed all the hypergroupoids of order $2, 3, 4$ and 5, we check the validity of this property to all of them and we count the ones that verify it. The package is given and explained in the appendix.
Its results are:

```
In[21]:= BinaryTest[2]
Out[21]= 3
In[22]:= BinaryTest[3]
Out[22]= 43
In[23]:= BinaryTest[4]
Out[23]= 2619
In[24]:= BinaryTest[5]
Out[24]= 602431
```

The counting of the hypergoupoids of orders $n \geq 6$ is time consuming, so we discontinued at $n = 5$.

4 Conclusions

This paper shows that the total hypergroup is the only hypergroup which can be produced by hypercomposition (1). Since it is a hypergoup it is also a semihypergroup and a quasihypergroup. No other semi- or quasi- hypergroups can be produced by (1). On the other hand there exist lots of hypergoupoids that can be produced by (1), the number of which is calculated with the use of Mathematica packages that are constructed for this purpose and consist part of the contents of this paper. The results of these calculations are given in the cumulative Table– 1 below for the orders $2, 3, 4$ and 5:

Table 1: Cumulative results

order →	2	3	4	5
Boolean matrices (BM)	16	512	65536	33554432
BM forming Hypergroupoids	3	73	6003	2318521
BM forming Weak-Associative Hypergroupoids	3	43	2619	602431
Nonisomorphic BM	10	104	3044	291968
Nonisomorphic BM forming Hypergroupoids	2	17	304	20660

A The Mathematica package

The Mathematica package referred to in section– 3, is given bellow.

```
BeginPackage["BinaryTest`"];
Clear["BinaryTest`*"];

BinaryTest::usage = "BinaryTest[n] counts the binary
relations of dimension n that form a hypergroupoid.
It also counts the Weak-associative binary hypergroupoids"

Begin["`Private`"];
Clear["BinaryTest`Private`*"];

BinaryTest[n_] :=
Module[{c, ch},
   c = Good1[n];
   ch = Table[AssociativityWeakTest[HyperGroupoid[c[[j1]]]],
             {j1, 1, Length[c]}];
   Return[Count[ch, True]]];

Good1[n_] :=
Module[{c, i1, z},
        c = Tuples[Tuples[{0, 1}, n], n];
        z = Table[Min[Flatten[c[[i1]].c[[i1]]]]
                  ,{i1, 1, 2^(n*n)}];
        Return[Select[Transpose[{c, z}], #[[2]] > 0 &][[All, 1]]]
      ]

AssociativityWeakTest[a_List] :=
Module[{i, j, k, test},
  i = 1; j = 1; k = 1; test = True;
  While[test && i <= Length[a],
    test = Intersection[
    Union[Flatten[Union[Extract[a,
                  Distribute[{a[[i, j]], {k}}, List]]]]],
    Union[Flatten[Union[Extract[a,
                  Distribute[{{i}, a[[j, k]]}, List]]]]]
                       ] != {};
    k = k + 1;
    If[k > Length[a], k = 1; j = j + 1;
                     If[j > Length[a], i = i + 1; j = 1];
      ];
        ];
   Return[test]
       ];

HyperGroupoid[a_List] :=
Table[Table[Intersection[
            Floor[(a[[j1, 1 ;; Length[a]]]
                 + a[[1 ;; Length[a], j2]])/2
```

```
                    ]*
              Table[j3, {j3, 1, Length[a]}],
              Table[j3, {j3, 1, Length[a]}]
                         ],
       {j1, 1, Length[a]}],
{j2, 1, Length[a]}];

End[];
EndPackage[];
```

The package consists of four functions. The three internal ones are:
`Good1:` that returns all the hypergroupoids associated to a binary relation of order n.
`HyperGroupoid:` that constructs the hypergroupoid associated to a given Boolean matrix of a binary relation.
`AssociativityWeakTest:` that tests property (2) by forming all n^3 possible products of all triplets of elements of H.
and the main one, which is:
`BinaryTest:` After calling `Good`, it constructs the table of the deriving hypergroupoids, using the function `HyperGroupoid`. Finally it counts the number of those, which satisfy the property of the weak associativity.

References

[1] J. Chvalina, Commutative hypergroups in the sense of Marty and ordered sets, Proceedings of the Summer School on General Algebra and Ordered Sets 1994, Olomouc (Czech Republic), pp 19–30, 1994.

[2] P. Corsini, Binary relations and hypergroupoids, Italian J. Pure & Appl. Math., 7, pp. 11–18, 2000.

[3] M. De Salvo and G. Lo Faro, Hypergroups and binary relations. Mult. Valued Log., 8, pp. 645–657, 2002.

[4] J. Jantosciak - Ch. G. Massouros, Strong Identities and fortification in Transposition hypergroups, J. Discrete Math. Sci. & Cryptography, 6 (2003), pp. 169–193.

[5] F. Marty, Sur une gènèralisation de la notion de groupe, Huitieme Congres des Mathematiciens Scand., Stockholm 1934, pp. 45–59.

[6] C.G. Massouros, On the semi-subhypergroups of a hypergroup, Inter. J. Math. & Math. Sci., 14 (1991) pp. 293–304.

[7] G.G. Massouros, An automaton during its operation. Proceedings of the 5th Internat. Cong. on Algebraic Hyperstructures and Applications. pp. 268–276, Iasi 1993. Hadronic Press 1994.

[8] J. Nieminen, Join space graphs. J. of Geometry, 33, pp 99–103, 1988.

[9] I. G. Rosenberg, Hypergroups and join spaces determined by relations, Italian J. Pure & Appl. Math., 4, pp. 93–101, 1998.

[10] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at `www.research.att.com/njas/sequences/`, 1996–2008.

[11] S. I. Spartalis and C. Mamaloukas, Hyperstructures associated with binary relations, Comput. Math. Appl., 51 (2006) pp. 41–50.

[12] Ch. Tsitouras and Ch. G. Massouros, On enumerating hypergroups of order 3, Submitted.

[13] S. Wolfram, The Mathematica Book, 5th ed., Wolfram Media, 2003.