Hybrid Hamilton–Webster and the Greek apportionment

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Abstract

The method of largest remainders (Hamilton) is used for allotment of 288 of the seats among the 56 Greek constituencies. This method possesses various paradoxes as was observed through two centuries of application. So we propose a modification where the seats are allocated at a first stage by the lower Hare-Quota and the additional ones using the method of major fractions (Webster) restricted simultaneously by the upper quota. This method may produce paradoxes but they are observed extremely rare. Extended simulations over the Greek electoral data indicate that the frequency that the new method violates monotonicity is by far less than the frequency that Webster method violates quota.

Keywords: Fair share, Hare Quota, Unbiased apportionment

1. Introduction

In Greece there are 300 Parliamentary Deputies for the entire country. Of these, 288 are elected from constituencies. The other 12 are elected as national representatives and are known as State Deputies. The number of parliamentary seats per constituency for general elections is based on the published results of the 18th March 2001 general census on population (citizen) numbers. The members of this 300-seat Greek legislature are elected for four-year terms by a system of bi-proportional representation [8] and voters may select the candidate or candidates of their choice by marking their name on the party ballot at each constituency. The remaining 12 seats are filled at large from nationwide party lists based on the overall proportion of the total vote each party receives.

The apportionment of the 288 seats is proportional to the legitimate population of the constituencies which is the number of citizens registered in them. Every citizen is obligatory registered in some constituency. So there are cases where more actual voters elect less representatives and vice versa. This happens since many voters are registered in outlying regions even if they resident

in Athens or Salonica. If they do not go voting for various reasons e.g. bad weather. For example in 2004 elections, not long after the general census, we had more voters in the 7-seat constituency of Salonika B' than every 8-seat constituency existing!

More information about the Greek elections may retrieved by the Ministry of Interior:

http://www.ypes.gr/en/Elections/

The method of Hamilton is used for distributing the seats to the 56 constituencies. It is known that it is a fair and "practically unbiased" method, [7]. But some peculiar performance is observed when its results are compared with previous apportionments. In the notorious Alabama paradox, the southern state would had 8 seats in a 299 member house of 1880 but only 7 seats in a 300 member house!

2. Apportionment–preliminaries and methods

The apportionment problem is specified by an $n-$vector of rational numbers (usually the number of voters at each constituency) $\mathbf{p} = [p_1, \ p_2 \cdots, p_n]$, all $p_i > 0$ and an integer house size $h > 0$, [1]. An apportionment of h among n is an integer $n-$vector $\mathbf{a} = [a_1, \ a_2 \cdots, a_n]$ with $\sum_{i=1}^{n} a_i = h$. A method of apportionment M is a multiple–valued function that assigns at least one apportionment to every problem $(\mathbf{p}, \ h)$. M may be multiple–valued due to the possibility of 'ties', [3]. There are also degenerate cases where the set of apportionments is empty. e.g. $\sum_{i=1}^{n} p_i < h$.

The quota of a constituency is the exact share of h to which its population would be entitled. Given a problem $(\mathbf{p}, \ h)$, the Hare quota used in Greece, is:

$$D = \lfloor \frac{\sum_{j=1}^{n} p_j}{h} \rfloor.$$

Thus the quota of constituency i is

$$q_i = \frac{p_i}{D}.$$

The lower quota is $\lfloor q_i \rfloor$ and the upper[1] quota is $\lceil q_i \rceil$.

Six methods have been attended major discussions over the years. First the method of Hamilton or greater remainders. According to this method each constituency receives firstly its lower quota. Then the remaining $h - \sum_{i \leq n} \lfloor q_i \rfloor$ seats are awarded to the constituencies with the greater remainders among $p_i - D \cdot \lfloor q_i \rfloor$, $i = 1, 2, \cdots, n$.

The other five methods of importance belong to the general family of divisor methods. So $i-$st constituency receives its $k-$st seat before $j-$st constituency receives its $v-$st seat if:

$$\frac{p_i}{f(k-1)} > \frac{p_j}{f(v-1)},$$

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[1] $\lfloor x \rfloor$ is the greatest integer less than x. $\lceil x \rceil$ is the smallest integer greater than x.

where $f(x)$ is a function that distinguishes each method. Specifically

- $f(x) = x$, the Adams method or method of smallest divisors

- $f(x) = x + 1/2$, the Webster method or method of major fractions

- $f(x) = x + 1$, the Jefferson method or method of greatest divisors

- $f(x) = \sqrt{x(x+1)}$, the Huntington-Hill method or method of equal proportions

- $f(x) = \frac{2x(x+1)}{2x+1}$, the Dean method

The methods of Adams, Dean and Huntington-Hill start apportioning one seat to every constituency. Thereupon begins the procedure described above.

Some methods referred here can be found in literature with different names. Such as D'Hondt stands for Jefferson and Sainte Laguë for Webster, [4].

3. Basic criteria and the hybrid method

Four criteria are important when checking a method, [2].

a *Staying within the quota.* The apportionment should guarantee to each constituency at least its lower quota and at most its upper quota.

b *No bias.* On average, over many problems, every constituency receives its fair share.

c *Population monotonicity.* When populations change, apportionment should not change by giving more seats to a constituency with relatively less population growth and less seats to a constituency with greater population growth.

d *House monotonicity.* If the house increases, with populations fixed, no constituency looses seats.

Unfortunately no method can satisfy all the above criteria, [2]. The first quota–criterion is satisfied only by Hamilton method which can not satisfy monotonicity criteria. All divisor methods satisfy criteria *c–d*. Only Webster method satisfies criteria *b, c* and *d*. The quota–method [2], satisfies criteria *a* and *d*.

The method of Webster seems to be the best one if criteria *b, c* and *d* are given more significance, [1]. But satisfaction of quota is hard to kept away from the play. The major reason that Webster is preferable is that Hamilton performs a major failure in criteria *c, d*. So a method satisfying quota and in addition fails rarely on the latter criteria is to be given.

Such a method is a hybrid construction gaining from the advantages of Hamilton and Webster. So we first award the lower quota $\lfloor q_i \rfloor$ for all $i = 1, 2, \cdots, n$. Then we allocate at most one seat to the constituencies with the

greatest major fractions, using Webster method for this part of the current procedure. Thus the new method may considered as Lower/Upper-quota restricted Webster method.

Minimum restriction methods have appeared in the literature. The direct-seat restricted Webster, is mentioned in [6]. It applies to German parliamentary elections and the number of direct seats won by a party is imposed as a minimum restriction there. Minimum restriction is also mentioned in [1] for apportionment lower bounds but it may not correlated to lower quota. A modification of Hagenbach-Bischoff system, using at first Hare quota and then standard rounding (Webster/Sainte Laguë) can be considered as Lower-quota restricted Webster.

Table 1: Cumulative comparison of Hamilton, Webster and Hybrid

popul.	quota	HAMILTON			DIVISORS		WEB	H-W
		1st	rem	total				
76000	1.68	1	*30889	2	2nd seat	50666	2	1
32000	0.71	0	*32000	1	1st seat	*64000	1	1
27000	0.60	0	27000	0	1st seat	*54000	1	1
271000	6.01	6	334	6	7th seat	41692	5	6
406000	9	7		9			9	9

An example comparing Hamilton, Webster and hybrid method is given in Table–1. Suppose we have $\mathbf{p} = [76000, 32000, 27000, 271000]$ and $h = 9$. The Hare quota is $D = \lfloor 406000/9 \rfloor = 45111$ (for the Greek version). The second column gives the corresponding quotas p_i/D. In the third column we recorded the seats awarded at the first stage of Hamilton (and hybrid) method. Two seats remain to be distributed. According to the remainders shown in the fourth column the first and second constituency gain one seat each. The final apportionment is given in the fifth column.

In the same table Webster divisors are shown in column 7. These numbers are produced dividing the populations in sequence by 1.5 for the first constituency, by 0.5 for the second and third constituencies and by 6.5 for the fourth. The Webster apportionment is shown in column 8. Observe that it failed to satisfy the first criterion. This, because the divisor for the 6th seat for the fourth constituency is $\lfloor 271000/5.5 \rfloor = 49272$ a little smaller from the divisor for allocating the second seat to the first constituency.

Finally the apportionment of the hybrid method is given in the last column. The first stage of this apportionment is shown in the third column. Then we observe that the second and third constituency are entitled to gain their first seat (with divisors 64000 & 54000 respectively) before the first constituency gain its second (with divisor 50666).

Another example that illustrates the upper-quota restriction is understood with given populations $\mathbf{p} = [48, 15, 14, 13, 12]$ and $h = 2$. Then by Webster method we get $\mathbf{a} = [2, 0, 0, 0, 0]$. The new method poses an upper restriction for

4

the first constituency and we get $\mathbf{a} = [1, 1, 0, 0, 0]$.

For applying the above methods to the Greek data we retrieved the legitimate population from the Ministry of Interior

`http://www.ypes.gr/en/Elections/NationalElections/DeputyElections/InterestCombination/.`

All three methods gave the same apportionment for this population.

4. Simulations

4.1. Simulations in the Greek data

We produced $50,000$ $56-$vectors $\mathbf{s}^{[j]}$, $j = 1, 2, \cdots, 50000$ for the simulated population of the 56 Greek constituencies. Each population was chosen uniformly in the region $0.9 p_i \leq \mathbf{s}_i^{[j]} \leq 1.1 p_i$, where p_i is the current population of the $i-$th constituency. This is a realistic simulation of the expected fluctuations of the population. We fix $h = 288$ since it seems difficult for this number to change. We checked all the methods presented above except quota method which is an artificial scheme designed to satisfy both quota and house monotonicity. All computations were done using [5] on a Pentium IV computer running Windows XP-SP2 Professional at 3.4GHz.

The first check concerns the satisfaction of the quota criterion. We distinguish the cases of lower and upper quota failures. No simultaneous double violation were observed. The results are recorded in Table–2.

Jefferson method is steadily favors big constituencies while Adams favours the small ones. Webster is the best of the divisor methods violating quota at about 1.5% of the populations. In 771 out of its 772 failures, the $42-$seat Athens–B was the constituency violating the quota. The $772-$th failure was the only simulation where the $17-$seat constituency of Athens–A (municipality of Athens) violated quota. As it was expected Hamilton and hybrid method satisfy the quota all the times.

We also applied the lower-quota restricted Webster method and found that it never violates lower quota but violates the upper quota the same 723 with the Webster method.

Table 2: Number of quota failures

	JEF	WEB	HAM	H-W	HUN	DEA	ADA
upper	50000	723	0	0	27	2	0
lower	0	49	0	0	867	5725	50000
total	50000	772	0	0	893	5727	50000

We proceed examining the detachment of the methods. For each method and population we recorded the maximum distance observed from quota. Then we recorded the average value for each method in in Table–3.

Jefferson awards Athens–B steadily with $3 - 4$ seats more. There was a case with 5.3 seats distance from the fair share. The exact opposite is done by Adams. Hamilton is the fairest and in all cases we observed $|a_i - q_i| \leq 0.647$.

Table 3: Mean of maximum distance from fair share

JEF	WEB	HAM	H-W	HUN	DEA	ADA
3.47	0.573	0.523	0.566	0.594	0.729	3.50

This bound was extended by the hybrid method to $|a_i - q_i| < 1$. The latter bound was violated by Webster in 1.5% of cases as we have seen above. So it was a little more unfair than Hamilton and hybrid. Huntington-Hill and Dean favor small constituencies. Results in Table–3 justify our feeling of the new method to be "practically unbiased".

Population monotonicity was not violated by divisor methods as expected from theory, see Table–4. Hamilton failed in a little more than 1% of the simulations. The result of main interest is that hybrid method failed only in one case out of 50,000! Since our experiments are realistic simulations of populations variation it is worth mentioning that hybrid method was about 750 times less faulty in population monotonicity than it was Webster in quota satisfaction.

Table 4: Number of population monotonicity failures

JEF	WEB	HAM	H-W	HUN	DEA	ADA
0	0	532	1	0	0	0

Setting $h = 289$, house monotonicity was not violated by divisor methods as shown in Table–5. But Hamilton gave unacceptable results. In the 8% of the cases some constituency received less seats than those possessed with $h = 288$. This high frequency was the reason that Alabama paradox was observed some years after adoption of Hamilton method for USA house apportionment. Hybrid method didn't fail in any of the 50,000 simulations. This is a coincidence but indicates the frequency of appearance of such a failure for the proposed method.

Table 5: Number of house monotonicity failures

JEF	WEB	HAM	H-W	HUN	DEA	ADA
0	0	3960	0	0	0	0

4.2. Simulations in the USA data

The number of seats in the 56 Greek constituencies and the corresponding numbers for the 50 USA states is in very tight correlation. Greek top 3 constituencies share 26% of the seats the same as top 3 states of the USA house. On the other hand there are eight one-seat constituencies in the Greek house and seven one-seat states in the USA house.

6

So the results observed in the previous subsection are in agreement with USA house results. Jefferson and Adams violate in the 100% of the cases the quota. California, Texas or NY are always overrepresented using Jefferson apportionment. On the other hand it is impossible for Adams not to steal some seats from these states on behalf of the one–seat states (now becoming two–seat). Hybrid method never violates (in simulation) the monotonicities while Hamilton violates them at the frequency it did for the Greek parliament. The main difference is that Webster justified previous results in quota violation, [1]. So only in 0.2% of the cases moved out of quota.

The explanation is that California represents 11.9% of the seats while Athens–B represents 14.6% of the Greek parliament. This difference in the top constituency percentage makes the disparity in performance. If the most populous state was only $6\% - 7\%$ of the total population then perhaps Webster would never violate quota.

5. Conclusion

In a house where there are some large constituencies (e.g., Greece and USA) it seems that Webster violates quota with a frequency that is not ignorable. A hybrid Hamilton–Webster method proposed here is forced to stay within the quota at a cost of violating monotonicities rarely if ever. Thus it needs attention since violating quota is a psychological criterion emerging easily the feeling of injustice even by those with less mathematical insight. Especially in Greece where Hare Quota is widely understood among voters. On the other hand, monotonicities are artificial criteria difficult for people to understand and deal with them.

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