Greatest Remainder Bi–proportional Rounding and the Greek Parliamentary Elections of 2007.

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Abstract

Matrix scaling is the problem of assigning values to the elements of a matrix that are proportional to a given input matrix. The assignment should fulfill a set of row- and column-sum requirements. We propose a new method that differs from divisor–type methods appeared until now in the literature. This method combines the largest remainder apportionment and bi–proportional rounding. Exhaustive application to the Greek parliamentary elections of 2007 justify our effort.

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1. Introduction

The Greek parliamentary elections of 9/16/2007 took place according to the electoral law which had been voted by a previous house back in 2004. This law was obligatory for the next two elections. The Vouli (the unicameral Greek Parliament) consists of 300 seats, 260 of which are elected by a Largest Remainder method and 40 bonus seats are assigned to the most voted party. These seats are allotted to the parties at 56 electoral regions.

Traditionally in these elections the country is separated in $n = 56$ regions that usually coincide with the limits of prefectures. Up to the elections 1996 a system of reinforcement proportion was applied. According to this, a first distribution of seats began in the regions. Afterwards, a remainder of seats were given in the apartments (groups of adjacent regions) at a second distribution. At this second distribution all the votes received by the parties were considered again. The allocation of seats was completed in the nation, where the remaining seats were distributed.

After the recent law was published any political coalition receives nationwide 3% the votes participates in the distribution of seats as following:

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The total of votes $v_i^N,\ i=1,2,\dots,m$ that it assembled in the nation by the $i-$party is multiplied with the number 260 and the product is divided with the sum of valid votes received by all the parties participating in the distribution of seats. The number

$$f_i^N = 260 \cdot \frac{v_i^N}{\sum_{k=1}^{m} v_k^N}$$

is the nation-wide fair share for each party. This is most probably a decimal number. So at first, each party receives $\lfloor f_i^N \rfloor$ seats. i.e. the greatest integer, smaller than f_i^N. Then the $\sum_{i=1}^{i=m}(f_i^N - \lfloor f_i^N \rfloor)$ parties having the largest remainders $f_i^N - \lfloor f_i^N \rfloor$ are assigned an additional seat.

This is actually an apportionment method discovered independently by A. Hamilton and T. Hare [2]. Finally the party that receives the greatest number of valid votes gets forty additional seats.

Under the current law the results of the elections of 2007, are given in Table–1. Details about these elections can be found in the following site of Ministry of interior

`http://ekloges-prev.singularlogic.eu/v2007/pages_en/index.html`

Table 1: Cumulative results of Greek elections in 2007.

parties	%	fair share	1st distrib.	2nd distrib.	bonus	total # of seats	nation list
ΝΔ	41.84	112.23	112	0	40	152	5
ΠΑΣΟΚ	38.10	102.20	102	0	0	102	5
ΚΚΕ	8.15	21.86	21	1	0	22	1
ΣΥΝ	5.04	13.52	13	1	0	14	1
ΛΑΟΣ	3.80	10.19	10	0	0	10	0
others	3.07	0	0	0	0	0	0
sums	100	260	258	2	40	300	12

In Table-1, we use the term others meaning various parties received below 3%. Their fair share is 0 because they do not receive any seat due to the restriction of the law.

The 300 seats given to the parties are recorded in the seventh column of Table-1. At first twelve of the seats are distributed in a nation-wide list according to the largest remainder proportional system described above. We write them in the last column of Table–1. The rest 288 seats are to be distributed in the 56 electoral regions. These regions elect a number of Parliament Members according to their nominal population [10]. Here we concentrate in optimal distribution of these 288 seats in the regions according to an established objective function.

2. Posing the problem.

The nomenclature of the paper can be found in Table–2. Since we are not interested about the distribution of the 12 seats of the last column of Table-1,

Table 2: Notation

m	number of parties received$\geq 3\%$	e_i	seats of $i-$party (nation-wide)
m^*	number of all parties	e_i^B	seats of $i-$party (2nd dist.)
n	number of regions, $n = 56$	k_{ij}	seats $i-$party at $j-$ region
v_{ij}	votes $i-$party at $j-$region	t_j	$= \sum_{i=1}^{m} v_{ij}$, $j = 1, 2, \ldots, n$
s_j	seats $j-$region, $\sum_{j=1}^{56} s_j = 288$	μ_j	$= t_j/s_j$ meter for $j-$region
v_i^N	votes of $i-$ party (nation-wide)	f_i^N	fair share of $i-$party (nation-wide)

observe that $e_1 = 147 = 152 - 5$ for the first party, $e_2 = 107 = 112 - 5$ for the second party, etc. Our objective is to allocate these $147, 107, 21, 13, 10$ seats to the 5 parties at the 56 regions. Everything but the values of k_{ij} is given in that table or can be evaluated straightforwardly. Our aim is not commenting the substance and the spirit of law but the way of distributing the seats to the regions. So we are interested in calculating optimally the prices k_{ij}.

The current law is focused in the distribution in small regions and the smaller parties. The first distribution begins by allocating to $i-$party at $j-$region which received there v_{ij} votes,

$$\hat{k}_{ij}^A = \lfloor \frac{v_{ij} s_j}{\sum_{k=1}^{m^*} v_{kj}} \rfloor$$

seats. Observe that usage of m^* indicates that all valid votes are counted for the evaluation of k_{ij}^A. In the regions with one, two or three seats the greatest remainder apportionment is applied.

After this, the $m-$th party (the smaller one from those receiving over 3% of votes) gets one additional seat in the regions it has the greatest remainder of votes

$$v_{mj} - \hat{k}_{mj}^A \cdot \lfloor \sum_{k=1}^{m^*} v_{kj}/s_j \rfloor \text{ for } j = 1, 2, \ldots, n,$$

in order to reach e_m seats. The second smaller party continues with the same process and finally the first party collects whatever remains. This procedure lacks any justification. For example, we could begin with the second party then the third, the first, etc. Perhaps the legislator wanted to avoid confusion if adding some mathematical insight. Actually, it is a transportation problem that legislator neglected its complete solution after finding a first random feasible solution.

The objective of an optimal distribution could be the minimization of distance of percentage of parties from the percentage of seats that they receive. That is to say the quantities

$$|\frac{v_{ij}}{t_j} - \frac{k_{ij}}{s_j}| \tag{1}$$

have to be as small as possible. More generally in $j-$ region it is desirable to keep small the quantities (1) for all the parties. Thus it is preferable to minimize the quantity:

$$\sum_{i=1}^{m}(\frac{v_{ij}}{t_j} - \frac{k_{ij}}{s_j})^2. \tag{2}$$

The power is used in the sense of least squares technique for avoiding the absolute value which is not differentiable. Squaring will produce a different apportionment than minimizing $\sum_{i=1}^{m}|\frac{v_{ij}}{t_j} - \frac{k_{ij}}{s_j}|$. Summing for all the regions we conclude to the requirement of minimizing:

$$\sum_{j=1}^{n}\sum_{i=1}^{m}(\frac{v_{ij}}{t_j} - \frac{k_{ij}}{s_j})^2.$$

We may interchange $\sum$ in the formula above. Thus we see that if we begin summing in parties we get the same result.

The regions elect different number of deputies. Thus the values (2) have different significance in a small region than in a large one. E.g. 10% is almost nothing in a one–seat region, but could affect four seats in a 40–seat region. Thus we use s_j^2 as weights and therefore we set

$$\epsilon = \sum_{j=1}^{n} s_j^2 \sum_{i=1}^{m}(\frac{v_{ij}}{t_j} - \frac{k_{ij}}{s_j})^2.$$

Other weights could be used but the least squares underlying sense dictates our choice.

Since $s_j > 0,\ j = 1, 2, \ldots n$, we have

$$\epsilon = \sum_{j=1}^{n}\sum_{i=1}^{m}(s_j\frac{v_{ij}}{t_j} - k_{ij})^2 = \sum_{j=1}^{n}\sum_{i=1}^{m}(\frac{v_{ij}}{\mu_j} - k_{ij})^2.$$

Our purpose here is to find

$$\min_{k_{ij}} \epsilon.$$

Concentrating now only to the parties receiving seats, we set

$$k_{ij}^A = \lfloor\frac{v_{ij}}{\mu_j}\rfloor,\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,$$

the seats gaining by each party at the first distribution. These numbers are set as lower bounds for k_{ij}. For implementing our technique, the quantities

$$k_{ij}^B = k_{ij} - k_{ij}^A$$

are set to be 0 or 1 according to the ability of $i-$party to gain an additional seat in the $j-$region using its $v_{ij} - k_{ij}^A\mu_j$ unused votes there. k_{ij}^B are the seats of the second distribution and are the actual parameters that we have to calculate

in the problem we are formulating. This restriction helps in the direction of converting to a kind of transportation problem.
We denote

$$r_{ij} = \frac{v_{ij}}{\mu_j} - \lfloor \frac{v_{ij}}{\mu_j} \rfloor,$$

the unused remainder as fraction of electoral measure. Then we get

$$\min_{k_{ij}} \epsilon = \min_{k_{ij}^B} \sum_{j=1}^{n} \sum_{i=1}^{m} (r_{ij} - k_{ij}^B)^2 = \sum_{j=1}^{n} \sum_{i=1}^{m} (r_{ij}^2 + (k_{ij}^B)^2) + 2 \min_{k_{ij}^B} \{ \sum_{j=1}^{n} \sum_{i=1}^{m} -r_{ij} k_{ij}^B \}.$$

Observe that

$$\sum_{j=1}^{n} \sum_{i=1}^{m} r_{ij}^2 = \text{constant},$$

and since $k_{ij}^B \in \{0, 1\}$, we have:

$$\sum_{j=1}^{n} \sum_{i=1}^{m} (k_{ij}^B)^2 = \sum_{j=1}^{n} \sum_{i=1}^{m} k_{ij}^B = \text{total seats of 2nd distribution} = \text{constant}.$$

Finally we conclude that we are interested in solving the binary linear program:

$$\epsilon^* = \min_{k_{ij}^B} \sum_{j=1}^{n} \sum_{i=1}^{m} -r_{ij} k_{ij}^B = \max_{k_{ij}^B} \sum_{j=1}^{n} \sum_{i=1}^{m} r_{ij} k_{ij}^B.$$

Thus we realize that what is asked is maximization of sum of remainders (as fractions of electoral measures) that are used for the allocation of seats at the 2nd distribution. We name the new method greatest remainder bi-proportional apportionment. Using the names of Hamilton or Hare instead of greatest remainders is also possible. As it is important to justify the optimization function, notice that there are several other ways to measure the disproportionality between votes and allotted seats (see e.g. Proposition 3.7 to 3.11 in [2, pp. 104-105]). Choosing different ways to measure can provide different allotments, all of them staying within the quota, so the names given above can also be applied to these methods.

This optimization function is reasonable enough and has an interesting property. Maximization of a certain measure of "unused voting power".

Determination of ϵ^* is a problem of linear programming because it is accompanied by restrictions for the prices k_{ij}^B. Concretely, we have the following $m + n$ restrictions:

$$\sum_{i=1}^{n} k_{ij}^B = e_i - \sum_{j=1}^{n} k_{ij}^A = e_i^B \text{ for } i = 1, 2, \ldots, m \tag{3}$$

and

$$\sum_{j=1}^{m} k_{ij}^B = s_j - \sum_{j=1}^{m} k_{ij}^A = s_j^B \text{ for } j = 1, 2, \ldots, n. \tag{4}$$

Table 3: Cost table for a 4–region, 3–party example.

party	A	B	C	D	e_i^B
	regions				
a	0.42	0.60	0.71	0.27	2
b	0.85	0.38	0.18	0.62	2
c	0.73	0.02	0.11	0.11	1
	2	1	1	1	$\leftarrow s_j^B$

The first $m-$constrains (3) follow from the total number of seats each party has to receive at the second distribution. The next $n-$ constrains (4) follow by the number each region offers at the 2nd distribution.

3. The method

Objective function ϵ^* with constrains (3–4) form a special problem of linear programming. Transportation problem, where $m-$ sources (the parties) transfer seats to the $n-$ destinations (the constituencies) with cost matrix the values r_{ij}.

Transportation problem [4, p. 207] has an interesting structure. All the coefficients in the constrains are 1 and every variable appears in exactly two equations. Two very important properties hold:

- The problem has a solution. As long as the supply equals demand there is a feasible solution.
- There is an integer solution. If supplies and demands are integers then every feasible solution (including the optimal) shares integer values [4, p. 212].

Thus we avoid the solution using the more difficult methods of integer programming. Simple use of linear programming techniques are capable of providing the solution.

Table 4: The modified tableau arrived from data given in Table–3.

party				A	B	C	D	row sums
				regions				
a	-	-	0	0.42	-	-	-	1
b	-	0	-	0.85	-	-	-	1
c	0	-	-	0.73	-	-	-	1
a	-	-	0	-	0.60	0.71	0.27	2
b	-	0	-	-	0.38	0.18	0.62	2
c	0	-	-	-	0.02	0.11	0.11	1
	1	1	1	2	1	1	1	<col sums

The problem is a variation of a capacitated transportation problem [4, p. 225–227]. Since k_{ij}^B can be only 0 or 1, then we may call it as *binary transportation problem*. Anyway a solution based in a modification of transportation problem method can be used here. It is based on a certain splitting of rows and columns of the corresponding tableau.

An illustrative example is shown in cost-tableau given in Table–3. In this tableau the numbers in the main grid represent the remainders r_{ij}. In the right foremost column we recorded the seats of each party for the second distribution. The final row shows the seats offered by each region. Classical transportation problem methods do not apply directly to this data. We can not ensure that some party will not receive two seats in region–A.

Thus we manipulate the tableau in the cases of the regions having more than one seat. In the working example we fix region–A. Thus the new modified tableau is given in Table–4. We added $m = 3$ slack columns and another three rows. Whatever is registered in the slack columns does not count. The seats in region–A are assigned in the three new rows. Empty places in tableau mean that no value can be given there since no k_{ij}^B corresponds there.

We observe that the splitting of rows and columns from the first table (Table 3) to the second table (Table 4) is unique. Then, as long as the solution of the transportation problem is unique we conclude the same for the presented method. But, if for example party–a is entitled to get 5 seats then our discussion is meaningless.

4. Application: Greek elections of 2007

Applying the above method we may improve the results of the Greek parliamentary elections of 2007. In [11] we gave briefly some results on the elections of 2004 but not the actual method presented here. The results were manipulated using MATLAB [5], and are summarized in Tables 5 and 6. In these tables first column presents the abbreviation for the constituencies. In the next five columns we recorded the fair share for the parties at each region taking into account only the first m parties, i.e. the numbers

$$f_{ij} = \frac{v_{ij}}{\sum_{k=1}^{m} v_{kj}} \cdot s_j, \;\; i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n.$$

Thus the row sum of these five columns is an integer number equal to s_j, $j = 1, 2, \ldots, n$. The column sums are decimals and may differ than the corresponding e_i, $i = 1, 2, \ldots, m$.

The next five columns show the distribution under the current low. In the columns $12 - 16$, the distribution under the proposed method appears. The stars appeared to the right of some numbers in columns representing current method show the seats lost. Thus the 2* in the row for region ΛΕΣ (region of Lesvos island) means that KKE would have lost its seat in favor of ΝΔ. It is obvious that no party looses any seat in total.

The basic difference is that for current law we calculate $\epsilon = 54.36$ while the proposed method gives $\epsilon = 35.56$. Thus a near 50% improvement in the value

Table 5: The results of the Greek elections in the first 28 regions in alphabetical order. N stands for NΔ, Π for ΠΑΣΟΚ, Κ for ΚΚΕ, Σ for ΣΥΝ and Λ for ΛΑΟΣ

	region quotas					current					new					Bazi				
	N	Π	Κ	Σ	Λ	N	Π	Κ	Σ	Λ	N	Π	Κ	Σ	Λ	N	Π	Κ	Σ	Λ
Α'ΑΘ	7.16	5.35	1.88	1.65	0.96	7	5	2	2	1	7	5	2	2	1	8	4	2	2	1
Α'ΘΕ	6.32	5.85	1.74	1.03	1.05	7	5	2	1	1	7	5	2	1	1	7	4	2	1	2
Α'ΠΕ	2.65	2.04	0.56	0.42	0.33	2	1*	1	1	1	3*	2*	1	0	0	3	1	1	1	0
ΑΙΤ	3.54	3.55	0.51	0.23	0.16	4	3	1	0	0	4	3	1	0	0	4	3	1	0	0
ΑΡΓ	1.50	1.15	0.15	0.11	0.10	2	1	0	0	0	2	1	0	0	0	2	1	0	0	0
ΑΡΚ	1.43	1.19	0.16	0.12	0.10	2	1	0	0	0	2	1	0	0	0	2	1	0	0	0
ΑΡΤ	1.44	1.18	0.20	0.12	0.05	2	1	0	0	0	2	1	0	0	0	2	1	0	0	0
ΑΤΤ	4.89	4.52	1.15	0.72	0.72	5	4	1	1	1	5	4	1	1	1	5	4	1	1	1
ΑΧΑ	3.41	4.25	0.66	0.45	0.23	3	4	1	1	0	4*	4	1	0	0	4	3	1	1	0
Β'ΑΘ	15.53	14.99	5.33	3.94	2.22	17*	14	5	4	2	16	15*	5	4	2	16	12	6	5	3
Β'ΘΕ	3.26	2.46	0.59	0.30	0.39	3	2	0	1	1	3	2	1*	0	1	3	2	1	0	1
Β'ΠΕ	2.62	3.17	1.22	0.54	0.46	2	3	1	1	1	3*	3	1	1	0	3	2	1	1	1
ΒΟΙ	1.69	1.66	0.33	0.18	0.14	3*	1	0	0	0	2	1	1*	0	0	2	2	0	0	0
ΓΡΕ	0.46	0.41	0.08	0.03	0.03	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
ΔΡΑ	1.50	1.15	0.14	0.10	0.12	2	1	0	0	0	2	1	0	0	0	2	1	0	0	0
ΔΩΔ	2.12	2.37	0.21	0.15	0.14	3	2	0	0	0	3	2	0	0	0	3	2	0	0	0
ΕΒΡ	1.97	1.63	0.18	0.09	0.14	3*	1	0	0	0	2	1	0	0	1*	3	1	0	0	0
ΕΥΒ	2.50	2.59	0.44	0.26	0.21	3	2	1	0	0	3	2	1	0	0	3	2	1	0	0
ΕΥΡ	0.45	0.47	0.03	0.02	0.02	0	1	0	0	0	1*	0	0	0	0	1	0	0	0	0
ΖΑΚ	0.39	0.43	0.11	0.04	0.02	0	1	0	0	0	1*	0	0	0	0	1	0	0	0	0
ΗΛΕ	2.57	2.84	0.29	0.18	0.12	4*	2	0	0	0	3	3*	0	0	0	3	3	0	0	0
ΗΜΑ	1.76	1.61	0.31	0.13	0.19	3*	1	0	0	0	2	1	1*	0	0	2	2	0	0	0
ΗΡΑ	2.73	4.45	0.36	0.35	0.11	3	4	0	1	0	3	4	0	1	0	3	4	0	1	0
ΘΕΣ	0.47	0.43	0.04	0.03	0.02	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
ΙΩΑ	2.23	1.99	0.41	0.24	0.12	2	2	1	0	0	3*	2	0	0	0	3	2	0	0	0
ΚΑΒ	1.85	1.61	0.26	0.14	0.13	3*	1	0	0	0	2	1	0	1*	0	2	2	0	0	0
ΚΑΡ	2.44	1.92	0.38	0.15	0.11	4*	1	0	0	0	3	2*	0	0	0	3	2	0	0	0
ΚΑΣ	1.11	0.66	0.08	0.08	0.07	1	1	0	0	0	2*	0	0	0	0	1	1	0	0	0

Table 6: Table–5 continued with another 28 regions.

	region quotas					current					new					Bazi				
	N	Π	K	Σ	Λ	N	Π	K	Σ	Λ	N	Π	K	Σ	Λ	N	Π	K	Σ	Λ
KEP	1.23	1.12	0.46	0.12	0.06	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0
KEΦ	0.41	0.38	0.14	0.04	0.03	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
KIΛ	1.44	1.13	0.22	0.07	0.13	2	1	0	0	0	2	1	0	0	0	2	1	0	0	0
KOZ	2.40	2.05	0.29	0.15	0.11	3	2	0	0	0	3	2	0	0	0	3	2	0	0	0
KOP	1.79	1.71	0.18	0.17	0.15	3*	1	0	0	0	2	1	0	1*	0	2	2	0	0	0
KYK	1.41	1.17	0.17	0.16	0.09	2	1	0	0	0	2	1	0	0	0	2	1	0	0	0
ΛAK	1.70	0.97	0.15	0.09	0.10	2	1	0	0	0	2	1	0	0	0	2	1	0	0	0
ΛAP	3.56	3.01	0.82	0.32	0.29	3	3	1	0	1	4*	3	1	0	0	4	3	1	0	0
ΛAΣ	0.75	1.07	0.07	0.09	0.02	1	1	0	0	0	1	1	0	0	0	1	1	0	0	0
ΛEΣ	1.21	1.16	0.43	0.12	0.08	1	1	1	0	0	2*	1	0	0	0	1	1	1	0	0
ΛEY	0.44	0.38	0.11	0.05	0.02	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
MAΓ	2.23	1.88	0.48	0.22	0.19	2	2	1	0	0	3*	1	1	0	0	2	2	1	0	0
MEΣ	2.67	1.64	0.32	0.24	0.14	4*	1	0	0	0	3	1	0	1*	0	3	2	0	0	0
ΞAN	1.23	1.50	0.11	0.08	0.08	1	2	0	0	0	2*	1	0	0	0	2	1	0	0	0
ΠEΛ	1.93	1.61	0.19	0.09	0.17	3*	1	0	0	0	2	1	0	0	1*	2	2	0	0	0
ΠIE	1.98	1.54	0.22	0.11	0.15	3*	1	0	0	0	2	1	0	0	1*	3	1	0	0	0
ΠPE	0.94	0.79	0.16	0.08	0.04	1	1	0	0	0	1	1	0	0	0	1	1	0	0	0
PEΘ	0.94	0.89	0.07	0.07	0.02	1	1	0	0	0	1	1	0	0	0	1	1	0	0	0
POΔ	1.40	1.40	0.09	0.06	0.05	1	2	0	0	0	2*	1	0	0	0	2	1	0	0	0
ΣAM	0.39	0.34	0.19	0.05	0.03	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
ΣEP	3.78	2.37	0.36	0.21	0.28	4	2	0	0	1	4	2	0	0	1	4	2	0	0	1
TPI	2.32	2.00	0.43	0.14	0.10	2	2	1	0	0	3*	2	0	0	0	3	2	0	0	0
ΦΘI	2.52	1.87	0.27	0.16	0.17	4*	1	0	0	0	3	2*	0	0	0	3	2	0	0	0
ΦΛO	0.98	0.80	0.11	0.06	0.05	1	1	0	0	0	1	1	0	0	0	1	1	0	0	0
ΦΩK	0.49	0.36	0.07	0.04	0.04	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
XAΛ	1.43	1.18	0.17	0.12	0.11	2	1	0	0	0	2	1	0	0	0	2	1	0	0	0
XAN	1.59	1.84	0.26	0.21	0.11	3*	1	0	0	0	2	2*	0	0	0	2	2	0	0	0
XIO	0.90	0.85	0.12	0.09	0.03	1	1	0	0	0	1	1	0	0	0	1	1	0	0	0

of ϵ is achieved by interchanging 28 seats (total number of stars in the columns $12 - 16$).

Another interesting improvement is that fair share is not violated by the new method while the other divisor-based bi-proportional methods do not share this property [12]. This means that no party receives seats out of the interval

$$[\lfloor f_{ij} \rfloor, \lfloor f_{ij} \rfloor + 1].$$

The current method violates this property 13 times. The violations are recorded by a star in columns $7 - 11$. Thus we see that ΠΑΣΟΚ receives only one seat in A' of Piraeus (A'ΠΕ) with a fair share of 2.04 while ΝΔ gets 17 seats in B' of Athens with fair share of only 15.53. Actually all the stars in the 7th column indicate the fair share violation in favor of the first party.

Divisor-based methods are in common use for addressing the problem we discussed here. In such a method the problem is solved by computing appropriate row- and column-divisors, and by rounding the quotients. The only known divisor-based method that provably solves the problem is the tie-and-transfer algorithm by Balinski and Demange [1].

The software BAZI [9] is the state of the art in this area. It is freely available from [3] and we applied it to our data. BAZI does not combine Hamilton/Hare method with bi-proportional apportionment. Thus we recorded in Tables–5 and 6 the combination of Webster/Sainte-Laguë method with bi-proportional apportionment. We got $\epsilon = 49.60$, a surprisingly large value. This was done since the method used with BAZI violates fair share many times. In the first three rows of Table-5, ΠΑΣΟΚ receives 4 seats in A' of Athens (A'ΑΘ), 4 seats in A' of Salonika (A'ΘΕ) and only 1 seat in A' of Piraeus (A'PE). Its corresponding fair shares were $5.35, 5.85$ and 2.04 respectively. This erratic behavior is extremely amplified by the definition of ϵ. Combination of the other divisor methods implemented in BAZI (e.g. Jefferson/D'Hondt or Huntington/Hill) gave no better results and the same problem with fair share was observed.

The method we presented here may apply very well for proportional allotment without bonus. But in the Greek elections the first party gets as bonus many seats. Thus, proportionality is lost. As the magnitude of the bonus is low then we experience no problems. Even with a bonus as high as $0.7 \cdot n$ we may avoid them. But in case of a large bonus then we may have $e_1^B > n$. Then the first party has to get two seats in some regions. In such a case our analysis does not hold. A way around this drawback is to give a seat ad hoc to the first party in all the regions so e_1^B becomes $e_1^B - n$ and s_j^B becomes $s_j^B - 1$, $j = 1, 2, \ldots, n$.

It is also true that some paradoxes can not be avoided. For example in regions of HPA (Heraklion) and KAB (Kavala) the fourth party receives a seat while the third party receives nothing. However all parties receive seat within fair share which is the basic restriction of our proposed methods. These paradoxes are present in the current method too. BAZI software also admits such paradoxes, see regions of B'ΘΕ or HPA. Anyway, even the last party has to gain some seats and this may happen only after the acceptance of some peculiar outcomes.

A problem similar to the one described for Greece is experienced under the recent Italian electoral law [7]. In Italy the system may end up by awarding a party more (or less) seats within the regions than those the same party is entitled to at the national level. In addition $e_1^B >> n$ for the Italian case. Thus our method does apply in this case only for the part of minimization of ϵ but we can not restrict the values k_{ij}^B to be in $0, 1$. A Mixed Integer Linear Problem procedure in the logic of Largest Remainders of maximum absolute difference (1) is proposed in [8] for tackling the problem.

5. Discussion

A new method of bi–proportional apportionment was presented in this paper. It seems to outperform divisor–type methods under a certain least–squares type criterion. Using a "linear transportation problem"–type method we may easily derive the results wanted. Application of the new method to the Greek parliamentary elections of 2007 gave very pleasant results.

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