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A spinning test body in the strong field of a Schwarzschild black hole

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Abstract. Binary pulsars could help us probe into strong gravitational field situations. One of the relativistic effects that could manifest itself in observations of such pulsars is the precession of the pulsars' spin axis due to spin–orbit coupling. This paper derives an analytical formula for the precessional frequency of the spin of a test particle in a circular orbit around a nonrotating massive black hole, that is not restricted to weak field regions only. Approximate analytical and numerical solutions of the spin precession in case of other, noncircular, bound orbits have also been investigated. Finally the observational implications of such a precession are discussed.

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1. Introduction

The recent expansion on the number of binary pulsars that have been detected by radioastronomers and the promising future detection of gravitational waves from coalescing compact binaries brings an imperative need for a full knowledge of the relativistic equations of motion for binary systems, especially for those consisting of spinning bodies. This knowledge is crucial both for extracting astronomical information by accurately timing binary pulsars and for detecting gravitational waves.

Unfortunately, the fully relativistic two-body problem still remains very difficult so that there is no hope for analytical treatment, but various special cases of the problem have already been investigated analytically by a number of authors. Although these special cases are often idealized situations, they serve as a useful source for gaining insight into realistic situations. One of these special cases is the topic of our present work. Namely, the spin evolution of a test body in the gravitational field of a nonrotating massive black hole.

Since 1951, when Papapetrou [1] first derived the equations of motion for a spinning test particle in a gravitational field, several people have considered the problem of the motion of spinning bodies in the gravitational field of a black hole or some other compact object such as a neutron star [2, 3], by means of post-Newtonian approximations. The fully relativistic equations of motion were not of much importance then, since all relevant observations and experiments were only dealing with weak-field situations.

This is no longer the case. A rotating neutron star or a small rotating black hole orbiting around a massive black hole will follow a precessional orbital motion, due to spin–orbit coupling (and additionally to spin–spin coupling if the massive black hole is rotating), while emitting gravitational waves. However, for close orbits this precessional motion cannot be correctly deduced from the post-Newtonian equations of motion, see [4].

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Therefore a binary pulsar in a very close orbit around a massive black hole will be out of sight, due to spin precession, in a different time period from the time period computed from the post-Newtonian equations of motion [5, 6].

The purpose of this paper is to study the relativistic precession of the spin axis of a spinning test body that is moving on a close orbit around a much more massive and nonrotating black hole. Because of astrophysical limitations on the spin's magnitude, the test body hardly deviates from a geodesic and thus this special low-mass-ratio case does not introduce any complications in the computation of gravitational waveforms. However, the precession of the spin could substantially affect the observations of the relevant binary pulsars.

The rest of the paper is organized as follows. In section 2 we review the equations of motion, derived by Papapetrou [1] for a spinning test particle in a gravitational field, and apply them to the gravitational field of a Schwarzschild black hole. We then show that in realistic situations the particle's spin is very small compared to its orbital angular momentum and thus all the higher spin terms can be neglected. On the one hand, this approximation simplifies considerably the equations of motion for the spin and on the other, it leaves the orbital motion of the particle almost planar. We also show that for tiny mass ratios, that is for the case we consider, the timescale for radiation reaction is much higher than the timescale for spin precession. This makes radiation reaction quite irrelevant to our calculations for the spin precession.

In section 3 we make an attempt to solve the equations of motion for the spin. Since the orbital precession is very small compared to the spin precession, we can assume that the spinning body moves on a geodesic orbit. In section 3.1 we focus on spinning test bodies that move on circular orbits. We then show that for this kind of orbit the spin precesses in a uniform fashion, and we derive an analytical formula for the precessional frequency. We also discuss the difference between this exact (as long as we are dealing with extremely low mass ratios) precessional frequency and the approximate frequency, that has been extensively used in the literature (see [3, 4]), and which is based on post-Newtonian equations of motion. As is expected, our formula for the frequency is approximately the same as the post-Newtonian formula in the weak-field region of the large gravitating body. In section 3.2 we extend our analysis to generic noncircular orbits. Now, since the spinning particle finds itself in different regions of the gravitational background field of the black hole, its spin precesses in a non-uniform fashion. Since we cannot derive an analytical expression to describe the precessional motion in this case, we discuss the basic characteristics of this motion and we analyse some limiting cases that admit an approximate analytical solution. For completeness, we investigate numerically this general case and present several numerical examples. In section 4 we summarize our results and discuss their implications on possible future observations of binary pulsars with massive non-spinning companions. Finally, in section 5 we conclude with the hypothetical situation of such a binary pulsar and present the observational effects related to its spin-induced precession.

In the appendix we examine the validity of the Papapetrou equation and conclude that it cannot be used for the comparable-mass case since the omission of higher-order spin terms introduces a higher-order error than the error due to the assumption of a fixed background geometry.

Throughout we use units where $G = c = 1$.

2. Equations of motion

In 1951, Papapetrou [1] derived the equations of motion for a spinning test particle in an arbitrary gravitational field. His conclusion was that such a particle deviates from its geodesic and moves on a different orbit according to

$$\frac{\mathrm{D}}{\mathrm{d}\tau}\left(mu^\alpha + u_\beta \frac{\mathrm{D}S^{\alpha\beta}}{\mathrm{d}\tau}\right) + \frac{1}{2} S^{\mu\nu} u^\sigma R^\alpha_{\nu\sigma\mu} = 0 \tag{1a}$$

while its spin moves according to

$$\frac{\mathrm{D}S^{\alpha\beta}}{\mathrm{d}\tau} + u^\alpha u_\rho \frac{\mathrm{D}S^{\beta\rho}}{\mathrm{d}\tau} - u^\beta u_\rho \frac{\mathrm{D}S^{\alpha\rho}}{\mathrm{d}\tau} = 0 \tag{1b}$$

where $\mathrm{D}/\mathrm{d}\tau$ denotes a covariant derivative along u^α, $S^{\alpha\beta}$ is the antisymmetric spin tensor of the particle, m and u^α are the particle's mass and 4-velocity, respectively, and $R^\alpha_{\nu\sigma\mu}$ is the Riemann tensor describing the gravitational field background on which the particle moves. In these equations the test particle is assumed so small in size and in mass that it does not alter the gravitational background and the radiation reaction has a negligible effect on the particle's orbit. One can see from equation (1a) that for vanishing spin the particle moves on a geodesic. By introducing the Pirani spin supplementary condition (SSC) [7]

$$S^{\alpha\beta} u_\beta = 0 \tag{2}$$

and the Pauli–Lubanski covariant spin vector

$$S_\sigma \equiv \tfrac{1}{2} \epsilon_{\rho\mu\nu\sigma} u^\rho S^{\mu\nu} \tag{3}$$

with an inverse solution $S^{\mu\nu} = \epsilon^{\alpha\beta\mu\nu} u_\alpha S_\beta$, and after some simple manipulations, the equations of motion (1) take the following form (cf [8]):

$$m\frac{\mathrm{D}u^\alpha}{\mathrm{d}\tau} = -S_\mu u_\nu \frac{\mathrm{D}^2 u_\beta}{\mathrm{d}\tau^2} \epsilon^{\alpha\mu\nu\beta} + \frac{1}{2}\epsilon^{\lambda\mu\rho\sigma} R^{\alpha\nu}{}_{\lambda\mu} u_\nu u_\sigma S_\rho \tag{4a}$$

and

$$\epsilon^{\mu\nu\alpha\beta} u_\mu \frac{\mathrm{D}S_\nu}{\mathrm{d}\tau} = (\epsilon^{\rho\nu\alpha\beta} u^\mu - \epsilon^{\mu\nu\rho\beta} u^\alpha - \epsilon^{\mu\nu\alpha\rho} u^\beta) u_\mu S_\nu \frac{\mathrm{D}u_\rho}{\mathrm{d}\tau}. \tag{4b}$$

In all the equations above, $\epsilon^{\alpha\beta\gamma\delta}$ is the completely antisymmetric Levi-Civita tensor. The replacement of the tensorial $S^{\alpha\beta}$ by the vectorial S_σ helps us to have a better understanding of the spin precession, since S^i, where $i = 1$–3, are the components of what we have been used to thinking of as the spin 3-vector. It is also easy to see that by construction (see equation (3)) S_α is kept orthogonal to the particle's 4-velocity u^α.

Now, in realistic situations, the magnitude of the spin for a black hole is strictly limited to $|\boldsymbol{S}| \leqslant m^2$. For a neutron star its spin depends somewhat on the uncertain nuclear equation of state. Most candidate equations of state yield an upper limit of $|\boldsymbol{S}| \lesssim 0.5m^2$ for uniformly rotating neutron stars, see [9]. In these expressions, m is the mass of the object. On the other hand, the orbital angular momentum of a small mass object orbiting around a nonrotating massive black hole is

$$|\boldsymbol{L}| = mM(r_0/M)^{1/2}(1 - 3M/r_0)^{-1/2} \tag{5}$$

where m and M are the masses of the orbiting object and the massive black hole, respectively, and r_0 is the orbital radius that corresponds to the minimum of the effective potential for the radial motion of the orbiting object in the Schwarzschild geometry of the massive black hole. In other words, r_0 is nearly the mean radial position of the orbiting object around which it undergoes radial oscillations. Now, since the lowest value that r_0 can attain is $6M$, it becomes obvious that the ratio of the spin to the orbital angular momentum of a spinning astrophysical compact object is

$$\frac{|\boldsymbol{S}|}{|\boldsymbol{L}|} \leqslant \frac{1}{\sqrt{12}} \frac{m}{M}. \tag{6}$$

This means, that for a very small ratio of masses m/M, as it is the case under consideration since we are dealing with a test particle, the spin is very small compared to the orbital angular momentum and thereby all the spin squared and higher terms could be omitted. Then the equations of motion (4) simplify to

$$m \frac{\mathrm{D}u^\alpha}{\mathrm{d}\tau} = \tfrac{1}{2}\epsilon^{\lambda\mu\rho\sigma} R^{\alpha\nu}{}_{\lambda\mu} u_\nu u_\sigma S_\rho + \mathcal{O}(S^2) \tag{7a}$$

and since $\epsilon^{\mu\nu\alpha\beta} u_\mu \mathrm{D}S_\nu/\mathrm{d}\tau = \mathcal{O}(S^2) \simeq 0$, after multiplying this equation by $\epsilon_{\rho\sigma\alpha\beta}u^\rho$ it can be shown that

$$\frac{\mathrm{D}S_\sigma}{\mathrm{d}\tau} = u_\sigma S_\lambda \frac{\mathrm{D}u^\lambda}{\mathrm{d}\tau} = \mathcal{O}(S^2) \simeq 0. \tag{7b}$$

In other words, the spin is to a good approximation parallel-transported along the particle's slightly nongeodesic orbit.

By specializing to the gravitational field of a Schwarzschild black hole in Schwarzschild coordinates, equations (7) transform to

$$m \frac{\mathrm{D}u^t}{\mathrm{d}\tau} = -\frac{3M \sin\theta}{r} u^r (S^\theta u^\phi - S^\phi u^\theta)(1 - 2M/r)^{-1} \tag{8a}$$

$$m \frac{\mathrm{D}u^r}{\mathrm{d}\tau} = -\frac{3M \sin\theta}{r} u^t (S^\theta u^\phi - S^\phi u^\theta)(1 - 2M/r) \tag{8b}$$

$$m \frac{\mathrm{D}u^\theta}{\mathrm{d}\tau} = -\frac{3M \sin\theta}{r} u^\phi (S^r u^t - S^t u^r) \tag{8c}$$

$$m \frac{\mathrm{D}u^\phi}{\mathrm{d}\tau} = -\frac{3M}{r \sin\theta} u^\theta (S^r u^t - S^t u^r) \tag{8d}$$

$$\dot{S}^t = -\frac{M}{r^2}(u^r S^t + u^t S^r)(1 - 2M/r)^{-1} \tag{8e}$$

$$\dot{S}^r = \left[\frac{M}{r^2}\left(-u^t S^t + \frac{u^r S^r}{(1-2M/r)^2}\right) + r u^\theta S^\theta + r \sin^2\theta u^\phi S^\phi\right](1 - 2M/r) \tag{8f}$$

$$\dot{S}^\theta = -\frac{1}{r}(u^r S^\theta + u^\theta S^r) + \sin\theta\cos\theta u^\phi S^\phi \tag{8g}$$

$$\dot{S}^\phi = -\frac{1}{r}(u^\phi S^r + u^r S^\phi) - \cot\theta(u^\phi S^\theta + u^\theta S^\phi) \tag{8h}$$

where r, θ, ϕ are the spherical polar coordinates of the test particle, u^μ is its 4-velocity, S^r, rS^θ, $r\sin\theta S^\phi$ are its polar spin components, S^t is its spin's coordinate-time component and an overdot represents '$\mathrm{d}/\mathrm{d}\tau$'. In equations (8) we did not write explicitly the covariant derivatives of the particle's 4-velocity since, as we shall show, these equations could be very well approximated by the corresponding geodesic equations. Without any loss of generality, let us assume that the particle is moving initially on a planar equatorial orbit ($\theta = \pi/2, \dot{\theta} = 0$). Then equation (8*d*) transforms to

$$\frac{\mathrm{d}|\boldsymbol{L}|}{\mathrm{d}\tau} = 0. \tag{9}$$

Thus the orbital angular momentum is conserved in magnitude and the only spin effect on it is a possible shift of the $\boldsymbol{L}$ direction. Now, the total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ should be conserved. (This cannot be proved as an exact formula from equations (8) since they are just an approximation, though a very useful one for our case, of the exact equations of motion (4)). Moreover, since $S/L \ll 1$ (see equation (6)), $\boldsymbol{L}$ and consequently the orbital plane hardly move. The low spin to orbital angular momentum ratio also guarantees that the particle's motion will hardly deviate from a geodesic, cf [11]. The only new effect, due to the presence of the spin, will be the one connected to the motion of the spin and it will be computed in the next section. (For further discussion on the validity of the approximations made on equations (7) for small ratio of masses see the appendix.)

One effect that has not been taken into account in our previous discussion is radiation reaction. The timescale for radiation reaction is

$$T_{\mathrm{radiation}} = \frac{E}{|\mathrm{d}E/\mathrm{d}t|} \sim \frac{M^2}{m}\left(\frac{r_0}{M}\right)^5. \tag{10}$$

However, as we will show in the next section, the timescale for the spin's motion (the period of its precession) is given (see equation (21)), as an order of magnitude, by

$$T_{\mathrm{precess}} \sim T_{\mathrm{orbital}}\left(\frac{r_0}{M}\right) \tag{11}$$

where T_{orbital} is the orbital period which is equal to $2\pi M(r_0/M)^{3/2}$. Therefore

$$\frac{T_{\mathrm{radiation}}}{T_{\mathrm{precess}}} \sim \frac{M}{m}\left(\frac{r_0}{M}\right)^{5/2} \gg 1. \tag{12}$$

This means that, as long as we are dealing with a very small ratio of masses, we are free to omit any radiation reaction effect from our calculations.

3. The solution

In this section, we attempt to derive a solution of equations (8*e*–*h*) by assuming that the particle moves on a planar orbit, say $\theta = \pi/2$; an assumption that was justified in the previous section. Equation (8*g*) then simplifies to

$$rS^{\theta} \equiv S_{\|} = \text{constant}. \tag{13}$$

We have denoted this constant spin component by $S_{\|}$ because it is parallel to the particle's orbital angular momentum which is assumed to be directed along the z-axis. Also, equations (8*a*, *b*) simplify to

$$\dot{t}(1 - 2M/r) = \mathrm{a} + \frac{MLS_{\|}}{m^2r^3} \tag{14a}$$

$$\frac{1}{2}\dot{r}^2 = b + x - \left(\frac{L^2}{2m^2M^2}\right)x^2 + \left(\frac{L^2 + LS_{\|}\mathrm{a}}{m^2M^2}\right)x^3 \tag{14b}$$

where $x = M/r$ and a, b are constants that arise from integration of the corresponding equations. These two constants also are interrelated by

$$\mathrm{a}^2 - 2b = 1 \tag{14c}$$

which is a consequence of $u^{\mu}u_{\mu} = -1$. In particular, a is of order unity, since for bound quasi-adiabatic orbits the orbiting particle never passes very close to the black hole horizon, but

$$\frac{MLS_{\|}}{m^2r^3} \leqslant \frac{ML}{r^3} \leqslant \left(\frac{m}{M}\right)\sqrt{2}\left(\frac{M}{r}\right)^{5/2}\left(\frac{r_0}{r}\right)^{1/2} \ll 1 \tag{15}$$

where we have taken into account the fact that r_0/r for bound orbits in a Schwarzschild geometry cannot exceed a numerical value of order unity. It is also true that

$$S_{\|}a \ll L \tag{16}$$

therefore the two terms in equations (14a, b) that contain $S_{\|}$ could be omitted. One then remains simply with the geodesic equations of motion without any spin influence on them.

Now, since the elements of the particle's motion are given and do not depend on the spin, it is very simple to analyse the spin motion. We shall start by considering spinning test particles in circular orbits which turn out to admit an analytic solution for the motion of their spin.

3.1. Circular orbits

For circular orbits, the equations of motion reduce to a simple set of equations:

$$r = r_0 = \text{constant} \qquad \theta = \pi/2 \qquad \dot{t} = \mathrm{a}(1-2x_0)^{-1} \qquad \dot{\phi} = \frac{L}{mr^2} \tag{17}$$

where $x_0 = M/r_0$. After plugging these results and the orthogonality relation $u^\alpha S_\alpha$ into equations (8f, h) and a few lines of algebra we obtain a simple coupled set of differential equations:

$$\mathrm{d}S^r/\mathrm{d}\tau = \frac{L}{mr_0^2}(1-3x_0)(rS^\phi) \qquad \mathrm{d}(rS^\phi)/\mathrm{d}\tau = -\frac{L}{mr_0^2}S^r \tag{18}$$

with the obvious solution

$$S^r = \mathcal{A}\sin(\omega\tau+\varphi) \qquad rS^\phi = \mathcal{B}\cos(\omega\tau+\varphi) \tag{19a}$$

where

$$\omega = \frac{L}{mr_0^2}\sqrt{1-3x_0} \tag{19b}$$

and

$$\mathcal{A} = \mathcal{B}\sqrt{1-3x_0}. \tag{19c}$$

In equation (19a) φ is a constant phase related to initial conditions. The use of a frame of reference not fixed to the moving particle has the disadvantage of seeing a spin contraction along the motion of the particle and thus a nonconservation of the spin 3-vector's magnitude. This explains the difference between the two amplitudes $\mathcal{A}$ and $\mathcal{B}$. By shifting to the comoving orthonormal frame of reference, where $S^{\hat{t}'} = 0$ and $\boldsymbol{S}' \cdot \boldsymbol{S}' = $ constant, one obtains a simple precessional spin motion

$$\boldsymbol{S}'_{\hat{\perp}} \equiv S'_{\hat{x}} + \mathrm{i}S'_{\hat{y}} = \mathcal{B}\mathrm{i}\sqrt{\frac{1-3x_0}{1-2x_0}}\exp\left[\mathrm{i}\left(\phi\left(1-\sqrt{1-3x_0}\right)-\varphi\right)\right] \tag{20}$$

where $\boldsymbol{S}'_{\hat{\perp}}$ is the spin component perpendicular to $\boldsymbol{L}$ and ϕ is simply $\int\dot{\phi}\,\mathrm{d}\tau = L\tau/mr_0^2$, while the spin component parallel to $\boldsymbol{L}$, $\boldsymbol{S}'_{\hat{\|}}$ remains constant and is given by equation (13). Then the precessional frequency of the spin is

$$\Omega_{\text{precess}} = \frac{1}{M}x_0^{3/2}\left(\frac{1}{\sqrt{1-3x_0}}-1\right) \tag{21}$$

where the magnitude of $\boldsymbol{L}$, given in equation (5), has been used. This result is in full agreement with the post-Newtonian result (cf [3]) for the precessional frequency of the spin as long as the particle moves in an orbit not in very close to the massive black hole. However, the two results deviate from each other for orbits in the strong-field region. This is shown in figure 1. It is clear that the post-Newtonian approximation may offer a quite accurate result for $r_0 \gtrsim 30M$ but not for closer orbits. The dashed line in figure 1 shows the relative error made if the post-Newtonian expression for the precessional frequency is used instead of equation (21).

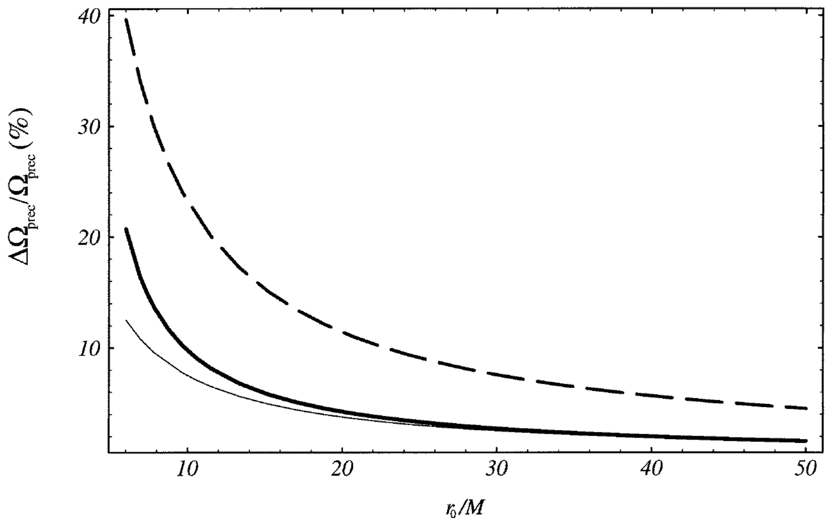

Figure 1. The two full curves show, in arbitrary units, Ω_{precess} as a function of the radius of the orbit (i) according to the post-Newtonian calculation (thin curve) and (ii) according to the exact equation (21) (thick curve). It is clear, from the figure, that the post-Newtonian result deviates substantially from the true one only for orbits that are very close to the massive black hole ($r_0 \lesssim 30M$). The dashed curve shows the percentage error made by using the post-Newtonian approximate formula instead of the exact one, given in equation (21).

3.2. Generic bound orbits

We now extend the analysis of the previous section to include generic noncircular bound orbits. These orbits are characterized by two frequencies, the azimuthal and the radial one. Thus, it is expected that in this case the spin precesses in a more complicated manner; faster near the periastron position where the field is stronger and slower near the apastron position where the field is weaker.

The equations of motion are given now in terms of elliptic functions

$$x \equiv \frac{M}{r} = x_0 + \delta_- + (\delta_+ - \delta_-)\,\mathrm{sn}^2\left(\sqrt{\frac{\delta_0 - \delta_-}{2}}\phi \,\middle|\, \sin^{-1}\sqrt{\frac{\delta_+ - \delta_-}{\delta_0 - \delta_-}}\right)$$

$$\theta = \pi/2 \qquad \dot{t} = \mathrm{a}(1 - 2x)^{-1} \qquad \dot{\phi} = \frac{L}{mr^2} \tag{22}$$

where $\delta_0 \geqslant \delta_+ \geqslant \delta_-$ are the three roots of the cubic equation $2z^3 - (1 - 6x_0)z^2 + x_0^2(1 - 4x_0) + 2x_0 b(1 - 3x_0) = 0$ that arises from equation (14*b*), after omitting the $S_\parallel$ term, substituting the value of L given in equation (5), replacing x by $x_0 + z$ and setting $\dot{r} = 0$. The physical meaning of these new parameters is the following: $x_0 + \delta_\pm$ correspond to the value of x when the particle is at the periastron or apastron position. Also, $\mathrm{sn}(y|m)$ is the corresponding Jacobian elliptic function.

Then equations ($8f, h$), transform to

$$\frac{\mathrm{d}S^r}{\mathrm{d}\phi} = (rS^\phi)(1-3x) \qquad \frac{\mathrm{d}(rS^\phi)}{\mathrm{d}\phi} = -S^r. \tag{23}$$

Now, x is no longer a constant, as in equation (18), but a complicated function of the azimuthal angle ϕ, given in the first of equations (22). These two coupled first-order differential equations lead to one decoupled second-order differential equation for rS^ϕ,

$$\frac{\mathrm{d}^2(rS^\phi)}{\mathrm{d}\phi^2} + (rS^\phi)[1-3x(\phi)] = 0 \tag{24}$$

while the other spin component is given by

$$S^r = -\frac{\mathrm{d}(rS^\phi)}{\mathrm{d}\phi}. \tag{25}$$

Equation (24) has roughly the form of Mathieu's equation, but instead of the simple sinusoidal term appearing in Mathieu's equation there is a periodic elliptic function here; see equation (22). Therefore, we can proceed analytically only for some limiting cases (for example, when the particle orbits far away from the black hole or when its orbit is slightly noncircular) and rather treat the general problem numerically.

3.2.1. Noncircular orbits far from the black hole. If the spinning particle moves in the weak-field region of a black hole, or in other words when $x_0 \to 0$, it can be shown that $\delta_0 = 0.5 - 3x_0 + \mathcal{O}(x_0^2)$, $\delta_+ = px_0 + \mathcal{O}(x_0^2)$ and $\delta_- = nx_0 + \mathcal{O}(x_0^2)$, where p/n is a positive/negative number of order unity. Hence, if we only keep terms up to $\mathcal{O}(x_0)$, then the $x(\phi)$ function, given in equation (22), will acquire its desirable sinusoidal form,

$$x = x_0 - \frac{p-n}{2}\cos[(1-(3+p)x_0)\phi] \tag{26}$$

and thus equation (24) can be brought to a Mathieu equation form,

$$\frac{\mathrm{d}^2P}{\mathrm{d}z^2} + (a - 2q\cos 2z)P = 0 \tag{27a}$$

where

$$\begin{aligned} &P \equiv rS^\phi \qquad 2z = (1-(3+p)x_0)\phi \\ &a = 4 + (12+8p)x_0 \qquad -2q = 6x_0(p-n). \end{aligned} \tag{27b}$$

The general solution of such an equation is

$$P = AF_\nu(z) + BF_\nu(-z) \tag{28a}$$

where $F_\nu(z)$ is a function that can be expanded in terms of the small parameter q [10] as

$$F_\nu(z) = c_0\left[\mathrm{e}^{\mathrm{i}\nu z} - q\left(\frac{\mathrm{e}^{\mathrm{i}(\nu+2)z}}{4(\nu+1)} - \frac{\mathrm{e}^{\mathrm{i}(\nu-2)z}}{4(\nu-1)}\right)\right] + \cdots \tag{28b}$$

and ν is a parameter that can be expressed as an expansion of $\sqrt{a}$ in terms of powers of q:

$$\nu = \sqrt{a} + \mathcal{O}(q^2) + \cdots. \tag{28c}$$

By substituting the values of the various parameters introduced in equation (27*b*) we obtain the following solution to equation (27*a*) up to order $\mathcal{O}(x_0)$:

$$P = P_0\left[\left(1 - \frac{p-n}{2}x_0\cos 2z\right)\sin[(2+(3+2p)x_0)z] \right. \\ \left. +((p-n)x_0\sin 2z)\cos[(2+(3+2p)x_0)z]\right] \tag{29}$$

where an extra phase that should show up in all the sinusoidal terms has been omitted for the sake of simplicity. (A suitable choice of the initial data could make this phase vanish.) The function appearing in equation (29) is not exactly periodic but its successive zeros are separated by the constant $\Delta z = \pi - (3+2p)\pi x_0/2 + \mathcal{O}(x_0^2)$ and thus $\Delta\phi = 2\pi(1 + 3x_0/2 + \mathcal{O}(x_0^2))$ can be considered as the period of rS^ϕ, which in its turn means that the precessional frequency of the spin is

$$\Omega_{\text{precess}} = \Omega_{\text{orbit}}\left[\frac{3x_0}{2} + \mathcal{O}(x_0^2)\right]. \tag{30}$$

This result is in agreement with the post-Newtonian result of Barker and O'Connell [3].

3.2.2. Slightly noncircular orbits. In this section we examine the equation of motion for the spin of a particle moving in a slightly noncircular orbit. The orbit, now, is parametrized by a small parameter ϵ which is connected to the low eccentricity of the orbit. The particle is assumed to oscillate radially between $r_0(1+\epsilon)$ and $r_0(1-\epsilon+\mathcal{O}(\epsilon^2))$, hence the three roots $\delta_0, \delta_+, \delta_-$ obtain the following values, up to first order with respect to ϵ:

$$\delta_0 = \frac{1-6x_0}{2} - \epsilon x_0 + \mathcal{O}(\epsilon^2) \qquad \delta_+ = \epsilon x_0 + \mathcal{O}(\epsilon^2) \qquad \delta_- = -\epsilon x_0 + \mathcal{O}(\epsilon^2). \tag{31}$$

Then, by following the same analysis as in section 3.2.1, we end up again with a Mathieu's equation, up to first order with respect to ϵ,

$$\frac{\mathrm{d}^2P}{\mathrm{d}z^2} + \left(4\frac{1-3x_0}{1-6x_0} + \epsilon\frac{12x_0}{1-6x_0}\cos 2z\right) = 0 \tag{32}$$

where now $z = \sqrt{1-6x_0}\phi/2$. As expected, the precessional frequency of the spin is approximately equal to the precessional frequency for a circular orbit, given in equation (21). As is always the case, the solution of a Mathieu equation is not exactly periodic but it has equally spaced roots. From this we find that the precessional frequency is given by

$$\Omega_{\text{precess}} = \frac{x_0^{3/2}}{M}\left[\frac{1}{\sqrt{1-3x_0}} - 1 + \epsilon\frac{x_0}{1-2x_0}\frac{\sin(2\pi\sqrt{(1-6x_0)/(1-3x_0)})}{2\pi} + \mathcal{O}(\epsilon^2)\right]. \tag{33}$$

3.2.3. General case. We now examine numerically the general solution of equation (24) for generic eccentric orbits in the vicinity of a black hole. The precession of the spin is no longer uniform around the axis of the angular momentum. In figure 2 we show various plots of one of the spin components along the orbital plane as a function of the particle's azimuthal angle ϕ. The plots are parametrized by the mean distance to the black hole r_0 while the eccentricity has been chosen to be *nearly* the maximum for each r_0 case. (We have avoided the use of the maximum possible eccentricity since then, for orbits with $r_0 \leqslant 12M$, the orbiting particle would spend a considerable part of its time period close to the black hole and thus the spin would precess for a long time with the fast rate that the strong field there dictates.) In each diagram, an additional curve is shown in order to compare the spin

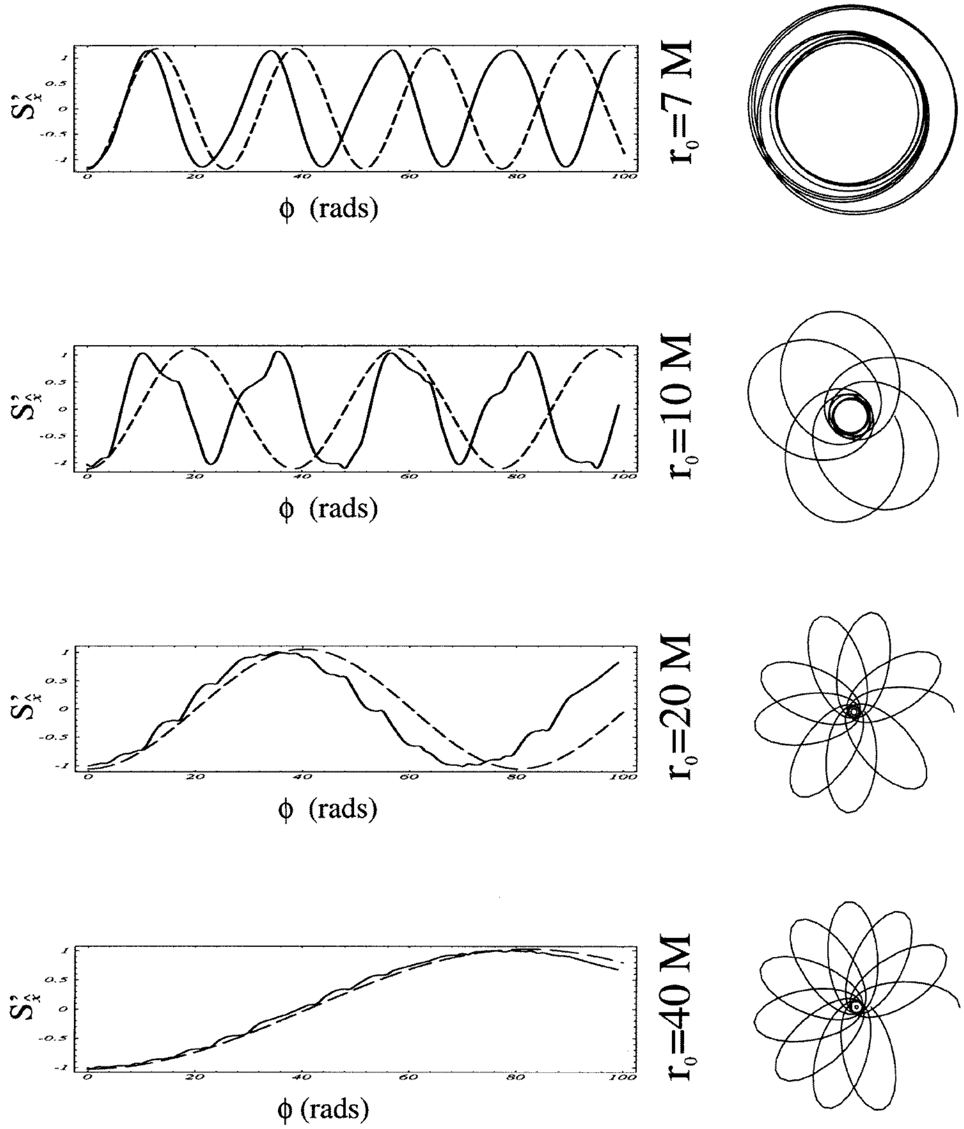

Figure 2. This is a series of diagrams showing the evolution of one of the components of the spin on the orbital plane for a highly noncircular orbit (full curves). The spin component $S'_{\hat{x}}$ that is plotted here refers to an orthonormal comoving frame of reference. The sinusoidal dashed curves show the corresponding spin component evolution for a circular orbit with the same r_0. On the right-hand side we have drawn a part of the orbit that corresponds to the parameters of each spin plot. The innermost circle in each of these orbital diagrams corresponds to $6M$.

precession for the very eccentric orbit (full curve) with the corresponding spin precession for a circular orbit with the same r_0 value (dashed curve).

From these figures the following remark can be drawn. By reducing r_0, for example for orbits that are closer and closer to the black hole, the spin precesses in a more complicated manner and deviates more from the spin precession of circular orbits. However, then, for very close orbits ($r_0 \approx 6M$) the orbits are again approximately circular and spin precession

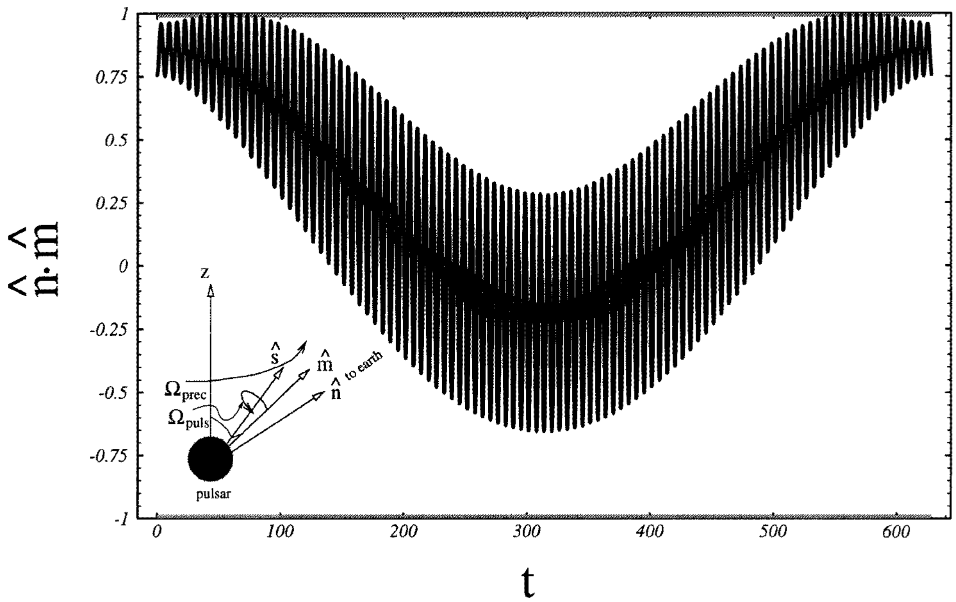

Figure 3. This is an illustrative example of a pulsar that comes in and out of sight because of its spin precession. The fast oscillations shown in the diagram are due to pulsar rotation. The envelope of these oscillations is changing because of the precessional motion of the spin. Two thin grey strips, one at the top and one at the bottom of the diagram, correspond to magnetic axis' orientations that miss the direction to Earth by no more than 10°. Therefore the pulsar is visible from the Earth whenever the oscillating curve enters one of these grey strips. Here the spin precessional frequency is assumed to be 1% of the pulsar frequency. The timescale is in arbitrary units, but it covers a full spin precession. In the inset we have drawn, for clarification, the corresponding pulsar with its orbital angular momentum oriented along the z-axis. Its magnetic axis $\hat{m}$ is rotating around its spin axis $\hat{s}$ which in its turn is precessing around the z-axis. The unit vector to Earth is denoted by $\hat{n}$.

is very uniform and approaches the spin precession of the circular orbits that is given analytically in equation (20).

4. Implications on observations

We have thoroughly analysed the precessional motion of the spin for any possible bound orbit around a Schwarzschild black hole. Now, we can infer its implications for pulsars orbiting a nonrotating massive black hole. According to equation (21) the spin axis of a pulsar precesses around its orbital angular momentum axis with a frequency Ω_{precess} (given in equation (21)) if its orbit is circular, and with about the same Ω_{precess} (give or take a factor of two) if its orbit is eccentric. This means that in a time period less than $2\pi/\Omega_{\text{precess}}$ the emission cone of a pulsar may go out of sight of the Earth and within a time period of $2\pi/\Omega_{\text{precess}}$ show up again. In figure 3 this situation is illustrated for some arbitrary geometry. As long as the oscillating curve enters the thin grey strip on the top and/or the bottom of the diagram the pulsar is visible from the Earth. For the rest of this time period the pulsar is completely invisible from the Earth. The light deflection by the black hole and the aberration effect may alter the details of the fast-oscillating curve in figure 3, but since they are periodic effects they do not alter the secular evolution of its envelope that is due

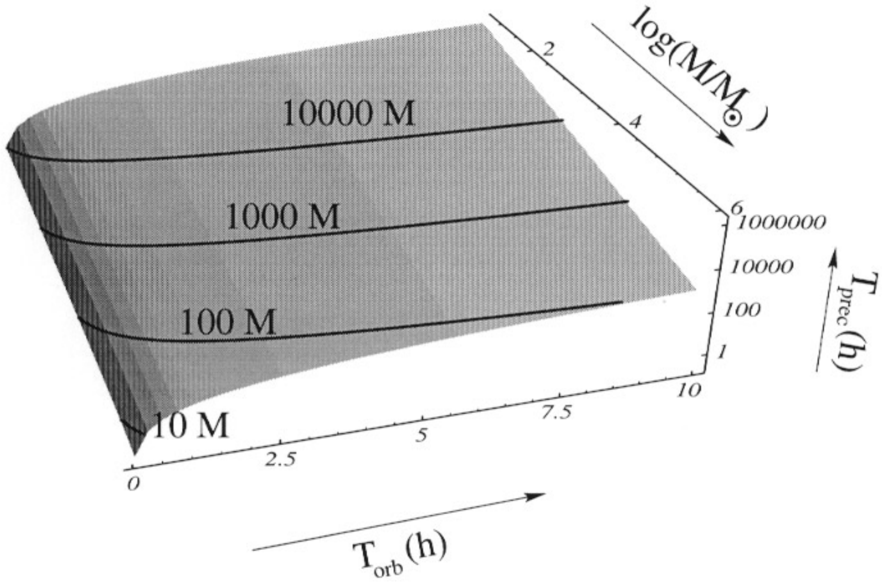

Figure 4. This 3D diagram shows the precessional period as a function of the orbital period and the black hole's mass. A few lines that correspond to points on the diagram with constant orbital radius are drawn on the 3D surface. The larger the mass of the black hole and the closer the orbit to the massive black hole the lower the precessional period of the pulsar's spin and the easier to detect the precessional effects. It should be noted that the formula used for the precessional period is based on equation (21) which is true for circular orbits. For noncircular orbits the time period is different but the numbers shown for the $T_{\rm prec}$ are still meaningful as an order of magnitude.

to precession.

For some special geometric configurations the pulsar may be visible from the Earth twice in this $2\pi/\Omega_{\rm precess}$ time period. This will happen if the envelope of the oscillating curve extends in both grey strips of the diagram; and on more physical grounds if both pulsar's magnetic poles come close to the direction to the Earth during the precessional motion of the spin.

In order to get a better feeling for this time period, in figure 4 we have plotted the period of precession for circular orbits $T_{\rm precess} = 2\pi/\Omega_{\rm precess}$ (in hours) as a function of the orbital period (in hours again) and the mass of the black hole. Also the distance from the black hole in black hole mass units is marked on the plot. From this plot it is clear that only for binary pulsars with massive black holes and very close orbits will the precession period be sufficiently small to see pulsars driven in and out of sight. As we have shown in previous sections, for noncircular orbits the precessional frequency is no longer given by equation (21), since the spin precession is no longer uniform. However, the precessional frequency for circular orbits gives the right order of magnitude for the precessional frequency of any orbit having the same r_0 as the circular one.

5. Conclusions

In this paper we have investigated the results of a highly relativistic effect on the observations of binary pulsars. For all the presently known binary pulsars the spin precessional effects are non-observable. For example the precessional period for PSR 1913+16 is of the order of 300 years (this number is based on post-Newtonian calculations of a binary system with comparable masses and therefore it does not agree with the result of our test body calculations), see [12]. Only for pulsars moving in close orbits around massive black holes will it be plausible to observe this strong-field effect. Even though the precessional effects may show up in ordinary neutron star–neutron star binaries, they are very tiny and difficult to draw conclusions from since the spins precess very slowly and they might interfere with other effects like radiation reaction or internal changes of the pulsar's emission mechanism. Thus only pulsars with massive black holes as companions (see figure 4) could be accurately timed for measuring their spin-induced precessional effects. For example, a pulsar orbiting a nonrotating $10^6 M_\odot$ in the centre of our Galaxy, in a circular orbit of $r_0 = 100M$, will have an orbital period of ~ 8.6 h and a precessional period of ~ 580 h. This means that once every 24 d (or 12 d for some special geometries) the pulsar could be observable from Earth for some time and then unobservable for the rest of time. Such observations could be cross checked with measurements from space-based gravitational-wave detectors such as LISA which is expected to be able to 'hear' the gravitational waves emitted from such a source. As we have shown in section 2, the presence of spin will hardly make the pulsar's orbit precess, thereby leading to gravitational waveforms without the complications of the spin-induced wave modulation considered in [4].

In the future we plan to extend our analysis to spinning test particles orbiting a massive rotating black hole.

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I would like to thank Gerhard Schäfer for suggesting this problem and for many useful discussions on the subject. I would also like to thank Norbert Wex for various informative discussions on pulsars.

Appendix. On the validity of the Papapetrou equations for massive spinning objects

Our whole analysis of the problem of spinning objects in the gravitational field of a nonrotating black hole has been based on the assumption that the spinning object has a far lower mass than the black hole. This allowed us to ommit essentially all the spin terms on the right-hand side of equations (7) and remain with the geodesic equations for the test particle and the equations of parallel transport for its spin. The question that arises then is whether we have the right to omit these terms in the case of comparable masses. The answer is, of course, no, but in that case the Papapetrou equations would not be valid in the first place. If the masses of the spinning object and the black hole were comparable, we could not assume that the spinning object is moving on the fixed gravitational background of the black hole, an assumption on which the Papapetrou equations have been based.

In this appendix we show that the errors on the equation of motion of a spinning object that are due to the assumption of a fixed gravitational background because of a low mass ratio are of higher order than the ones made by omitting the spin terms on the right-hand side of equation (7*a*) for the same low ratio of masses. Our proof will be based on the post-Newtonian equations of motion for a binary with a low ratio of masses with one of the bodies, in our case the less massive one, spinning.

Following the ordering of Will [13] for the post-Newtonian expansion terms of the relative accelaration between the bodies we obtain

$$\boldsymbol{a} = -\frac{M}{r^2}\hat{\boldsymbol{n}} - \frac{M}{r^2}\left[\hat{\boldsymbol{n}}\left(-4\frac{M}{r} + v^2\right) - 4\dot{r}\boldsymbol{v}\right] + \frac{m}{M}\frac{M}{r^2}\left[\hat{\boldsymbol{n}}\left(2\frac{M}{r} - 3v^2 + \frac{3}{2}\dot{r}^2\right) - 2\dot{r}\boldsymbol{v}\right]$$
$$+\frac{m}{M}\frac{M^2\alpha}{r^3}[6\hat{\boldsymbol{n}}[(\hat{\boldsymbol{n}}\times\boldsymbol{v})\cdot\hat{\boldsymbol{S}}] - 3\boldsymbol{v}\times\hat{\boldsymbol{S}} + 3\dot{r}(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{S}})] + \cdots \tag{34}$$

where M and m are the masses of the black hole and the spinning body, respectively, $\hat{\boldsymbol{S}}$ is the unit vector along the spin of the less massive body, α is the spin parameter ($\alpha = S/m^2 \leqslant 1$ for realistic astrophysical compact bodies), r is the distance between the two bodies, $\hat{\boldsymbol{n}}$ is the unit vector along r, $\boldsymbol{v}$ is the relative orbital velocity of the bodies and v its magnitude, and an overdot represents '$\mathrm{d}/\mathrm{d}\tau$'. All the higher post-Newtonian terms and corrections of m/M have been omitted. Though a truncated post-Newtonian expansion, equation (34) is a good guide to check the relative importance of the mass ratio on the deviation of the orbit due to the motion of the massive companion (third term in the post-Newtonian expansion) and due to the spin–orbit term (fourth term in the expansion), from the orbit on a fixed background (first two terms in the expansion). It is obvious that the spin term is half a post-Newtonian order ($\sim v$) higher than the m/M correction on the acceleration in the absence of spin.

Although a post-Newtonian expansion is a good approximation only for weak-field low-velocity situations, we have no reason to believe that the relative strength between the two m/M terms will be much different in the strong-field high-velocity region. Thus our omission of the spin terms in equations (7) is justified under the assumption of extremely low mass ratios. For comparable masses the Papapetrou equation is not adequate and it should be replaced by the complete Einstein equation for the two-body problem.

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