Revising the Multipole Moments of Numerical Spacetimes and its Consequences

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Identifying the relativistic multipole moments of a spacetime of an astrophysical object that has been constructed numerically is of major interest, both because the multipole moments are intimately related to the internal structure of the object, and because the construction of a suitable analytic metric that mimics a numerical metric should be based on the multipole moments of the latter one in order to yield a reliable representation. In this Letter, we show that there has been a widespread delusion in the way the multipole moments of a numerical metric are read from the asymptotic expansion of the metric functions. We show how one should read correctly the first few multipole moments (starting from the quadrupole mass moment) and how these corrected moments improve the efficiency of describing the metric functions with analytic metrics that have already been used in the literature, as well as other consequences of using the correct moments.

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Introduction.—In the beginning of the 1970s, Geroch [1] and Hansen [2] defined the multipole moments of an asymptotically flat spacetime in the static and stationary case in analogy to the Newtonian ones. In 1989, Fodor *et al.* [3] found a concise and practical way to compute the multipole moments of a spacetime that is additionally axially symmetric, taking advantage of the insightful Ernst-potential formalism. The definitions of alternative relativistic multipole moments by Simon [4] and by Thorne [5] should be noted as well: the former one ends up to be the same moment as Geroch and Hansen's, while Thorne's moments are coordinate dependent, and thus they could be used as a criterion to choose the appropriate coordinates to read the usual Geroch-Hansen moments (for a review see [6]).

The advent of technological developments that offer us the capability to observe gravitational wave signals gave a further boost to the study and use of the notion of relativistic multipole moments during the last two decades. The multipole moments uniquely characterize the gravitational field of a compact object; thus, Ryan [7] wrote formulas that relate the moments of a spacetime with the observable frequencies and the number of cycles of the gravitational wave signal that is emitted by a low mass object inspiraling adiabatically into such a spacetime. It should be emphasized that both the moments and the frequencies in Ryan's paper are invariant quantities that do not depend on the coordinates used to describe the background metric. Besides Ryan's attempt to connect the multipole moments with astrophysical observables, Shibata and Sasaki (SS) have given analytic relations of the radius of the innermost stable circular orbit (ISCO) to the moments [8], and Laarakkers and Poisson have attempted to relate the multipole moments of a neutron star with the equation of state (EOS) of the matter it consists of [9].

Exploring such interconnections of observables to multipole moments associated to a particular metric, expressed either in an analytic form or through a numerical grid, brought to our attention a systematic deviation of the way the multipole moments are read from the asymptotic expansions of various metrics. This systematic error arises mainly from erroneously assuming that a metric, which is expressed in a given coordinate system, has an asymptotic behavior similar to a Schwarzschild metric up to some order.

We have tried to correct such errors by relying on the coordinate-invariant expressions of Ryan [7]. Thus, one could obtain the correct moments by computing the gravitational-wave spectrum $\Delta\tilde{E}$ (the energy emitted per unit logarithmic frequency interval) of a test particle that is orbiting on a circular equatorial orbit in an asymptotically flat, stationary, and axially symmetric spacetime. Note that we do not assume that the astrophysical object we study is actually surrounded by such test particles emitting gravitational radiation; we just use this hypothetical configuration to relate quantities (frequencies and moments) which are independent from the coordinates in which the metric is presented.

First, we corrected the quadrupole-moment values of the rotating neutron star models, which were constructed by the numerical code of Stergioulas [10]. Then, we showed that if one tries to approximate the metric functions by a three-parameter analytic metric, like the metric of Manko *et al.* [11], the numerical metric is, in most cases, almost an order of magnitude better approximated by that particular analytic metric than what was initially found in [12]. This conclusion ensures that a suitable analytic metric with only a few free parameters (three or four [13]) could be quite faithful to represent with very good accuracy the gravitational field of a realistic neutron star.

The rest of the Letter is organized as follows: First we compare the asymptotic expressions for the metric functions derived by Butterworth and Ipser (BI) [14] to the corresponding asymptotic expressions introduced by Komatsu, Eriguchi, and Hechisu (KEH) [15]. The KEH formalism was later implemented numerically by Cook, Shapiro, and Teukolsky [16] and by Stergioulas and Friedman [10] to build numerical models of neutron stars. Next, we apply Ryan's method for the BI metric and get a direct relation of the asymptotic term coefficients with the first multipole moments. At this point, we explain why a generic isolated body in quasi-isotropic coordinates does not have the same asymptotic metric behavior as the corresponding Schwarzschild metric. Finally, we discuss the improvement produced in matching the numerical spacetime of a rotating neutron star with an analytic metric, like the one of Manko *et al.* [11], when the right quadrupole moment of the neutron star model is used in the analytic metric instead of the one that was used in the past in similar comparisons. We end up by giving a short list of other consequences of not using the right multipole moments in various astrophysical explorations of compact objects.

Asymptotic expansion of a metric.—In 1976, BI wrote the relativistic equations for the structure and the gravitational field of a uniformly rotating fluid body. They assumed that the line element has the following form:

$$ds^2 = -e^{2\nu}dt^2 + r^2\sin^2\theta B^2 e^{-2\nu}(d\phi - \omega dt)^2 + e^{2(\zeta-\nu)}(dr^2 + r^2 d\theta^2), \quad (1)$$

where ν, B, ω, and ζ are the four metric functions, all functions of the quasi-isotropic coordinates r, θ (the other two coordinates t and ϕ do not show up in the metric functions since the geometry is assumed stationary and axially-symmetric). By writing down the field equations and the equations of motion for the fluid, they obtained differential equations for the metric functions [see Eqs. (4)–(7) of [14]] through which they constructed the asymptotic expansion of the three metric functions (ν, ω, B), while the last metric function ζ could be easily computed from the rest by a suitable integration. We copy here these asymptotic expansions since the various coefficients are intimately related to the multipole moments of the central object, as will be shown later on.

$$\nu \sim \left\{-\frac{M}{r} + \frac{\tilde{B}_0 M}{3r^3} + \ldots\right\} + \left\{\frac{\tilde{\nu}_2}{r^3} + \ldots\right\}P_2 + \ldots, \quad (2)$$

$$\omega \sim \left[\frac{2J}{r^3} - \frac{6JM}{r^4} + \left(8 - \frac{3\tilde{B}_0}{M^2}\right)\frac{6JM^2}{5r^5} + \ldots\right]\frac{dP_1}{d\mu} + \left[\frac{\tilde{\omega}_2}{r^5} + \ldots\right]\frac{dP_3}{d\mu} + \ldots, \quad (3)$$

$$B \sim \sqrt{\frac{\pi}{2}}\left[\left(1 + \frac{\tilde{B}_0}{r^2}\right)T_0^{1/2} + \frac{\tilde{B}_2}{r^4}T_2^{1/2} + \ldots\right]. \quad (4)$$

In the formulas above, P_l are the Legendre polynomials expressed as functions of $\mu = \cos\theta$, $T_l^{1/2}$ are the so-called Gegenbauer polynomials (similar to the Legendre polynomials, also functions of μ), and M and J are the first two multipole moments (the mass and the spin) of the spacetime. The rest of the coefficients are related to the higher multipole moments.

In 1989, KEH proposed a different scheme for integrating the field equations using Green's functions. The line element they assumed was a bit different than the previous one:

$$ds^2 = -e^{2\nu}dt^2 + r^2\sin^2\theta e^{2\beta}(d\phi - \omega dt)^2 + e^{2\alpha}(dr^2 + r^2 d\theta^2).$$

The new metric functions are related to the metric functions of BI by the following simple relations:

$$\nu_{\rm BI} = \nu_{\rm KEH} = \nu, \quad B_{\rm BI}e^{-\nu} = e^{\beta_{\rm KEH}}, \quad \zeta_{\rm BI} = \nu + \alpha_{\rm KEH}. \quad (5)$$

The combinations of $\nu_{\rm KEH}$ and $\beta_{\rm KEH}$,

$$\gamma = \nu_{\rm KEH} + \beta_{\rm KEH}, \qquad \rho = \nu_{\rm KEH} - \beta_{\rm KEH}, \quad (6)$$

along with ω could be expressed as power series in $1/r$ in the same manner as in Eqs. (2) and (4)

$$\rho = \sum_{n=0}^{\infty}\left(-2\frac{M_{2n}}{r^{2n+1}} + \text{higher order}\right)P_{2n}(\mu), \quad (7)$$

$$\omega = \sum_{n=1}^{\infty}\left(-\frac{2}{2n-1}\frac{S_{2n-1}}{r^{2n+1}} + \text{higher order}\right)\frac{P^1_{2n-1}(\mu)}{\sin\theta}, \quad (8)$$

$$\gamma = \sum_{n=1}^{\infty}\left(\frac{D_{2n-1}}{r^{2n}} + \text{higher order}\right)\frac{\sin(2n-1)\theta}{\sin\theta}. \quad (9)$$

In 1997, in another paper of Ryan [17] it was explained that the Hansen-Geroch (GH) mass and current-mass moments, $M_{2n}^{\rm GH}$ and $S_{2n-1}^{\rm GH}$, first show up in asymptotic expansions in the terms of order $1/r^{2n+1}$. Although correct, this statement could be easily misinterpreted: one could be misled to assume that $M_2^{\rm GH} = M_2 = -\tilde{\nu}_2$ and $S_3^{\rm GH} = S_3 = \frac{3}{2}\tilde{\omega}_2$, where the last equalities result from a direct comparison between the above expansions and the corresponding ones of BI.

The numerical scheme of KEH was applied in the numerical codes of [10,16], which were then used by various people in order to construct realistic models of neutron stars. The values M_{2n} and S_{2n-1} of these numerical neutron stars were then read from the coefficients of the above asymptotic expansions.

Identifying the multipole moments through Ryan's method.—In 1995, Ryan wrote coordinate-independent expressions that relate the energy of a test body that is orbiting in the stationary and axially symmetric spacetime of an isolated compact body along a circular equatorial orbit to the Hansen-Geroch multipole moments of the spacetime itself [7]. We wrote similar expressions for the asymptotic expansions of the metric functions of Eqs. (2)–(4) and then related the series coefficients to the multipole moments of the corresponding spacetime. Following the procedure of Ryan, we first wrote the orbital frequency of Ω of the test body through the dimensionless parameter $v = (M\Omega)^{1/3}$ as a power series in $x = (M/r)^{1/2}$ and then inverted it to obtain x as a series in v. Following the same procedure, we then calculated the energy per mass $\tilde{E}$ of a test particle in a circular equatorial orbit as a function of x [the expression for $\tilde{E}$ as a function of the metric is given in Eqs. (10) and (11) of [7]]. Finally, the energy change per logarithmic interval of the rotational frequency $\Delta\tilde{E} = -d\tilde{E}/d\log\Omega$ was expressed as a power series in v with coefficient terms written as polynomials of the metric coefficients

$$\Delta\tilde{E} = \frac{v^2}{3} - \frac{v^4}{2} + \frac{20jv^5}{9} - \frac{(89 + 32b + 24q)}{24}v^6 + \frac{28jv^7}{3} - \frac{5(1439 + 896b - 256j^2 + 672q)v^8}{432} + \frac{[(421 + 64b - 60q)j - 90w_2]v^9}{10} + O(v^{10}), \tag{10}$$

where $j = J/M^2$, $q = \tilde{\nu}_2/M^3$, $w_2 = \tilde{\omega}_2/M^4$, $b = \tilde{B}_0/M^2$. By equating the coefficients of the previous power series with the corresponding ones of Ryan [Eq. (17) of [7]], we yield directly the right relations between the coefficients of BI (or of KEH) and the multipole moments of the spacetime. More specifically, from the coefficients of v^6 and v^9 terms of the two series we yield the following values for the quadrupole mass moment and the octupole current mass moment:

$$M_2^{\rm GH} = M_2 - \frac{4}{3}\left(\frac{1}{4} + b\right)M^3, \tag{11}$$

$$S_3^{\rm GH} = S_3 - \frac{12}{5}\left(\frac{1}{4} + b\right)jM^4, \tag{12}$$

respectively, where M_2 and S_3 are the quantities that were mistakenly identified by various authors as the corresponding moments of numerical models. By replacing these values in the terms of order v^7 and v^8 of Ryan's Eq. (17) [7], we recover exactly the corresponding term coefficients of Eq. (10), as expected. Henceforth, we will omit the superscript GH in M_2 and S_3 when we refer to the correct Hansen-Geroch multipole moments.

The last terms in Eqs. (11) and (12), which were missing up to now in the literature, are the ones that are causing the discrepancy between the estimated incorrect moments and the true moments. Both these extra terms would be zero for $b = -1/4$. This was pointed out by Laaarkkers and Poisson [9]. These correcting terms were considered harmless, though, since the Schwarzschild metric corresponds to $B = 1 - M^2/4r^2$, that is, to $b = -1/4$. This, as argued by Laarakkers and Poisson, should correspond to the lowest order term of the metric function B [the term in front of $T_0^{1/2}$ in Eq. (4)], of any axisymmetric isolated body. This is not true, though, if r is the isotropic coordinate radius. The lowest order asymptotic term of any metric describing a stationary isolated object is simply the 1 in the first order term of B; the higher orders are generally expected to deviate from their Schwarzschild corresponding terms.

In the recent book by Friedman and Stergioulas [18], this discrepancy was noted and was corrected by a transformation of the r coordinate, leading to exactly the same correction for M_2 as in our Eq. (11). The analysis in our Letter, though, is general and coordinate independent. It could be used to treat any set of coordinates and find the relation between the corresponding coefficients and the true multipole moments. Moreover, in our Letter we give the right formula for S_3 as well.

As a final remark, we note that the Kerr metric, expressed in isotropic coordinates [19], yields a B function $B = 1 - (M^2 - a^2)/4r^2$, hence $b^{\rm Kerr} = -(1/4)(1 - j^2)$. This result is a clear manifestation of the erroneous assumption that all stationary axisymmetric metrics correspond to $b = -1/4$.

Consequences of evaluating the right moments.—We now present a list of the various effects caused by computing correctly the multipole moments of the numerical models of neutron stars and give a short account, whenever possible, of the subsequent quantitative alterations in recent scientific conclusions related to studies of the exterior field of neutron stars.

(i) We start with the attempt of constructing analytic vacuum solutions of Einstein's equations that could then be used to fit the various numerical models of rotating neutron stars. Berti and Stergioulas [12] tried to match a three-parameter analytic solution [11] to a wide diversity of uniformly rotating neutron-star models. Each analytic solution was constructed so that its first three multipole moments were equal to the corresponding moments (mass, spin, and quadrupole) of the particular neutron star, where these moments were read directly from the asymptotic expansion of the corresponding numerical metric. Their conclusion was that this type of analytic solution was quite good to describe the external metric of all kinds of fast rotating neutron stars. Since the specific metric cannot assume low values of quadrupole moment, the metric is not adequate to describe the slowly rotating neutron stars. Whenever an analytic solution could be constructed, the matching was such that the two metrics (analytic and numerical) did not differ by more than 6% at the surface of the star.

We attempted the same comparison as in [12], assuming the same analytic spacetime and using all numerical models with EOS's AU, FPS, and L, while the quadrupole moment that was inserted in the analytic solution was the one corrected according to Eq. (11). In order to correct it, we used the asymptotic expansion of the metric function B of the numerical metric. Although the quadrupole moment was not affected by more than ~20% and for some numerical models by a much lower fraction, the improvement it evoked in matching the numerical metric was almost an order of magnitude. We computed the overall mismatch between the numerical and the analytic metric exterior to the star, which was defined as $\sigma_{ij} = [\int_{R_S}^{\infty} (g_{ij}^n - g_{ij}^a)^2 dr]^{1/2}$, where n and a indicate the numerical and the analytic metric components, respectively, and R_S is the surface radius. The improvement we gained in the overall mismatch was a factor of ~2 to ~8 for the g_{tt} and ~2 to ~15 (in most cases) for the $g_{t\phi}$. In a few cases, the improvement in the mismatch of $g_{t\phi}$ (and in one model of g_{tt}) was either marginal or worse after the correction of M_2. This happened because, while we correct the M_2 of the metric of Manko *et al.*, its octupole S_3 is altered in such a way that the analytic metric finally raises its overall mismatch. However, even these unimproved cases have an extremely good overall mismatch (less than 0.004 after the worsening). For a full presentation of the numerical results of the mismatch and the fractional deviation of M_2 and S_3 due to correction, see the Supplemental Material (SM) [20]. Figure 1 shows an example of the mismatch between the analytic and numerical metric functions before and after the correction.

Whenever an analytic metric is used to mimic the gravitational field of a realistic neutron star [13] and then this metric is used to reproduce various observables related to neutron stars [21], it is important to use the right moments in order to build a faithful analytic metric; otherwise, any physical conclusion inferred by the analytic metric would be off [22]. If quantitative conclusions are drawn while using wrong moments in the analytic metric, they should be taken into account with some reservation.

(ii) Another important effect of altering the values of the multipole moments of numerical neutron-star models is in relating the higher moments of the compact object with its spin j. This was attempted by Laarakkers and Poisson [9] for the quadrupole moment, but because of the omission of the b-related term, the parameters estimated when relating M_2 to j^2 were somewhat off. We repeated their analysis with the corrected values of M_2 and we got slightly different fitting parameters. In the SM [20], one can find our parameters; however, they cannot be directly compared to the ones in [9] since the sequences of models we used are not exactly equivalent to the ones in [9]. Having the right relations between various moments could help us interpret future accurate observations with connection to the internal structure of neutron stars. At this point, we should note that the correcting factor b deviates from $-1/4$ following a quadratic relation with j as well, thus preserving the quadratic fit of M_2 with j that was found in [9]. Furthermore, we found a similar empirical relation for S_3 with j, namely, $S_3 = a_3 j^3$, with a_3 a constant parameter depending on the EOS and the mass of the neutron star (see SM [20]).

(iii) The ISCO radius, which is significant for a number of astrophysical observations, is intimately related to the exact relativistic moments of the central object since it lies close to it. In [23], the ISCO of various numerical models of neutron stars were compared to the ISCO computed for a Hartle-Thorne (HT) analytic metric [24] that was suitably constructed to match the behavior of the numerical models. What they found was that, although the relative difference in quadrupole between HT and numerical models was oddly high even at slow rotations (where one would expect almost perfect match), the corresponding ISCOs were extremely close (at low spins). This intriguing disagreement disappears for the right quadrupoles of the numerical models. On the other hand, the ISCOs were originally in good

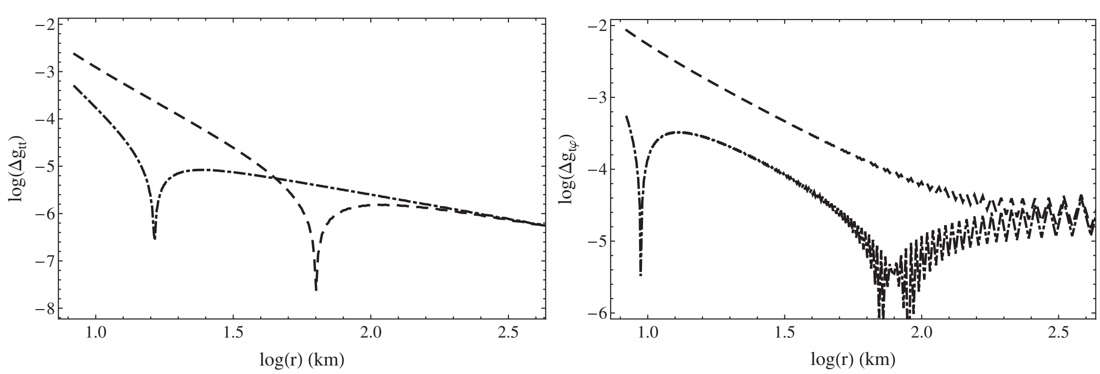

FIG. 1. A typical log-log plot of the relative difference between the numerical and the analytic metric $[(g_{ij}^n - g_{ij}^a)/g_{ij}^n]$ for a specific numerical model (model 16 of EOS FPS of the SM [20]), before (dashed curve) and after (dashed-dotted curve) the correction of M_2. The left plot is for g_{tt} and the right one for $g_{t\phi}$. We note that the corresponding overall improvement for this particular neutron star model was 6.6 in g_{tt} and 15.1 in $g_{t\phi}$. This was a model with a medium improvement in g_{tt}, compared to the whole set of models which were examined. The dips in the curves are simply due to coincidental crossings between the numerical and the analytic metrics.

agreement since they are not affected by such misidentification of M_2 and were accurately computed for both metrics. However, that would not be the case if the approximate formula of SS [8] was used to compute the ISCO, since it is a function of the various moments. We repeated the comparison of ISCOs [12] between the numerical one and that obtained from SS, and we found a partial improvement but not an impressive one. The reason is that at low quadrupoles (corresponding to low spins), SS's formula is extremely accurate and quite insensitive to small corrections of M_2, while at high quadrupoles the correction of ISCO through M_2 is significant, but then the formula deviates a lot from the true ISCO. For example, the relative difference of ISCO drops from 0.92% to 0.61% (due to correction of M_2) for low M_2, while it drops from 17.3% to 15.6% for a large value of M_2 (for a sequence of models of FPS).

(iv) Finally, in [25] the apparent surface area of a rotating neutron star, due to its quadrupole deformation, is computed as a function of this deformation. The deformation parameter, though, is read from the quadratic fit of [9], which is somewhat distorted due to wrong identification of moments. Hence, the quadrupole deformation η assigned to the graphs and tables of their models should be recomputed according to the corrected quadrupole moment values.

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