

How to Observe a Non-Kerr Spacetime Using Gravitational Waves

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We present a generic criterion which can be used in gravitational-wave data analysis to distinguish an extreme-mass-ratio inspiral into a Kerr background spacetime from one into a non-Kerr spacetime. We exploit the fact that when an integrable system, such as the system that describes geodesic orbits in a Kerr spacetime, is perturbed, the tori in phase space which initially corresponded to resonances disintegrate so as to form Birkhoff chains on a surface of section. The KAM curves of the islands in such a chain share the same ratio of frequencies, even though the frequencies themselves vary from one KAM curve to another inside an island. However the KAM curves, which do not lie in a Birkhoff chain, do not share this characteristic property. Such a temporal constancy of the ratio of frequencies during the evolution of the gravitational-wave signal will signal a non-Kerr spacetime.

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Introduction.—While ground-based gravitational-wave detectors are already in operation, and are trying to detect gravitational waves from stellar mass compact objects, LISA—the space-borne detector [1] which is planned to be launched during the forthcoming decade—is expected to observe much more massive sources of gravitational waves with high signal-to-noise ratio. Such signals will offer us an opportunity to map the strong field around massive astrophysical objects [2,3], and to test the conventional wisdom, partly supported by astrophysical observations [4,5], according to which the massive compact objects harbored at galactic centers should be highly spinning Kerr black holes. Such an astrophysical fact is further enforced by the no-hair theorem, which states that when a black hole is formed, its multipole moments, and, consequently, its gravitational field, are determined by only two parameters; its mass and spin (assuming its charge is negligible). In this Letter we suggest a clear and generic observable signal that distinguishes an extreme-mass-ratio inspiral (EMRI) [6] related to a non-Kerr black hole spacetime from a corresponding Kerr one.

Our suggestion comes from the fact that a Kerr spacetime leads to an integrable system that describes geodesic orbits, while any other generic, stationary, and axisymmetric non-Kerr spacetime is not expected to be integrable. Thus assuming that a slightly perturbed Kerr metric, that supposedly describes the neighborhood of an axisymmetric rotating compact object, is governed by a nonintegrable system of geodesic equations of motion, any qualitative new characteristics of the orbit of the test body, which could in principle be observed through gravitational waves, can be used to identify such a source.

According to the KAM theorem [7] almost all KAM tori in the phase space of a perturbed integrable system are not destroyed; they simply become slightly deformed.

However, among the KAM tori of the initial integrable system, there are the so called resonant tori that are characterized by commensurate ratios of frequencies [8,9]. In the perturbed system these tori disintegrate, and according to the Poincaré-Birkhoff theorem [10] they form a chain of islands, on a surface of section, inside which the ratio of the corresponding frequencies remains equal to the rational number of the corresponding initial resonant torus. The width of this chain of islands is a monotonically growing function of the perturbative parameter that measures the system's deviation from the corresponding integrable one, at least for small perturbations.

In our case, an EMRI in a perturbed Kerr spacetime will be described by an adiabatically changing geodesic orbit, which will sweep a finite range of KAM tori in the course of time, while the frequencies of the orbital oscillations on the polar plane ρ - z in the Weyl-Papapetrou cylindrical coordinate system [11] will change continuously. When the system enters a Birkhoff chain of islands the ratio of frequencies will remain strictly constant, while the frequencies will continue to change. This ratio is not just approximately constant, due to a slow passage through a resonance, as implied in [12] for the Kerr spacetime. Therefore the appearance of a plateau in the ratio of frequencies during the evolution of an EMRI will definitely signal the presence of a non-Kerr spacetime. These frequencies will be encoded in the gravitational wave that is radiated from the corresponding source. Thus they can in principle be monitored. The question is how probable is for an orbit to cross such a chain of islands during its evolution that is monitored by the LISA detector. We argue [13] that this happens quite often, since the orbits of EMRI's that develop in the neighborhood of the supermassive compact object at galactic centers are initially quite eccentric and inclined [14]. Depending on the value of its physical

parameters, the system will eventually enter the region of such an island and then the plateau in the ratio of frequencies will show up. Furthermore, since the rational ratios of frequencies are dense in any finite interval, during the whole inspiral phase there is a non-negligible probability for observing more than one plateaus in frequency ratios. Only the low-order resonances though are probable to be observed since the width of the corresponding islands is usually much lower for the higher resonances.

In order to exhibit our criterion for observing a non-Kerr spacetime, we have used a Manko-Novikov (MN) metric [11,15] as an example of a non-Kerr metric. The specific MN metric is characterized by one parameter Q (in [11] this parameter is noted as q), besides the mass and the spin parameters, which measures the deviation of its quadrupole moment from the corresponding Kerr metric. This particular metric was analyzed in [11] and was found to behave like an integrable system with respect to regularity of its curves in the outer allowed region of phase space. Based on the general applications of the KAM theorem, we performed a more thorough analysis of the orbits in the same metric and found that the expected Birkhoff chains of islands are present on a surface of section, although they are very thin to be observed in a coarse study of orbits.

Also, we investigated the effect on the ratio of frequencies of polar oscillations when an orbit is moving adiabatically in and out of such an island while the corresponding EMRI radiates energy and angular momentum away. We have shown that the plateau in the ratios of frequencies could last for a few hours to a few weeks, depending on the masses involved in the EMRI, as well as the value of the Q parameter of the background spacetime. Actually, the aforementioned plateaus will be more distinctly observed in lower mass-ratios ($\mu/M \lesssim 10^{-5}$ where μ and M are the reduced and total mass of the binary, respectively) since then the system spends more time within an island. These effective plateaus could be somehow integrated in the schemes used in data analysis of LISA to look for non-Kerr EMRI's.

We emphasize that the existence of a plateau is a generic result for any nonintegrable perturbed Kerr metric. Thus the MN metric does not imply any restrictions in the general application of the proposed observational criterion. The MN metric is merely an example of a perturbed Kerr metric with anomalous moments.

Resonances in a non-Kerr MN spacetime.—The MN spacetime, which was used in [11] to explore the orbits in a non-Kerr metric, is a family of solutions of the Einstein equations in vacuum, that has been built on the foundations of a Kerr metric and thus it has as a special case the pure Kerr metric [15]. For the specific metric this could actually be done by setting the Q parameter equal to zero. The higher mass and mass-current multipoles are also affected by Q [11].

The Kerr metric is a very special metric for two main reasons. (i) It is a highly symmetric metric since apart from the three integrals of geodesic motion that all axially symmetric, stationary, and asymptotically flat metrics share, it is characterized by an extra integral of motion, the so called Carter constant. (ii) It is a metric related to realistic physical objects. According to the “no hair theorem”, all spinning objects that collapse to form a black hole are described by such a metric. Since highly compact objects that move very close to each other are very powerful sources of gravitational radiation, an EMRI of a stellar-mass compact object around a massive Kerr black hole is a highly promising source for the LISA detector.

In order to look for qualitative new characteristics in non-Kerr compact sources that could exist in nature, we have computed numerically the geodesic orbits in a MN metric. Since this is a metric which is built on the basis of a Kerr metric, while it does not share its special symmetry that leads to Carter constant, it could be considered as a perturbed Kerr; therefore it is not expected to be described by an integrable system, in contrast to what happens in a Kerr spacetime. The fact that this is not an integrable system could already be deduced from the fact that orbits in the inner allowed region of the MN spacetime exhibit chaotic behavior [11,13]. In the outer region (where the gravitational field is weaker) the MN spacetime looks like a perturbation of Kerr and thus most of KAM tori just modify their shape. However, the resonant tori disintegrate and intersect a surface of section forming regions of finite thickness, instead of single closed KAM curves. These regions form sets of islands, known as Birkhoff chains of islands; each one of them bounded externally and internally by KAM curves [16]. The existence of such chains of islands is characteristic of a system that is nearly integrable. A thorough analysis of geodesic orbits in the MN spacetime revealed two such Birkhoff chains of islands that correspond to the resonances $2/3$ and $1/2$. Discovering such islands on a Poincaré surface of section is not an easy task since these islands are quite thin (cf. Fig. 1); a very fine sweep of initial conditions is needed to get such an island. Fortunately, there is an alternative tool to approach the corresponding orbit much faster, the “rotation number” [16]. This number can be easily computed for every orbit and by changing the initial conditions and monitoring the corresponding rotation number, we could arrive at the desirable rational value of the rotation number.

A phase orbit in an integrable system (like the Kerr case) is wound around a nonresonant torus filling the whole torus densely in the course of time, while for a resonant torus the orbit is periodic, repeating itself after a few windings [7]. Thus, on a surface of section which intersects transversally the corresponding tori (we have used the plane $z = 0$, on which we mark the intersecting points when $\dot{z} > 0$, thus our orbits are purely nonequatorial), the crossing points constitute a set of points that densely fill a closed curve in the

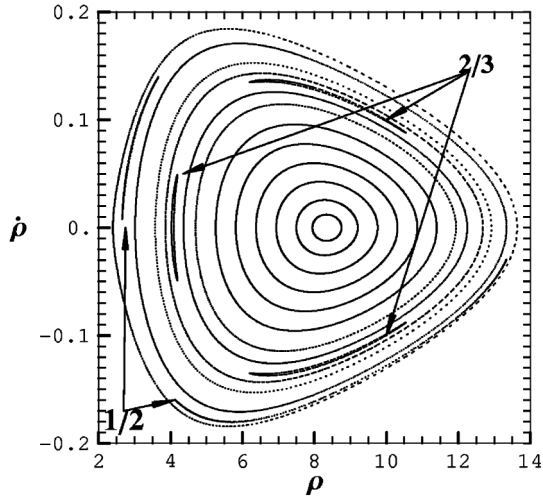


FIG. 1. The surface of section of the outer region on the $(\rho, \dot{\rho})$ plane for the parameter set $E = 0.95$, $L_z = 3M$, $\chi = 0.9$, $Q = 0.95$ of MN. E , L_z are the energy and the angular momentum of the inspiral, respectively, while χ is the spin of the central body (for details about these quantities see [11]).

former case, and a finite number of fixed points in the latter case. This number is the number of oscillations along the spatial axis that intersects transversally the surface of section (the z axis in our case) that correspond to a finite number of oscillations along the other spatial axis (the ρ axis). When the system deviates from being integrable, these fixed points of the periodic orbit evolve to an equal number of Birkhoff islands.

By means of the sequence of crossing points described above it is easy to define the aforementioned rotation number. Either for an integrable or for a non integrable system of 2 degrees of freedom, this is defined as follows: Let \mathcal{A} be the fixed point on a surface of section and \mathcal{B}_i be the i th point of intersection of the phase orbit with the surface of section. The vector $\overrightarrow{\mathcal{A}\mathcal{B}_i}$ rotates through an angle $\Delta\theta_i$ as it moves from the i th intersecting point to the next one. The rotation number is $\nu = \frac{1}{2\pi} \lim_{N \rightarrow \infty} \frac{1}{N} \times \sum_{i=1}^N \Delta\theta_i$ [16]. The rotation number is the fraction of the two corresponding fundamental frequencies of the system, each multiplied by an integer. Thus for a geodesic orbit in a MN metric it will be $\nu = \frac{m f_\rho}{n f_z}$ with m , n integer numbers. Whenever the two frequencies are commensurate to each other, the rotation number is rational. Actually the resonant tori of the integrable system that form Birkhoff islands on a surface of section when the system gets perturbed, are the ones with commensurate frequencies, and consequently with rational rotating numbers. Thus we can alter the initial conditions until we obtain a specific rational value for ν and thus reveal the location of a Birkhoff island. In our case of geodesic orbits in MN we have searched for Birkhoff islands when $f_\rho/f_z = 2$, or 1, which correspond to $\nu = 2/3$, or $1/2$, respectively. Moreover, the thickness of the

islands grows with the perturbation parameter Q , in agreement with simple perturbed integrable Hamiltonian toy models that have already been studied in the literature [16–18].

Evolving orbits due to radiation reaction.—Each chain of Birkhoff islands, separate the surface of section in an interior and an exterior region. While one moves from the former to the latter region, the rotation number, and consequently the ratio of frequencies, drops monotonically, except if we land on the thin window of an island; then the ratio of frequencies does not change, even though the frequencies themselves change. Thus by analyzing the spectrum of a gravitational-wave signal and monitoring the ratio of the fundamental frequencies (cf. Fig. 2), whenever a plateau shows up, it will signal a non-Kerr type of background. The plateau will be more prominent (i) if the deviation from Kerr is larger, (ii) if the ratio of mass is more extreme, since then the deviation from geodesic orbit, due to energy and angular momentum loss, is slower and the system needs more time to cross an island, and (iii) the island corresponds to a resonance of low-integer ratios, since these resonances correspond to wider islands.

We have run such adiabatic orbits in MN spacetimes numerically, following approximate formulae for the evolution of energy and angular momentum along the axis of symmetry (for a generic spacetime there is no reliable way to compute the losses due to radiation) [19]. Our numerical explorations show that these plateaus are visible for ratios

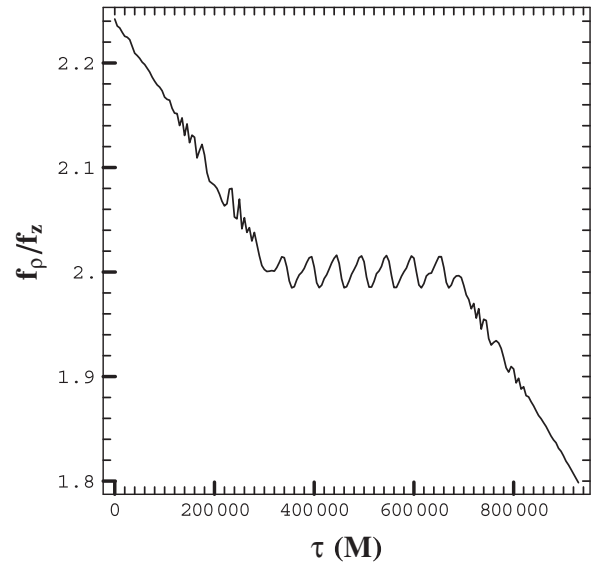


FIG. 2. The evolution of the ratio of fundamental frequencies f_ρ/f_z for a MN EMRI with $Q = 0.95$ and ratio of masses $\mu/M = 5 \times 10^{-5}$. The initial conditions were chosen to get quite a long plateau. The fundamental frequencies were computed numerically from the Fourier analysis of the orbital motion in time intervals of length $5000M$. The oscillations of the ratio—mainly at the plateau interval—are artifacts due to Fourier analyzing a finite part of the orbit.

of masses below some threshold that depends on the resonance we are dealing with and the value of Q assumed for the MN metric. For example for $Q = 0.95$ the higher ratio of masses for which we can tell the presence of such a plateau is $\mu/M \cong 10^{-4}$. If the ratio of masses is above these thresholds, the system evolves so fast that the corresponding plateaus are not discernible. Below the threshold, the actual duration of the plateau is more extended for lower values of μ/M , although it varies a lot depending on the specific trajectory of the phase orbit through the Birkhoff island. The aforementioned thresholds are actually within the range of masses expected for an EMRI signal detectable by LISA [20].

Another crucial point with respect to observability of the plateau effect is the following: Since the chains of Birkhoff islands are numerous, and the phase orbit is forced to move adiabatically from one KAM curve to the next, it is forced to pass through many such islands. Thus, while the effect is always present, an actual system will exhibit an unambiguous plateau whenever it has sufficient time to cross a strong resonance, like the $2/3$ or the $1/2$ one. This happens if the geometric characteristics of the orbit when it enters the window of sensitivity of LISA are such that during the evolution of the orbit, it will cross one of these resonances. Thus, the orbit should have initial eccentricity and inclination within a suitable range. These ranges are quite wide; therefore it is anticipated that a large fraction of such suitable EMRI's will leave their imprints on their signal through an apparent plateau of the ratio of the observed frequencies. Moreover, even if the evolution of the signal is such that it correlates well with a Kerr EMRI, we could focus our search in the particular period when the ratio of the corresponding peaks in the spectrum that are related to the radial and precessional oscillation are close to a resonant ratio ($f_\rho/f_z = 2$ for the $2/3$ resonance and 1 for the $1/2$ resonance). A statistically important persistence of such a ratio would be a clear "smoking gun" for a non-Kerr metric. An apparent persistence of a corresponding orbit in Kerr [12] is qualitatively different since it is simply due to slow crossing of a resonance.

Finally, we should note that a possible positive signal from a non-Kerr EMRI, could be further explored for other consequences of such a peculiar source, like the instabilities that are expected to show up before the corresponding final plunge [11], or a possible transit of the orbit to chaotic

behavior through entrance in the interior region of allowed orbits of a MN-like metric [11,13].

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