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2008 Class. Quantum Grav. 25 228002

(http://iopscience.iop.org/0264-9381/25/22/228002)

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Class. Quantum Grav. **25** (2008) 228002 (6pp) doi:10.1088/0264-9381/25/22/228002

COMMENTS, REPLIES AND NOTES

Faithful transformation of quasi-isotropic to Weyl–Papapetrou coordinates: a prerequisite to compare metrics

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Received 5 March 2008, in final form 24 August 2008
Published 3 November 2008
Online at stacks.iop.org/CQG/25/228002

Abstract

We demonstrate how one should transform correctly quasi-isotropic coordinates to Weyl–Papapetrou coordinates in order to compare the metric around a rotating star, which has been constructed numerically in the former coordinates, with an axially symmetric stationary metric, which is given through an analytical form in the latter coordinates.

PACS numbers: 95.30.Sf, 04.25.D−, 04.40.Dg

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Recently, there has been quite a few attempts to describe the geometry around an astrophysical object, such as a rotating neutron star, a strange star or a black hole surrounded by an accretion disk, through various types of analytical solutions of the vacuum Einstein equations [1–5].

A large variety of analytical solutions of vacuum Einstein equations are nowadays available to be used as candidate metrics for the exterior spacetime of axisymmetric astrophysical objects. Ernst [6] formulated the Einstein equations in the case of axisymmetric stationary spacetimes long time ago, while Manko *et al* and Sibgatullin [7–12], and Neugebauer [13] have used various analytical methods for producing such space-times parametrized by various parameters that have a different physical context depending on the type of each solution. One has to choose among all these a specific solution that relates better to the particular astrophysical type of object.

Also, during the last decade various groups (see [14, 15]) have constructed the metric inside and outside such astrophysical objects by solving numerically the full Einstein equations in stationary cases.

To compare an analytical solution with a metric that has been constructed numerically, one should first transform faithfully the coordinates of the metrics to each other. A problem arises

0264-9381/08/228002+06$30.00

though when one attempts to transform the quasi-isotropic coordinates, which are usually used to describe a numerical metric both inside and outside the astrophysical object, to Weyl–Papapetrou coordinates that are usually used in the analytical expressions of the stationary axisymmetric metrics. Although the transformation of coordinates is straightforward in the vacuum region, the same type of transformation leads to erroneous coordinates when it is used in the matter region. If one insists on using the Weyl–Papapetrou coordinates ρ and z inside matter, one could no longer use them on an equal basis as in the usual vacuum axisymmetric stationary metric; instead one more function $\Lambda(\rho, z)$ should be introduced to describe the induced metric of the two-dimensional surface spanned by ρ and z, namely

$$\mathrm{d}s^2_{(\rho,z)} = \Omega^2(\mathrm{d}\rho^2 + \Lambda(\rho, z)\,\mathrm{d}z^2), \tag{1}$$

cf [16].

Thus, we suggest that in order to translate the former to the latter coordinates, the corresponding integration path that is used to compute the z-coordinate (cf section 2) should avoid entering the matter region. Moreover, only the exterior region is of real interest to us as an arena to probe the source of the gravitational field.

The rest of this note is organized as follows: in section 2 we present the best path of integration to compute the z-coordinate. In section 3, we give an estimate of the errors that arise from numerical integration of z, and the corresponding errors that are induced in the metric components.

All physical quantities used in this note are in geometrized units ($G = c = 1$).

2. Transformation to Weyl–Papapetrou coordinates

In section 4.1 of [1] one can find all relations connecting the isotropic coordinates with the Weyl–Papapetrou ones, along with the corresponding metric components. In particular, equations (41, 42) of [1] define the z-coordinate and constitute a set of Cauchy–Riemann conditions. Through these Cauchy–Riemann relations, it is easy to write the expressions for $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$,

$$\frac{\partial z}{\partial r} = \cos\theta B + \sin\theta \frac{\partial B}{\partial \theta}, \tag{2}$$

$$\frac{\partial z}{\partial \theta} = -r^2 \sin\theta \left(\frac{\partial B}{\partial r} + \frac{B}{r}\right), \tag{3}$$

and from them derive the corresponding derivatives of z with respect to $\mu \equiv \cos\theta$ (not to be confused with the corresponding metric function in quasi-isotropic coordinates) and $x \equiv r/(r + r_e)$ (r_e is the equatorial radius of the star; see section 2.1 of [1]):

$$\frac{\partial z}{\partial x} = \frac{r_e}{(1-x)^2}\left(\mu B + (\mu^2 - 1)\frac{\partial B}{\partial \mu}\right), \tag{4}$$

$$\frac{\partial z}{\partial \mu} = r_e\left(x^2 \frac{\partial B}{\partial x} + B\frac{x}{1-x}\right), \tag{5}$$

where B is one of the metric functions defined in section 2.1 of [1].

Equipped with these expressions, we may now choose a suitable path to integrate them in order to compute z numerically. In realistic rotating stars the equatorial radius r_e is the maximum radius of the star. Thus we could simply start from an equatorial grid point ($\theta = \pi/2, r = r_0$), just outside the surface of the star where $z = 0$, and follow the grid points along the meridian $r = r_0$ (or $x = x_0$), until we reach whatever angle θ, and then move radially—either outward, or inward up to the surface—to obtain the numerical value of z at every grid point in the vacuum region where the metric is known. We should stress at this point that along the meridian of constant $x = x_0$ the optimum integration is achieved by using expression (5), while along the radial direction at constant θ (or μ) the optimum integration is achieved by using expression (2) instead of (4), because the corresponding integrant is well behaved in the former case as $r \to \infty$, in contrast to what is happening in the latter case as $x \to 1$. Thus the z value at the grid point (x, μ) is the outcome of the integral

$$z(r,\mu) = r_e \left(x_0^2 \int_0^{\mu} \frac{\partial B}{\partial x}\, \mathrm{d}\mu' + \frac{x_0}{1-x_0} \int_0^{\mu} B\, \mathrm{d}\mu' \right) + \left(\mu \int_{r_0}^{r} B\, \mathrm{d}r' + (\mu^2 - 1) \int_{r_0}^{r} \frac{\partial B}{\partial \mu}\, \mathrm{d}r' \right). \tag{6}$$

Of course one could choose any other path in the vacuum region. In the following section we will show why the path suggested above is expected to be efficient with respect to numerical errors. The basic argument in favor of this path is the fact that if we follow to move along another meridian $x = \text{const}$ which is far outside the surface of the star, the error in the numerical computation of $\partial B/\partial x$ will be much greater, since $r(x)$ has a rapidly increasing derivative as $x \to 1$.

Following the path described above, we have replotted figure 6 of [1] in figure 1. By direct inspection we find that the matching between the numerical metric and the analytical metric of Manko *et al* (cf [8]) is even better than what is inferred by Berti and Stergioulas [1]. The right transformation of the z-coordinate leads to about two orders of magnitude better matching between the two metrics than that presented in [1], even right at the surface of the star. This renders the metric introduced by Manko *et al* a very good candidate to describe the spacetime around a rotating neutron star.

3. Estimate of numerical errors

Let us say then that we want to compare two metrics, a numerical one $g^{(N)}_{\alpha\beta}(r,\theta)$, which corresponds to a rotating neutron star with specific internal physical characteristics, and an analytical one $g^{(A)}_{\alpha\beta}(\rho, z)$. The difference between the two metrics could be attributed either to a true difference between the two spacetimes described by the corresponding metrics, or to errors induced by numerical transformation of coordinates. In order to probe these errors we will first use the example of a Schwarzschild metric which is known analytically in both coordinates. Specifically, the metric function B of Schwarzschild spacetime is

$$B = 1 - \frac{M^2}{4r^2} = 1 - \frac{M^2}{4r_e^2}\left(\frac{1-x}{x}\right)^2, \tag{7}$$

while the Weyl–Papapetrou z-coordinate along the axis $\rho = 0$ as a function of r is

$$z(r, \mu = 1) = r\left(1 + \frac{M^2}{4r^2}\right). \tag{8}$$

The z-coordinate along this axis, computed numerically, will be compared to its true value. This is more demanding for numerical transformation than to any other angle, since the integration path from the equator to the z-axis is longer. Even though the Schwarzschild metric could not be considered suitable to check the errors in a realistic rotating neutron star case (since the Schwarzschild metric is spherically symmetric, B is not a function of μ and thus the computation of the first two integrals in equation (6) does not contribute any error), it gives at least a minimum estimate of the order of magnitude of errors. In the spherically symmetric

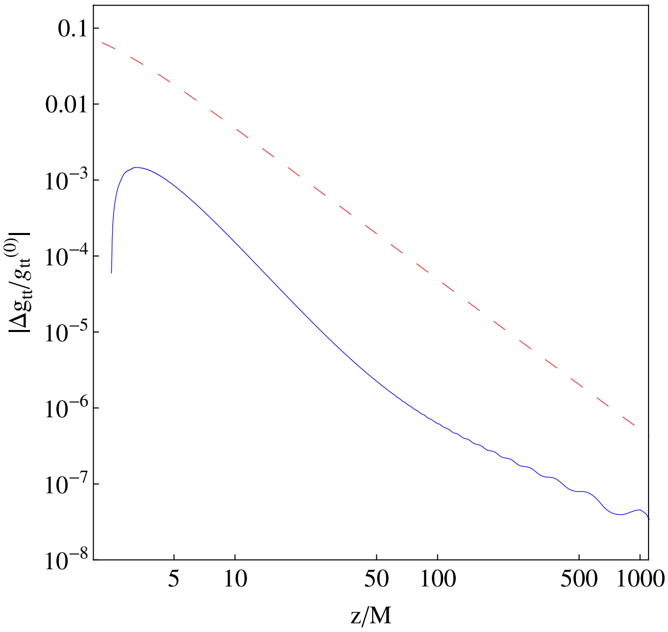

Figure 1. This is a replot of figure 6 of [1] (upper curve) with somewhat altered dimensions along the horizontal axis (it is dimensionless here). The same difference between exactly the same two metrics is plotted for contrast (lower curve) when the transformation of coordinates is done according to our proposed method.

case the errors arise from the numerically estimated value of $\partial B/\partial x$ in the first integral of equation (6), and the numerical computation of the third integral. Although the z-coordinate that we compute numerically gets shifted more and more dramatically as z increases, due to cumulative errors along the integration path, the error induced in the metric itself does not increase so much with z. This is expected since the metric becomes less sensitive to z as we recede from the neutron star. Consequently since the metrics themselves are the ones that we want to compare, the cumulative error in z at large values of z is not disturbing.

In figure 2 we have plotted the logarithmic relative error in the numerical computation of the z-coordinate in the Schwarzschild metric along the z-axis and the corresponding logarithmic relative error in g_{tt}, that is due only to erroneous numerical integration of the z-coordinate, along the z-axis. The specific metric used to estimate the numerical error is a Schwarzschild metric that has the same gravitational mass as the most rapidly rotating model of neutron star in table 3 of [1]; namely $M = 1.864M_{\odot}$ ($M_B = 2.105M_{\odot}$), while the radial distances in the grid of our numerical integrations are scaled by $r_e = 10.755$ Km, which corresponds to the equatorial radius of the same rotating neutron star model. The number of grid points assumed is the same with all models of rotating neutron stars used in [1], and a simple trapezoid method of integration is implemented. As shown in the plot the relative error in g_{tt} does not even exceed the $\sim 10^{-6}$, which is actually just about one order of magnitude higher than the level of accuracy of the numerical metrics produced by the numerical code of Stergioulas [14]. Thus we conclude that if the size of the remaining errors which are coming from the μ-dependent terms of a realistic numerical model does not exceed the errors arising in the simple Schwarzschild case, there is no need to worry about any numerical errors induced to metrics caused by transformation of coordinates.

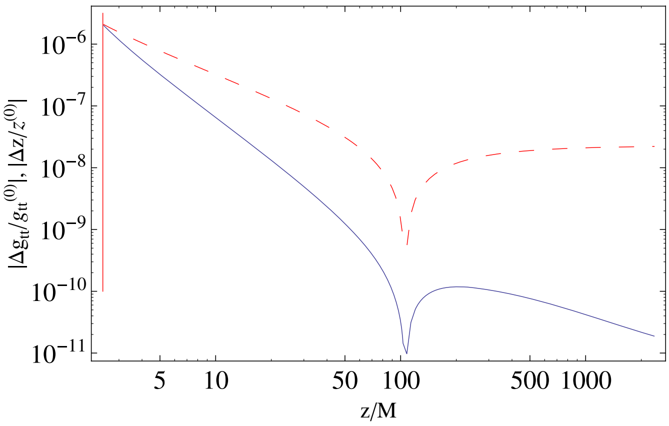

Figure 2. This is a log–log plot of the relative difference between the exact z value ($z^{(0)}$) and the z-coordinate produced through numerical integration ($z^{(N)}$) at fixed grid points (upper thick dashed curve), as well as the relative difference between the exact metric component $g_{tt}^{(0)}$ and that computed from transforming (r, θ) to (z, ρ) coordinates (lower thin solid curve) for our Schwarzschild model. It is clear that the errors are at most of the order of $\sim 10^{-6}$ in g_{tt}. The deep wells in the plots are due to opposite signs of the errors accumulated along μ- and r-integration in the region $r > r_0$ that ends up nullifying the difference between $z^{(0)}$ and $z^{(N)}$ at some point. Finally the vertical solid line on the leftmost part of the plot marks the polar surface location of the rotating neutron star model cited above in order to indicate the minimum z-value at which there is any physical meaning to look for numerical errors due to coordinate transformation.

In the remaining part of this section we will argue that this is exactly the case with a generic numerical metric, as long as the number of grid points is sufficiently high (of the order of 300×300 for the μ and the x coordinates in the vacuum region). To show this we shall again appeal to the same fastest rotating neutron star model, as previously, to estimate the magnitude of the error in computing the z-coordinate. Of course, in the case of a metric that is given in a tabulated form we do not have a true value of the z coordinate to compare with, as in the simple example analyzed previously. On the other hand by a simple plot of $B(x_0, \mu)$ and $(\partial B(x_0, \mu)/\partial x)_N$ as a function of μ (the subscript $_N$ refers to the fact that the derivative is computed numerically), we conclude that both these functions are quite constant along the same meridian (B and $(\partial B/\partial x)_N$ do not change by more than 0.1% and 25%, respectively, over the whole range of μ), and thus the error in computing the first two integrals in (6), even by the simple trapezoid rule is of the order

$$(\Delta z)_{\mu\text{-integration}} = \frac{1}{h}\frac{h^3}{12} f''(\xi) < \frac{h^2}{10} \times \max_{\mu} f''(\mu), \tag{9}$$

where h denotes the step size of μ used in numerical integration ($1/h$ is the total number of steps), while ξ is some value of μ in the interval [0, 1]. It is easy to verify that this error in numerical integration over μ is less than $\Delta z/M \simeq 3 \times 10^{-7}$. This is systematically lower, or at most of the same order of magnitude, than the errors related to r-integration and numerical computation of $\partial B/\partial x$, which were estimated previously by means of the Schwarzschild example (cf figure 2). Therefore, the plot of figure 2 summarizes quite well the overall numerical errors in computing the values of z and the corresponding errors induced in computing the g_{tt} component of a numerically constructed metric.

We thus conclude that relative differences of metrics at the level of 10^{-6} and higher suggest true differences in metrics, and only these should be taken seriously into consideration when a proposed analytical metric is used as a faithful representation of a metric that is constructed numerically. The remaining discrepancies between metrics could easily be attributed to inexact transformation of coordinates.

In a forthcoming paper we examine another similar candidate analytical metric (the one described in [7, 17]) to describe the spacetime around any kind of neutron star, either rotating or not, since the analytical metric used in [1] had the disadvantage that it could not be adjusted to describe very slow rotating stars.

For an extended and analytical version of the present note see [18].

Acknowledgments

We would like to thank Nikolaos Stergioulas for many useful discussions and for providing us with the access to his numerical code. This work was supported partly by the 'PYTHAGORAS' research funding program grant no 70/3/7396 and partly by the research funding program 'Kapodistrias' with grant no 70/4/7672.

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