Tracing the geometry around a massive, axisymmetric body to measure, through gravitational waves, its mass moments and electromagnetic moments

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The geometry around a rotating massive body, which carries charge and electrical currents, could be described by its multipole moments (mass moments, mass-current moments, electric moments, and magnetic moments). When a small body is orbiting around such a massive body, it will move on geodesics, at least for a time interval that is short with respect to the characteristic time of the binary due to gravitational radiation. By monitoring the waves emitted by the small body we are actually tracing the geometry of the central object, and hence, in principle, we can infer all its multipole moments. This paper is a generalization of previous similar results by Ryan. Ryan explored the mass and mass-current moments of a stationary, axially symmetric, and reflection symmetric, with respect to its equatorial plane, metric, by analyzing the gravitational waves emitted from a test body which is orbiting around the central body in nearly circular equatorial orbits. In our study we suppose that the gravitating source is endowed with intense electromagnetic field as well. Because of its axisymmetry the source is characterized now by four families of scalar multipole moments: its mass moments M_l, its mass-current moments S_l, its electrical moments E_l, and its magnetic moments H_l, where $l = 0, 1, 2, \ldots$. Four measurable quantities, the energy emitted by gravitational waves per logarithmic interval of frequency, the precession of the periastron, the precession of the orbital plane, and the number of cycles emitted per logarithmic interval of frequency, are presented as power series of the Newtonian orbital velocity of the test body. The power series coefficients are simple polynomials of the various moments. If any of these quantities are measured with sufficiently high accuracy, the lowest moments, including the electromagnetic ones, could be inferred and thus we could get valuable information about the internal structure of the compact massive body. The fact that the electromagnetic moments of spacetime can be measured demonstrates that one can obtain information about the electromagnetic field purely from gravitational-wave analysis. Additionally, these measurements could be used as a test of the no-hair theorem for black holes.

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I. INTRODUCTION

While, it has been well known for quite some time that the geometry of the vacuum around a massive object is related directly to the multipole moments [1–4] of the central object, as in Newtonian gravity, Ryan [5] was the first who attempted to map the spacetime geometry of a central axisymmetric body, through its mass and mass-current moments, on a few measurable physical quantities that are related to the kinematics of a hypothetical test body that is orbiting around the central object while emitting gravitational waves. Later, Ryan [6] used the outcome of his work to perform the analysis on the output of gravitational-wave detectors, in order to extract the moments of the central body around which a much lower-mass body is orbiting and emitting waves.

Fortunately, we live in an era where technology may give us the opportunity to observe astrophysical phenomena that are related to the highly distorted geometry of a compact massive central object. In particular, the detection of gravitational waves from binaries will provide us with data that are strongly dependent on the geometry itself. Apart from the Earth-based detectors which are restricted to detect moderate mass binaries (e.g., 1 to 300 $M_\odot$ for the Laser Interferometer Gravitational Wave Observatory, LIGO) [7], the Laser Interferometer Space Antenna, LISA, when it flies in space and starts operating, is expected to explore the geometry of very massive objects with exceptional high precision [8]. Besides, telescopes with higher and higher resolution, operating at various regions of the electromagnetic spectrum, are able to probe the close neighborhood of compact massive objects, as in accretion disks and jets of active galactic nuclei. All this information which is related, by one way or another, to the motion of small objects in the curved geometry of massive astrophysical objects could somehow shed light on the internal structure of the central object. Of course, it is not expected to fully determine its structure from the knowledge of its moments (this is not possible even in Newtonian gravity). However, knowing the moments of the central body could set restrictions on the various models that are assumed to describe the interior of the central body. Moreover, the no-hair theorem in the case of a large black hole at the role of the central object could be fully tested, if from our moment-extraction formulas we get the values of electric and magnetic moments.

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The binaries we have considered in our paper are idealized with regard to the following quite realistic assumptions.

(i) The central object is assumed to be stationary and axisymmetric, and characterized by reflection symmetry with respect to its equatorial plane. This is expected to be true for a quiescent massive object with internal fluid motions that are strictly toroidal. The axisymmetry gives us the freedom to describe the spacetime geometry with scalar, instead of tensorial, multipole moments (see Ref. [2]). The same is true in Newtonian gravity as well, although in that case all other moments except the mass moments do not show up in the expansion of the gravitational potential. Since we are taking into account the electromagnetic content of the central object as well, we are considering four families of multipole moments to characterize the geometry around the central body: its mass moments $M_0, M_2, M_4, \ldots$, its mass-current moments $S_1, S_3, \ldots$, its electric charge moments $E_0, E_1, E_2, \ldots$, and its magnetic moments $H_0, H_1, H_2, \ldots$. In particular, the $M_0 \equiv M$ moment is the mass of the object, S_1 is its angular momentum, $E_0 \equiv E$ is its charge, and H_1 is its magnetic dipole moment. In every family of moments, each moment appears in steps of 2, and this holds true for the electromagnetic moments as well [9]. This property is due to the reflection symmetry of the metric itself (cf., [2]). The geometry of such objects can be described by the Papapetrou metric which consists of only two dynamical functions. The third one (see Sec. II) could be easily inferred from the first two.

(ii) Although we plan to extend our exploration in a generic geodesic motion around such a central object, in the present paper we only take into consideration nearly equatorial and nearly circular geodesic orbits of test bodies in the fixed geometry of a central massive object. We know that gravitational radiation from a test body that is far from its innermost stable circular orbit tends to circularize the orbit [10,11], and therefore the orbit could safely be considered circular if it has a sufficiently long time to evolve without being perturbed by other objects. Also, we know that at least for not extremely fast rotating Kerr black holes the evolution of nonequatorial orbits due to radiation reaction is such that their inclination remains almost constant while their radius decreases [12].

(iii) The energy emitted in the form of gravitational radiation will be assumed to be given by the quadrupole formula, since there is no known way to fully analyze the wave emission in a generic geometrical background. Furthermore, we assume that this energy is carried away by waves at infinity, and there is no energy loss through any horizon, or due to thermal heating of the surface of the central object from the impact of gravitational waves.

The rest of the paper is organized as follows: First, in Sec. II we define the observable quantities that will be used to measure the moments of the metric. These quantities are the periastron precession Ω_ρ, the precession of the orbital plane Ω_z, the energy emitted at infinity per logarithmic frequency change $\Delta E/\mu$, and the number of cycles of the primary gravitational waves per logarithmic frequency change ΔN. In particular, the latter one, which can be measured with high accuracy by the broadband wave detectors that are operating now, or will be built in the near future, is computed assuming that the phase of the waves is coming simply from the dominant frequency, $f = \Omega/\pi$. We also show how one defines the mass and electromagnetic multipole moments of an axially symmetric body with reflectional symmetry, and how these moments uniquely determine the metric of the space around the object. For the equatorial plane, and slightly out of this plane, we write the metric as a power series of the Weyl radial coordinate ρ. The coefficients of the power series are polynomials of the various moments of spacetime. We end this section by explaining the implications of reflection symmetry of the metric on the electromagnetic fields. Having in hand all these expressions that connect the observable quantities to the metric and hence to the moments, we proceed in Sec. III to write down expressions for the four astrophysical quantities, as power series of $v \equiv (M\Omega)^{1/3}$, where M is the mass of the central object and Ω is the orbital frequency of the test body, observed at infinity. This quantity is the Newtonian orbital velocity of the orbiting body, and is a measure of the gravitational field strength. Following the analysis of Ryan [5], we present the power expansion of ΔN with coefficients that include only the leading order contribution of each central-body multipole moment. It is argued that these are the only terms we could get without getting into the complicated analysis of wave emission. Finally, in Sec. IV we comment on how the gravitational-wave analysis could inform us about the moments of the central object, the accuracy with which these moments could be computed, and the implications they could have on the observational verification (or not) of the full no-hair theorem (when charges are included). Throughout the paper units are chosen so that $G = c = 1$.

II. OBSERVABLE QUANTITIES AND MOMENTS

In this section we briefly present and discuss the measurable quantities that Ryan has used as tracers to measure moments. A thorough presentation and analysis of them can be found in Ref. [5]. Then, we present all formulas that determine the various moments and relate them with the metric describing the geometry around a central object. Finally, we discuss what kind of electromagnetic fields are consistent with the symmetries assumed for the metric.

A. Quantities that can be measured through gravitational-wave analysis

As is explained in [5] there are four physical quantities in a binary with high-mass ratio, that can, in principle, be measured through the gravitational radiation emitted by the binary, and are straightforwardly related to the spacetime metric of the massive body. It is exactly these quantities that we will use here as basic information in order to extract the various multipole moments of the central object. These are: (i) Ω_ρ, the periastron precession of the low-mass body, (ii) Ω_z, the orbital-plane precession of the low-mass body, (iii) ΔE, the energy emitted as gravitational waves per logarithmic interval of frequency, and most important (iv) ΔN, the number of gravitational-wave cycles per logarithmic interval of frequency. The first two quantities are computed by analyzing the geodesic motion of nearly circular, nearly equatorial orbits of a test body on a fixed background metric, and can be measured through the modulation they induce on the gravitational waves emitted by the binary. The third one, although it is directly related to gravitational radiation emitted by the binary, can be easily inferred by the functional relation of the energy of the test body, which changes adiabatically, to its orbital frequency, which is simply half the primary wave frequency. Finally, the fourth one is the best measurable quantity, since the phase matching used in data analysis leads to a highly accurate estimation of the frequency dependence of the phase, assuming detection has been established. The computation of this quantity involves a number of approximations, since it is directly related to the mechanism of gravitational-wave emission on a complicated metric background.

Here, for the sake of completeness, we rewrite the expressions of Ryan [5] that relate all these observable quantities with the metric without further comments on how these are computed. Both precession frequencies are given by

$$\begin{aligned}\Omega_\alpha = \Omega - \bigg(-\frac{g^{\alpha\alpha}}{2}\bigg[&(g_{tt}+g_{t\phi}\Omega)^2\bigg(\frac{g_{\phi\phi}}{\rho^2}\bigg)_{,\alpha\alpha} \\ &-2(g_{tt}+g_{t\phi}\Omega)(g_{t\phi}+g_{\phi\phi}\Omega)\bigg(\frac{g_{t\phi}}{\rho^2}\bigg)_{,\alpha\alpha} \\ &+(g_{t\phi}+g_{\phi\phi}\Omega)^2\bigg(\frac{g_{tt}}{\rho^2}\bigg)_{,\alpha\alpha}\bigg]\bigg)^{1/2}, \end{aligned} \tag{1}$$

where α stands for ρ, or z. Actually, the frequencies written above correspond to the difference between the orbital frequency and the frequency of perturbations in ρ, or z, since these differences are expected to show up in gravitational waves as a modulating frequency. Of course these frequencies are accurate only for orbits that are slightly noncircular and slightly nonequatorial; otherwise the frequencies would depend not only on the metric but on specific characteristics of the orbit, its eccentricity and inclination.

The energy per unit test-body mass for an equatorial circular orbit in an axially symmetric spacetime is

$$\frac{E}{\mu} = \frac{-g_{tt} - g_{t\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}, \tag{2}$$

and thus, the specific energy released as gravitational radiation per logarithmic interval of frequency is

$$\frac{\Delta E}{\mu} = -\Omega\frac{d(E/\mu)}{d\Omega}. \tag{3}$$

The expression above assumes that all the energy lost from the test body has been emitted at infinity as radiation, and is neither wasted as thermal energy on the fluid of the central object, nor is it "lost" through any horizon.

The number of gravitational-wave cycles spent in a logarithmic interval of frequency is

$$\Delta N = \frac{f\Delta E(f)}{dE_{\mathrm{wave}}/dt}, \tag{4}$$

where dE_{wave}/dt is the gravitational-wave luminosity, which is assumed to be exactly the rate of energy loss of the orbiting test body. As Ryan has analytically shown the main contribution of dE_{wave}/dt comes from the mass quadrupole radiative moment of the binary. More specifically, up to fourth order of $v \equiv (M\Omega)^{1/3}$ after the leading order, the gravitational-wave luminosity is accurately computed from the quadrupole formula

$$\frac{dE_{\mathrm{wave}}}{dt}\bigg|_{I_{ij}} = \frac{32}{5}\mu^2\rho^4\Omega^6, \tag{5}$$

plus a contribution of the current quadrupole radiative moment, due to the motion of the central object around the center of mass,

$$\begin{aligned}\frac{dE_{\mathrm{wave}}}{dt}\bigg|_{J_{ij}} =& \frac{32}{5}\bigg(\frac{\mu}{M}\bigg)^2 v^{10} \\ &\times\bigg[\frac{v^2}{36} - \frac{S_1 v^3}{12M^2} + \frac{S_1^2 v^4}{16M^4} + \mathcal{O}(v^5)\bigg];\end{aligned} \tag{6}$$

see Ref. [5]. The expression above should be complemented by additional contributions of dE_{wave}/dt, due to post-Newtonian corrections. The corresponding contributions, up to v^4, are simply numerical if one computes them from perturbation analysis in a Schwarzschild background. A comparison though, between the final formula (55) of [5] for dE/dt, and formula (3.13) of [13], which is based on perturbations on a Kerr background, shows that at least up to v^4 order, the terms of [5] that include S_1 and M_2, which come from the corresponding contributions that are given by Eqs. (5) and (6), in the case of a Kerr metric, are equal to the ones of Ref. [13]. This agreement indicates that up to v^4 order we can simply add the following numerical post-Newtonian terms (the corresponding terms of [13] if we set $q = 0$) to the rest of the contributions of dE_{wave}/dt,

$$\frac{dE_{\rm wave}}{dt}\bigg|_{PN} = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 v^{10}\times\left[-\frac{1247}{336}v^2 + 4\pi v^3 - \frac{44711}{9072}v^4 + \mathcal{O}(v^5)\right]. \tag{7}$$

Finally, in order to compute the number of cycles ΔN, one has to add up all contributions of $dE_{\rm wave}/dt$ and combine them with the expression for $\Delta E/\mu$, which is given above [see Eq. (3)].

To get expressions for all these quantities that are straightforwardly connected to the moments of the central object and can be observationally measured, first one has to reexpress the metric functions in terms of all moments. Also, the relation between the radius ρ and the orbital frequency of the test body Ω, through moments, is necessary so as to finally express the four measurable quantities as power series of v.

B. Moments describing spacetime

In our paper, we consider only stationary axisymmetric objects that are symmetric with respect to their equatorial plane. These symmetries are more or less realistic assumptions for a quiescent massive rotating astrophysical body around which much smaller bodies orbit. The metric of such a central object alone could be written in (t, ρ, z, ϕ) coordinates, in the form of the Papapetrou metric [14],

$$ds^2 = -F(dt - \omega d\phi)^2 + \frac{1}{F}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2], \tag{8}$$

where F, ω, and γ are the three functions that fully determine a specific metric. These are functions of ρ and $|z|$ only, due to axisymmetry and reflection symmetry. Einstein's equations in vacuum guarantee that once F and ω are given, γ can be easily computed (see [15]). Once we incorporate an electromagnetic field in the vacuum around the compact object, the source of which is the compact object itself, which allows spacetime to have the same symmetries, the metric above still describes the electrovacuum spacetime, but now the metric and the electromagnetic field should satisfy the Einstein-Maxwell equations. In order to fully compute the metric functions, one more complex function, Φ, which is related to the electromagnetic field, is necessary. F, ω, and Φ themselves can be determined by solving the so-called Ernst equations [16,17], which are nothing more than the Einstein-Maxwell equations in a different form. It is a system of nonlinear complex differential equations of second order:

$$[\mathrm{Re}(\mathcal{E}) + |\Phi|^2]\nabla^2\mathcal{E} = (\boldsymbol{\nabla}\mathcal{E} + 2\Phi^*\boldsymbol{\nabla}\Phi)\cdot\boldsymbol{\nabla}\mathcal{E}, \tag{9}$$

$$[\mathrm{Re}(\mathcal{E}) + |\Phi|^2]\nabla^2\Phi = (\boldsymbol{\nabla}\mathcal{E} + 2\Phi^*\boldsymbol{\nabla}\Phi)\cdot\boldsymbol{\nabla}\Phi, \tag{10}$$

where $\boldsymbol{\nabla}$ denotes the gradient in a Cartesian 3D space (ρ, z, ϕ), and Re(...), Im(...), here and henceforth, denote the real and imaginary part, respectively, of the complex function in parentheses. An asterisk $*$ denotes complex conjugate. The third metric function, γ, is then easily computed by integrating the partial derivatives $\partial\gamma/\partial\rho$, $\partial\gamma/\partial z$, which are given as functions of derivatives of all other functions,

$$\frac{\partial\gamma}{\partial\rho} = \frac{1}{4}\frac{\rho}{g_{tt}^2}\left[\left(\frac{\partial g_{tt}}{\partial\rho}\right)^2 - \left(\frac{\partial g_{tt}}{\partial z}\right)^2\right] - \frac{1}{4}\frac{g_{tt}^2}{\rho}\left[\left(\frac{\partial(g_{t\phi}/g_{tt})}{\partial\rho}\right)^2 - \left(\frac{\partial(g_{t\phi}/g_{tt})}{\partial z}\right)^2\right] - \frac{\rho}{g_{tt}}\left[\left(\frac{\partial\mathrm{Re}(\Phi)}{\partial\rho}\right)^2 - \left(\frac{\partial\mathrm{Re}(\Phi)}{\partial z}\right)^2\right] - \frac{\rho}{g_{tt}}\left[\left(\frac{\partial\mathrm{Im}(\Phi)}{\partial\rho}\right)^2 - \left(\frac{\partial\mathrm{Im}(\Phi)}{\partial z}\right)^2\right], \tag{11}$$

$$\frac{\partial\gamma}{\partial z} = \frac{1}{2}\frac{\rho}{g_{tt}^2}\frac{\partial g_{tt}}{\partial\rho}\frac{\partial g_{tt}}{\partial z} - \frac{1}{2}\frac{g_{tt}^2}{\rho}\frac{\partial(g_{t\phi}/g_{tt})}{\partial\rho}\frac{\partial(g_{t\phi}/g_{tt})}{\partial z} - 2\frac{\rho}{g_{tt}}\frac{\partial\mathrm{Re}(\Phi)}{\partial\rho}\frac{\partial\mathrm{Re}(\Phi)}{\partial z} - 2\frac{\rho}{g_{tt}}\frac{\partial\mathrm{Im}(\Phi)}{\partial\rho}\frac{\partial\mathrm{Im}(\Phi)}{\partial z}. \tag{12}$$

The relation of the two complex functions, $\mathcal{E}$ and Φ, with the metric functions is the following,

$$\mathcal{E} = (F - |\Phi|^2) + i\varphi, \tag{13}$$

where φ is related with $g_{t\phi}$ through

$$g_{t\phi} = F\omega = F\int_\rho^\infty d\rho' \frac{\rho'}{F^2}\left[\frac{\partial\varphi}{\partial z} + 2\mathrm{Re}(\Phi)\frac{\partial\mathrm{Im}(\Phi)}{\partial z} - 2\mathrm{Im}(\Phi)\frac{\partial\mathrm{Re}(\Phi)}{\partial z}\right]\bigg|_{z=\rm const}. \tag{14}$$

Note that there is a sign difference in Eq. (22) of [5], which has been corrected in a later paper of Ryan [18], and comes from an odd convention of ω used by Ernst (see relevant comment of [19]).

Instead of $\mathcal{E}$ and Φ, one could use two new complex functions $\tilde{\xi}$ and $\tilde{q}$, that play the role of gravitational potential and Coulomb potential, respectively, and are more directly connected to the mass and electromagnetic moments of the central body. These potentials are related to the Ernst functions by

$$\mathcal{E} = \frac{\sqrt{\rho^2 + z^2} - \tilde{\xi}}{\sqrt{\rho^2 + z^2} + \tilde{\xi}}, \tag{15}$$

$$\Phi = \frac{\tilde{q}}{\sqrt{\rho^2 + z^2} + \tilde{\xi}}, \tag{16}$$

and can be written as power series expansions at infinity,

$$\tilde{\xi} = \sum_{i,j=0}^{\infty} a_{ij}\bar{\rho}^i\bar{z}^j, \qquad \tilde{q} = \sum_{i,j=0}^{\infty} b_{ij}\bar{\rho}^i\bar{z}^j, \tag{17}$$

where

$$\bar{\rho} \equiv \frac{\rho}{\rho^2 + z^2}, \qquad \bar{z} \equiv \frac{z}{\rho^2 + z^2}, \tag{18}$$

and a_{ij}, b_{ij} are coefficients that vanish when i is odd. This reflects the analyticity of the potentials on the z-axis. The tilded quantities, here and henceforth, are the conformally transformed ones, which are essential for calculating the moments (see [1]).

Because of Ernst equations (9) and (10) the above power expansion coefficients a_{ij} and b_{ij} are interrelated through the following complicated recursive relations:

$$\begin{aligned}(r+2)^2 a_{r+2,s} = {}& -(s+2)(s+1)a_{r,s+2} \\ & + \sum_{k,l,m,,n,p,g} (a_{kl}a^*_{mn} - b_{kl}b^*_{mn})[a_{pg}(p^2 \\ & + g^2 - 4p - 5g - 2pk - 2gl - 2) \\ & + a_{p+2,g-2}(p+2)(p+2-2k) \\ & + a_{p-2,g+2}(g+2)(g+1-2l)],\end{aligned} \tag{19}$$

and

$$\begin{aligned}(r+2)^2 b_{r+2,s} = {}& -(s+2)(s+1)b_{r,s+2} \\ & + \sum_{k,l,m,n,p,g} (a_{kl}a^*_{mn} - b_{kl}b^*_{mn})[b_{pg}(p^2 \\ & + g^2 - 4p - 5g - 2pk - 2gl - 2) \\ & + b_{p+2,g-2}(p+2)(p+2-2k) \\ & + b_{p-2,g+2}(g+2)(g+1-2l)],\end{aligned} \tag{20}$$

where $m = r - k - p$, $0 \leq k \leq r$, $0 \leq p \leq r - k$, with k and p even, and $n = s - l - g$, $0 \leq l \leq s + 1$, and $-1 \leq g \leq s - l$. Essentially, these relations are simply an algebraic version of Einstein-Maxwell equations for the coefficients of the power expansion of the metric and the electromagnetic field tensor. The recursive relations (19) and (20) could be used to build the whole power series of $\tilde{\xi}$ and $\tilde{q}$ from a full knowledge of the metric on the axis of symmetry

$$\tilde{\xi}(\bar{\rho} = 0) = \sum_{i=0}^{\infty} m_i\bar{z}^i, \qquad \tilde{q}(\bar{\rho} = 0) = \sum_{i=0}^{\infty} q_i\bar{z}^i. \tag{21}$$

In [20] a method of calculating the complex multipole moments of the central object in terms of the m_i's and q_i's is presented. In brief, the gravitational moments are given by

$$P_n = \frac{1}{(2n-1)!!}S_0^{(n)}, \tag{22}$$

where $S_a^{(n)}$ are computed recursively by

$$\begin{aligned} & S_0^{(0)} = \tilde{\xi}, \qquad S_0^{(1)} = \frac{\partial}{\partial\bar{z}}S_0^{(0)}, \qquad S_1^{(1)} = \frac{\partial}{\partial\bar{\rho}}S_0^{(0)}, \\ & S_a^{(n)} = \frac{1}{n}\bigg\{a\frac{\partial}{\partial\bar{\rho}}S_{a-1}^{(n-1)} + (n-a)\frac{\partial}{\partial\bar{z}}S_a^{(n-1)} + a\bigg[(a+1-2n)\gamma_1 - \frac{a-1}{\bar{\rho}}\bigg]S_{a-1}^{(n-1)} + (a-n)(a+n-1)\gamma_2 S_a^{(n-1)} \\ & \qquad + a(a-1)\gamma_2 S_{a-2}^{(n-1)} + (n-a)(n-a-1)\bigg(\gamma_1 - \frac{1}{\bar{\rho}}\bigg)S_{a+1}^{(n-1)} - \bigg(n - \frac{3}{2}\bigg)[a(a-1)\tilde{R}_{11}S_{a-2}^{(n-2)} + 2a(n-a)\tilde{R}_{12}S_{a-1}^{(n-2)} \\ & \qquad + (n-a)(n-a-1)\tilde{R}_{22}S_a^{(n-2)}]\bigg\}. \end{aligned} \tag{23}$$

The electromagnetic moments Q_n are computed from exactly the same formulas, by simply replacing the initial term $S_0^{(0)}$ with $\tilde{q}$ instead of $\tilde{\xi}$. $\tilde{R}_{11}$, $\tilde{R}_{12}$, and $\tilde{R}_{22}$ are given by

$$\begin{aligned}\tilde{R}_{ij} = {}& (\bar{r}^2\tilde{\xi}^*\tilde{\xi} - \bar{r}^2\tilde{q}^*\tilde{q} - 1)^{-2}(D_i\tilde{\xi}D_j\tilde{\xi}^* + D_i\tilde{\xi}^*D_j\tilde{\xi} \\ & - D_i\tilde{q}D_j\tilde{q}^* - D_i\tilde{q}^*D_j\tilde{q} + \tilde{s}_i\tilde{s}^*_k + \tilde{s}^*_i\tilde{s}_k),\end{aligned} \tag{24}$$

where

$$\begin{aligned} & \bar{r}^2 = \bar{\rho}^2 + \bar{z}^2, \qquad D_1 = \bar{z}\frac{\partial}{\partial\bar{\rho}} - \bar{\rho}\frac{\partial}{\partial\bar{z}}, \\ & D_2 = \bar{\rho}\frac{\partial}{\partial\bar{\rho}} + \bar{z}\frac{\partial}{\partial\bar{z}} + 1, \qquad \tilde{s}_i = \bar{r}\,\tilde{\xi}\,D_i\tilde{q} - \bar{r}\,\tilde{q}\,D_i\tilde{\xi}, \end{aligned} \tag{25}$$

and $\gamma_1 \equiv \gamma_{,\rho}$ and $\gamma_2 \equiv \gamma_{,z}$ can be expressed in term of $\tilde{R}_{ij}$ as

$$\gamma_1 = \frac{1}{2}\bar{\rho}(\tilde{R}_{11} - \tilde{R}_{22}), \qquad \gamma_2 = \bar{\rho}\tilde{R}_{12}. \tag{26}$$

The mass moments M_n and the mass-current moments S_n

are related to P_n by

$$P_n = M_n + iS_n, \tag{27}$$

whereas the electric moments E_n and the magnetic moments H_n are related to Q_n by

$$Q_n = E_n + iH_n. \tag{28}$$

Since this algorithm can be used to evaluate the moments in terms of the m_i's and q_i's, one can invert these relations and express the m_i's and q_i's in terms of the moments:

$$\begin{aligned} m_n &= a_{0n} = M_n + iS_n + \text{LOM}, \\ q_n &= b_{0n} = E_n + iH_n + \text{LOM}, \end{aligned} \tag{29}$$

where "LOM" stands for lower order multipole moments of any type. Thus, we can use the recursive relations (19) and (20) to evaluate the a_{ij} and b_{ij} coefficients in terms of the moments. Finally, following the procedure presented in the beginning of this subsection [Eqs. (11)–(17)] we can express the metric functions and their first and second derivatives as power series of ρ and z with coefficients that are simple algebraic functions of the moments of the massive body. Since in our study we have confined the motion of the test particle on the equatorial plane, we actually need to compute everything at $z = 0$ which makes calculations far simpler than what they seem.

C. Reflection symmetry and electromagnetic moments

Following Ernst [16,17] and using Papapetrou's metric (8) we end up with the following Einstein-Maxwell equations:

$$\nabla \cdot [\rho^{-2} F(\nabla A_3 - \omega \nabla A_4)] = 0, \tag{30}$$

$$\nabla \cdot [F^{-1} \nabla A_4 + \rho^{-2} F \omega (\nabla A_3 - \omega \nabla A_4)] = 0, \tag{31}$$

$$\nabla \cdot [\rho^{-2} F^2 \nabla \omega - 4\rho^{-2} F A_4 (\nabla A_3 - \omega \nabla A_4)] = 0, \tag{32}$$

$$\begin{aligned} F\nabla^2 F = {} & \nabla F \cdot \nabla F - \rho^{-2} f^4 \nabla \omega \cdot \nabla \omega + 2F \nabla A_4 \cdot \nabla A_4 \\ & + 2\rho^2 F^3 (\nabla A_3 - \omega \nabla A_4) \cdot (\nabla A_3 - \omega \nabla A_4), \end{aligned} \tag{33}$$

where A_3 and A_4 denote the A_ϕ and A_t components of the electromagnetic 4-potential, respectively, and ∇ is the three-dimensional divergence operator in Weyl coordinates. As we have already mentioned, the three metric functions F, ω, and γ are functions of ρ and $|z|$ only, due to the assumed symmetries. From the equations above one cannot easily tell which symmetries, if any, are inherited in A_3 and A_4. It is obvious though that A_3 and A_4 being either both odd or both even functions of z is consistent with the reflection symmetry of the metric functions. We will argue that these are the only reasonable types of the electromagnetic field.

The Ernst potential Φ is defined by Ernst through the following relations:

$$\rho^{-1} f(\nabla A_3 - \omega \nabla A_4) = \hat{n} \times \nabla A_3', \tag{34}$$

$$\Phi = A_4 + iA_3', \tag{35}$$

where $\hat{n}$ is the unit vector in the azimuthal direction. From Eq. (14), we obtain the metric function $g_{t\phi}$. Since we want to end up with metric functions that are even functions of z, then the whole intergrand should be even as well. This could be accomplished if φ is an odd function of z and the real and imaginary parts of Φ are either even and odd or odd and even functions of z, respectively. If none of the above holds then, in order to get an even function for $g_{t\phi}$, one must impose a restraining functional relation between φ and Φ. But φ and Φ should be independent in order to describe a generic spacetime with the symmetries mentioned above. Therefore, by virtue of Eqs. (34) and (35), either A_4 and A_3 are both even or both odd functions of z. Consequently, for both cases the $\mathrm{Re}(\mathcal{E})$ is an even function and the $\mathrm{Im}(\mathcal{E})$ is an odd function of z.

Now, we can use this information to conclude that, by virtue of Eq. (15), the action of reflection symmetry leaves $\mathrm{Re}(\tilde{\xi})$ invariant and changes the sign of $\mathrm{Im}(\tilde{\xi})$. Thus only even-order mass moments and odd-order current mass moments will occur [21]. For the electromagnetic moments, things are not univocal since we have two discernible electromagnetic cases that are consistent with the symmetries of the metric. If both A_3 and A_4 are odd functions of z then the action of the reflection symmetry leaves $\mathrm{Re}(\tilde{q})$ invariant and reverses the sign of $\mathrm{Im}(\tilde{q})$, which means that only even-order electric field moments and odd-order magnetic field moments will occur. On the other hand, if both A_3 and A_4 are even functions of z then the action of the reflection symmetry leaves $\mathrm{Im}(\tilde{q})$ invariant, but reverses the sign of $\mathrm{Re}(\tilde{q})$, which means that in this case only odd-order electric field moments and even-order magnetic field moments will occur.

III. THE POWER EXPANSION FORMULAS

Combining all formulas that are given in Sec. II, eventually we can express all four measurable quantities as power series of $v \equiv (M\Omega)^{1/3}$ with coefficients that have explicit dependence on all four types of moments. The choice of v as a dimensionless parameter to expand all physical quantities is warranted from the fact that the inspiral phase of a binary, the best exploitable part in gravitational-wave analysis [22], involves comparatively low magnitudes of v. All measurable quantities have been transformed to a dimensionless form as well, for example, by dividing the two frequencies Ω_ρ and Ω_z by the orbital frequency Ω.

Since the metric functions and their derivatives are expressed as functions only of ρ at the equatorial plane, in order to express all measurable quantities as power series of v, we need also a power series expansion of ρ with respect to v, or equivalently Ω. Thus we have to invert the function $\Omega(\rho)$, at least as a power expansion. From an elementary analysis of circular geodesics on the equatorial plane (see [5]) we know that

$$\Omega = \frac{-g_{t\phi,\rho} + \sqrt{(g_{t\phi,\rho})^2 - (g_{tt,\rho})(g_{\phi\phi,\rho})}}{g_{\phi\phi,\rho}}. \tag{36}$$

In the following part of this section we explain the algorithm that one should follow, in order to obtain the power series for Ω_ρ/Ω, Ω_z/Ω, $\Delta E/\mu$, and ΔN. One starts with a power series of $\tilde{\xi}$ and $\tilde{q}$ of the form given by Eq. (17). Since no higher than second derivatives of the metric functions with respect to z are necessary, one should keep a_{ij}'s and b_{ij}'s with $0 \le j \le 2$, and as many values of i as one needs to carry the power series expansion of the measurable quantities at a desirable order. In our paper where all quantities are written up to no higher than v^{11} order, we only need i's in the interval $0 \le i \le 4$. All quantities that are expressed as power series of $\bar{\rho}$ and $\bar{z}$ are evaluated at $\bar{z} = z = 0$ at the end, and thus, all expressions are finally power series of $\bar{\rho} = 1/\rho$, due to Eq. (18). Although the a_{ij}'s and b_{ij}'s are polynomials of various moments, from the practical point of view it is preferable to keep them as they are, and replace them by their moment's dependences only at the final expressions. Then from $\tilde{\xi}$ and $\tilde{q}$ we construct $\mathcal{E}$, Φ, and F, φ [cf., Eqs. (13) and (15)]. These are sufficient to build all metric functions through Eqs. (11), (12), and (14). Next, following the procedure described above, we expand Ω as a power series of $1/\rho$, by virtue of Eq. (36). This series is inverted and in this way we obtain $1/\rho$ as a power series of Ω, which then can easily be turned into a power series of the dimensionless parameter v.

Now, the power series representing $1/\rho$ will replace all $1/\rho$ terms appearing at the expansions of the metric, its derivatives, and all other physical quantities depending on them [Eqs. (1)–(5)]. Finally, one has to rewrite the a_{ij} and b_{ij} terms appearing at the coefficients of all these power series as polynomials of the various moments. The recursive relations (19) and (20) relate all a_{ij} and b_{ij} with $m_k \equiv a_{0k}$ and $q_k \equiv b_{0k}$, which are directly related to the scalar moments of spacetime through Eqs. (24, 25) of Ref. [20].

The algorithm described in the previous two paragraphs has been carried out with MATHEMATICA, and has been checked for the following two subcases: (i) When all electromagnetic fields are turned off, by erasing all electromagnetic moments ($E_l = H_l = 0$), our expressions for Ω_ρ, Ω_z, $\Delta E/\mu$, ΔN are identical to the ones computed by Ryan [5]. (ii) For the Kerr-Newman metric it is quite easy to compute Ω, Ω_ρ, Ω_z, and $\Delta E/\mu$ for a quasiequatorial, quasicircular orbit. Actually, there is no need to use Weyl coordinates to describe the metric; one could simply work with the metric in the usual Boyer-Lindquist coordinates (see Eq. (33.2) of [23]), and compute everything according to the formulas given above, by replacing the derivatives with respect to z, with the corresponding derivatives with respect to θ around $\theta = \pi/2$. The expressions for ρ, though, should be replaced with $(g_{t\phi}^2 - g_{tt}g_{\phi\phi})^{1/2}$. Thus, if in the power series for Ω_ρ, Ω_z, and $\Delta E/\mu$ that are written below [cf., Eqs. (38)–(40)], one makes the following substitutions for the moments,

$$\begin{aligned} M_{2l} &= (-1)^l M a^{2l}, \qquad & S_{2l+1} &= (-1)^l M a^{2l+1}, \\ E_{2l} &= (Q/M) M_{2l}, \qquad & H_{2l+1} &= (Q/M) S_{2l+1}, \end{aligned} \tag{37}$$

according to [20], the expressions we obtain are identical to the ones obtained directly from the Kerr-Newman metric.

There is one more thing that should be pointed out before we write down the power series for all four observable quantities. As is explained in Sec. II C there are two possible cases for the electromagnetic field that lead to reflection-symmetric spacetimes. The first case (with odd A_3 and A_4 as functions of z) is the one that describes an electric field that is reflection-symmetric (like in a monopole electric field), and a magnetic field that is reflection antisymmetric (like in a magnetic dipole field). Henceforth we shall call this case the electric-symmetric case, (es). In that case only the even electric moments and the odd magnetic moments show up in the moment analysis of spacetime. Thus b_{0l} is real for even l's and purely imaginary for odd l's. The other case (with even A_3 and A_4 as functions of z) is the one that describes an electric field that is reflection antisymmetric (like in a dipole electric field), and a magnetic field that is reflection-symmetric (like in a magnetic quadrupole field). Henceforth we shall call this case the magnetic-symmetric case, (ms). In that case only the odd electric moments and the even magnetic moments show up in the moment analysis of spacetime. Thus b_{0l} is real for odd l's and purely imaginary for even l's. Although classically we do not expect the central object to carry any magnetic monopole, the zeroth-order magnetic moment shows up formally in the terms of a generic magnetic-symmetric case, and thus we have not omitted it.

The power series expansion for Ω_ρ, Ω_z, and $\Delta E/\mu$ takes the following form:

$$\frac{\Omega_\rho}{\Omega} = \sum_{n=2}^{\infty} R_n v^n, \tag{38}$$

$$\frac{\Omega_z}{\Omega} = \sum_{n=3}^{\infty} Z_n v^n, \tag{39}$$

$$\frac{\Delta E}{\mu} = \sum_{n=2}^{\infty} A_n v^n, \tag{40}$$

while the corresponding coefficients, up to ninth order for the two frequencies and up to 11th order for the radiated energy, in the two distinct electromagnetic cases (es) and (ms) are

$$R_2^{(\mathrm{es})} = \left(3 - \frac{1}{2}\frac{E^2}{M^2}\right), \qquad R_3^{(\mathrm{es})} = -4\frac{S_1}{M^2}, \qquad R_4^{(\mathrm{es})} = \frac{9}{2} - \frac{3}{2}\frac{M_2}{M^3} - 2\frac{E^2}{M^2} - \frac{13}{24}\frac{E^4}{M^4},$$

$$R_5^{(\mathrm{es})} = -10\frac{S_1}{M^2} - \frac{10}{3}\frac{S_1E^2}{M^4} + 5\frac{H_1E}{M^3},$$

$$R_6^{(\mathrm{es})} = \frac{27}{2} - 2\frac{S_1^2}{M^4} - \frac{21}{2}\frac{M_2}{M^3} - \frac{33}{4}\frac{E^2}{M^2} - \frac{1}{8}\frac{E^4}{M^4} - \frac{35}{48}\frac{E^6}{M^6} - \frac{11}{4}\frac{M_2E^2}{M^5} + 3\frac{E_2E}{M^4},$$

$$R_7^{(\mathrm{es})} = -48\frac{S_1}{M^2} - 5\frac{S_1M_2}{M^5} + 9\frac{S_3}{M^4} - \frac{5}{3}\frac{S_1E^2}{M^4} - \frac{91}{18}\frac{S_1E^4}{M^6} + 19\frac{H_1E}{M^3} + \frac{11}{2}\frac{H_1E^3}{M^5},$$

$$R_8^{(\mathrm{es})} = \frac{405}{8} + \frac{2243}{84}\frac{S_1^2}{M^4} - \frac{661}{14}\frac{M_2}{M^3} - \frac{21}{8}\frac{M_2^2}{M^6} + \frac{15}{4}\frac{M_4}{M^5} - \frac{81}{2}\frac{E^2}{M^2} + \frac{53}{16}\frac{E^4}{M^4} - \frac{19}{216}\frac{E^6}{M^6} - \frac{11339}{10368}\frac{E^8}{M^8} - \frac{251}{21}\frac{M_2E^2}{M^5} - \frac{85}{16}\frac{M_2E^4}{M^7} + \frac{75}{4}\frac{E_2E}{M^4} + \frac{35}{6}\frac{E_2E^3}{M^6} - \frac{29}{28}\frac{H_1^2}{M^4} - \frac{103}{9}\frac{S_1^2E^2}{M^6} + \frac{13}{3}\frac{H_1S_1E}{M^5},$$

$$R_9^{(\mathrm{es})} = -243\frac{S_1}{M^2} - 16\frac{S_1^3}{M^6} + 4\frac{S_1M_2}{M^5} + 45\frac{S_3}{M^4} + 54\frac{S_1E^2}{M^4} - \frac{125}{12}\frac{S_1E^4}{M^6} - \frac{35}{4}\frac{S_1E^6}{M^8} - \frac{41}{2}\frac{S_1M_2E^2}{M^7} + 8\frac{S_1E_2E}{M^6} + \frac{165}{2}\frac{H_1E}{M^3} + \frac{20}{3}\frac{H_1E^3}{M^5} + \frac{205}{24}\frac{H_1E^5}{M^7} + \frac{15}{2}\frac{H_1M_2E}{M^6} - \frac{7}{2}\frac{H_1E_2}{M^5} - \frac{21}{2}\frac{H_3E}{M^5} + \frac{25}{2}\frac{S_3E^2}{M^4},$$

$$R_2^{(\mathrm{ms})} = \left(3 - \frac{1}{2}\frac{H^2}{M^2}\right), \qquad R_3^{(\mathrm{ms})} = -4\frac{S_1}{M^2}, \qquad R_4^{(\mathrm{ms})} = \frac{9}{2} - \frac{3}{2}\frac{M_2}{M^3} - 2\frac{H^2}{M^2} - \frac{13}{24}\frac{H^4}{M^4},$$

$$R_5^{(\mathrm{ms})} = -10\frac{S_1}{M^2} - \frac{10}{3}\frac{S_1H^2}{M^4} - 5\frac{E_1H}{M^3},$$

$$R_6^{(\mathrm{ms})} = \frac{27}{2} - 2\frac{S_1^2}{M^4} - \frac{21}{2}\frac{M_2}{M^3} - \frac{33}{4}\frac{H^2}{M^2} - \frac{1}{8}\frac{H^4}{M^4} - \frac{35}{48}\frac{H^6}{M^6} - \frac{11}{4}\frac{M_2H^2}{M^5} + 3\frac{H_2H}{M^4},$$

$$R_7^{(\mathrm{ms})} = -48\frac{S_1}{M^2} - 5\frac{S_1M_2}{M^5} + 9\frac{S_3}{M^4} - \frac{5}{3}\frac{S_1H^2}{M^4} - \frac{91}{18}\frac{S_1H^4}{M^6} - 19\frac{E_1H}{M^3} - \frac{11}{2}\frac{E_1H^3}{M^5},$$

$$R_8^{(\mathrm{ms})} = \frac{405}{8} + \frac{2243}{84}\frac{S_1^2}{M^4} - \frac{661}{14}\frac{M_2}{M^3} - \frac{21}{8}\frac{M_2^2}{M^6} + \frac{15}{4}\frac{M_4}{M^5} - \frac{81}{2}\frac{H^2}{M^2} + \frac{53}{16}\frac{H^4}{M^4} + \frac{19}{216}\frac{H^6}{M^6} - \frac{11339}{10368}\frac{H^8}{M^8} - \frac{251}{21}\frac{M_2H^2}{M^5} - \frac{85}{16}\frac{M_2H^4}{M^7} + \frac{75}{4}\frac{H_2H}{M^4} + \frac{35}{6}\frac{H_2H^3}{M^6} - \frac{29}{28}\frac{E_1^2}{M^4} - \frac{103}{9}\frac{S_1^2H^2}{M^6} - \frac{13}{3}\frac{E_1S_1H}{M^5},$$

$$R_9^{(\mathrm{ms})} = -243\frac{S_1}{M^2} - 16\frac{S_1^3}{M^6} + 4\frac{S_1M_2}{M^5} + 45\frac{S_3}{M^4} + 54\frac{S_1H^2}{M^4} - \frac{125}{12}\frac{S_1H^4}{M^6} - \frac{35}{4}\frac{S_1H^6}{M^8} - \frac{41}{2}\frac{S_1M_2H^2}{M^7} + 8\frac{S_1H_2H}{M^6} - \frac{165}{2}\frac{E_1H}{M^3} - \frac{20}{3}\frac{E_1H^3}{M^5} - \frac{205}{24}\frac{E_1H^5}{M^7} - \frac{15}{2}\frac{E_1M_2H}{M^6} + \frac{7}{2}\frac{E_1H_2}{M^5} + \frac{21}{2}\frac{E_3H}{M^5} + \frac{25}{2}\frac{S_3H^2}{M^4},$$

$$Z_3^{(\mathrm{es})} = 2\frac{S_1}{M^2}, \qquad Z_4^{(\mathrm{es})} = \frac{3}{2}\frac{M_2}{M^3}, \qquad Z_5^{(\mathrm{es})} = -\frac{H_1E}{M^3} + 2\frac{S_1E^2}{M^4}, \qquad Z_6^{(\mathrm{es})} = 7\frac{S_1^2}{M^4} + 3\frac{M_2}{M^3} + \frac{5}{2}\frac{M_2E^2}{M^5} - \frac{3}{2}\frac{E_2E}{M^4} - \frac{1}{2}\frac{H_1^2}{M^4},$$

$$Z_7^{(\mathrm{es})} = 11\frac{S_1M_2}{M^5} - 6\frac{S_3}{M^4} + \frac{8}{5}\frac{S_1E^2}{M^4} + \frac{8}{3}\frac{S_1E^4}{M^6} - \frac{8}{5}\frac{H_1E}{M^3} - \frac{4}{3}\frac{H_1E^3}{M^5},$$

$$Z_8^{(\mathrm{es})} = \frac{153}{28}\frac{S_1^2}{M^4} + \frac{153}{28}\frac{M_2}{M^3} + \frac{39}{8}\frac{M_2^2}{M^6} - \frac{15}{4}\frac{M_4}{M^5} + \frac{81}{14}\frac{M_2E^2}{M^5} + \frac{25}{6}\frac{M_2E^4}{M^7} - \frac{15}{4}\frac{E_2E}{M^4} - 3\frac{E_2E^3}{M^6} + \frac{46}{3}\frac{S_1^2E^2}{M^6} - \frac{27}{28}\frac{H_1^2}{M^4} - \frac{H_1^2E^2}{M^6} - \frac{26}{3}\frac{H_1S_1E}{M^5},$$

$$Z_9^{(\mathrm{es})} = 26\frac{S_1^3}{M^6} + 31\frac{S_1M_2}{M^5} - 15\frac{S_3}{M^4} + \frac{34}{15}\frac{S_1E^2}{M^4} + \frac{8}{3}\frac{S_1E^4}{M^6} + \frac{S_1E^6}{M^8} + 32\frac{M_2S_1E^2}{M^7} - 13\frac{S_1E_2E}{M^6} - \frac{34}{15}\frac{H_1E}{M^3} - \frac{8}{3}\frac{H_1E^3}{M^5} - 2\frac{H_1E^5}{M^7} - 3\frac{H_1^2S_1}{M^6} - \frac{1}{2}\frac{H_1E_2}{M^5} - \frac{17}{2}\frac{H_1M_2E}{M^6} - 10\frac{S_3E^2}{M^6} + \frac{9}{2}\frac{H_3E}{M^5},$$

$$Z_3^{(\mathrm{ms})} = 2\frac{S_1}{M^2}, \qquad Z_4^{(\mathrm{ms})} = \frac{3}{2}\frac{M_2}{M^3}, \qquad Z_5^{(\mathrm{ms})} = \frac{E_1H}{M^3} + 2\frac{S_1H^2}{M^4}, \qquad Z_6^{(\mathrm{ms})} = 7\frac{S_1^2}{M^4} + 3\frac{M_2}{M^3} + \frac{5}{2}\frac{M_2H^2}{M^5} - \frac{3}{2}\frac{H_2H}{M^4} - \frac{1}{2}\frac{E_1^2}{M^4},$$

$$Z_7^{(\mathrm{ms})} = 11\frac{S_1M_2}{M^5} - 6\frac{S_3}{M^4} + \frac{8}{5}\frac{S_1H^2}{M^4} + \frac{8}{3}\frac{S_1H^4}{M^6} + \frac{8}{5}\frac{E_1H}{M^3} + \frac{4}{3}\frac{E_1H^3}{M^5},$$

$$Z_8^{(\mathrm{ms})} = \frac{153}{28}\frac{S_1^2}{M^4} + \frac{153}{28}\frac{M_2}{M^3} + \frac{39}{8}\frac{M_2^2}{M^6} - \frac{15}{4}\frac{M_4}{M^5} + \frac{81}{14}\frac{M_2H^2}{M^5} + \frac{25}{6}\frac{M_2H^4}{M^7} - \frac{15}{4}\frac{H_2H}{M^4} - 3\frac{H_2H^3}{M^6} + \frac{46}{3}\frac{S_1^2H^2}{M^6} - \frac{27}{28}\frac{E_1^2}{M^4} - \frac{E_1^2H^2}{M^6} + \frac{26}{3}\frac{E_1S_1H}{M^5},$$

$$Z_9^{(\mathrm{ms})} = 26\frac{S_1^3}{M^6} + 31\frac{S_1M_2}{M^5} - 15\frac{S_3}{M^4} + \frac{34}{15}\frac{S_1H^2}{M^4} + \frac{8}{3}\frac{S_1H^4}{M^6} + 4\frac{S_1H^6}{M^8} + 32\frac{M_2S_1H^2}{M^7} - 13\frac{S_1H_2H}{M^6} + \frac{34}{15}\frac{E_1H}{M^3} + \frac{8}{3}\frac{E_1H^3}{M^5} + 2\frac{E_1H^5}{M^7} - 3\frac{E_1^2S_1}{M^6} + \frac{1}{2}\frac{E_1H_2}{M^5} + \frac{17}{2}\frac{E_1M_2H}{M^6} - 10\frac{S_3H^2}{M^6} - \frac{9}{2}\frac{E_3H}{M^5},$$

$$A_2^{(\mathrm{es})} = \frac{1}{3}, \qquad A_3^{(\mathrm{es})} = 0, \qquad A_4^{(\mathrm{es})} = -\left(\frac{1}{2} - \frac{2}{9}\frac{E^2}{M^2}\right), \qquad A_5^{(\mathrm{es})} = \frac{20}{9}\frac{S_1}{M^2}, \qquad A_6^{(\mathrm{es})} = -\frac{27}{8} + \frac{M_2}{M^3} + \frac{3}{2}\frac{E^2}{M^2} + \frac{1}{3}\frac{E^4}{M^4},$$

$$A_7^{(\mathrm{es})} = \frac{28}{3}\frac{S_1}{M^2} - \frac{28}{9}\frac{H_1E}{M^3} + \frac{56}{27}\frac{S_1E^2}{M^4},$$

$$A_8^{(\mathrm{es})} = -\frac{225}{16} + \frac{80}{27}\frac{S_1^2}{M^4} + \frac{70}{9}\frac{M_2}{M^3} + \frac{20}{9}\frac{M_2E^2}{M^5} + \frac{15}{2}\frac{E^2}{M^2} + \frac{5}{9}\frac{E^4}{M^4} + \frac{140}{243}\frac{E^6}{M^6} - \frac{20}{9}\frac{E_2E}{M^4},$$

$$A_9^{(\mathrm{es})} = \frac{81}{2}\frac{S_1}{M^2} + 6\frac{S_1M_2}{M^5} - 6\frac{S_3}{M^4} + 8\frac{S_1E^2}{M^4} + 4\frac{S_1E^4}{M^6} - 17\frac{H_1E}{M^3} - 4\frac{H_1E^3}{M^5},$$

$$A_{10}^{(\mathrm{es})} = -\frac{6615}{128} + \frac{115}{18}\frac{S_1^2}{M^4} + \frac{935}{24}\frac{M_2}{M^3} + \frac{35}{12}\frac{M_2^2}{M^6} - \frac{35}{12}\frac{M_4}{M^5} + \frac{525}{16}\frac{E^2}{M^2} + \frac{70}{81}\frac{E^6}{M^6} + \frac{770}{729}\frac{E^8}{M^8} - \frac{140}{9}\frac{E_2E}{M^4} - \frac{140}{27}\frac{E_2E^3}{M^6} + \frac{370}{27}\frac{M_2E^2}{M^5} + \frac{140}{27}\frac{M_2E^4}{M^7} + \frac{700}{81}\frac{S_1^2E^2}{M^6} + \frac{11}{18}\frac{H_1^2}{M^4} - \frac{259}{27}\frac{H_1S_1E}{M^5},$$

$$A_{11}^{(\mathrm{es})} = 165\frac{S_1}{M^2} + \frac{1408}{243}\frac{S_1^3}{M^6} + \frac{968}{27}\frac{S_1M_2}{M^5} - \frac{352}{9}\frac{S_3}{M^4} + \frac{418}{27}\frac{S_1E^2}{M^4} + \frac{440}{27}\frac{S_1E^4}{M^6} + \frac{6160}{729}\frac{S_1E^6}{M^8} + \frac{616}{27}\frac{S_1M_2E^2}{M^7} - \frac{352}{27}\frac{S_1E_2E}{M^6} - \frac{88}{9}\frac{S_3E^2}{M^6} - \frac{1903}{27}\frac{H_1E}{M^3} - \frac{484}{27}\frac{H_1E^3}{M^5} - \frac{616}{81}\frac{H_1E^5}{M^7} - \frac{88}{9}\frac{H_1M_2E}{M^6} + \frac{22}{9}\frac{H_1E_2}{M^5} + \frac{22}{3}\frac{H_3E}{M^5},$$

$$A_2^{(\mathrm{ms})} = \frac{1}{3}, \qquad A_3^{(\mathrm{ms})} = 0, \qquad A_4^{(\mathrm{ms})} = -\left(\frac{1}{2} - \frac{2}{9}\frac{H^2}{M^2}\right), \qquad A_5^{(\mathrm{ms})} = \frac{20}{9}\frac{S_1}{M^2}, \qquad A_6^{(\mathrm{ms})} = -\frac{27}{8} + \frac{M_2}{M^3} + \frac{3}{2}\frac{H^2}{M^2} + \frac{1}{3}\frac{H^4}{M^4},$$

$$A_7^{(\mathrm{ms})} = \frac{28}{3}\frac{S_1}{M^2} + \frac{28}{9}\frac{HE_1}{M^3} + \frac{56}{27}\frac{S_1H^2}{M^4},$$

$$A_8^{(\mathrm{ms})} = -\frac{225}{16} + \frac{80}{27}\frac{S_1^2}{M^4} + \frac{70}{9}\frac{M_2}{M^3} + \frac{20}{9}\frac{M_2H^2}{M^5} + \frac{15}{2}\frac{H^2}{M^2} + \frac{5}{9}\frac{H^4}{M^4} + \frac{140}{243}\frac{H^6}{M^6} - \frac{20}{9}\frac{H_2H}{M^4},$$

$$A_9^{(\mathrm{ms})} = \frac{81}{2}\frac{S_1}{M^2} + 6\frac{S_1M_2}{M^5} - 6\frac{S_3}{M^4} + 8\frac{S_1H^2}{M^4} + 4\frac{S_1H^4}{M^6} + 17\frac{HE_1}{M^3} + 4\frac{E_1H^3}{M^5},$$

$$A_{10}^{(\mathrm{ms})} = -\frac{6615}{128} + \frac{115}{18}\frac{S_1^2}{M^4} + \frac{935}{24}\frac{M_2}{M^3} + \frac{35}{12}\frac{M_2^2}{M^6} - \frac{35}{12}\frac{M_4}{M^5} + \frac{525}{16}\frac{H^2}{M^2} + \frac{70}{81}\frac{H^6}{M^6} + \frac{770}{729}\frac{H^8}{M^8} - \frac{140}{9}\frac{H_2H}{M^4} - \frac{140}{27}\frac{H_2H^3}{M^6} + \frac{370}{27}\frac{M_2H^2}{M^5} + \frac{140}{27}\frac{M_2H^4}{M^7} + \frac{700}{81}\frac{S_1^2H^2}{M^6} + \frac{11}{18}\frac{E_1^2}{M^4} + \frac{259}{27}\frac{E_1S_1H}{M^5},$$

$$A_{11}^{(\mathrm{ms})} = 165\frac{S_1}{M^2} + \frac{1408}{243}\frac{S_1^3}{M^6} + \frac{968}{27}\frac{S_1M_2}{M^5} - \frac{352}{9}\frac{S_3}{M^4} + \frac{418}{27}\frac{S_1H^2}{M^4} + \frac{440}{27}\frac{S_1H^4}{M^6} + \frac{6160}{729}\frac{S_1H^6}{M^8} + \frac{616}{27}\frac{S_1M_2H^2}{M^7} - \frac{352}{27}\frac{S_1H_2H}{M^6} - \frac{88}{9}\frac{S_3H^2}{M^6} + \frac{1903}{27}\frac{E_1H}{M^3} + \frac{484}{27}\frac{E_1H^3}{M^5} + \frac{616}{81}\frac{E_1H^5}{M^7} + \frac{88}{9}\frac{E_1M_2H}{M^6} - \frac{22}{9}\frac{E_1H_2}{M^5} - \frac{22}{3}\frac{HE_3}{M^5}.$$

We note that the higher order terms we have computed are by one order lower than the corresponding higher order terms of Ryan. This is due to computing power restrictions, since as one goes to higher order terms our coefficients become far richer in moments than Ryan's. The fact that we have two new sets of moments (the electromagnetic ones) allows many more combinations of moments in high order terms. Actually, from a practical point of view these expansions are far more advanced than what will be used in gravitational-wave data analysis in the near future. On the other hand, the expressions above present an important feature: in every new order term a new moment shows up. This suggests that a very accurate observational estimation of the series could in principle reveal any moment.

A glance at the corresponding terms of the two electromagnetic cases shows that each combination of moments for the (es) case is numerically equal to the corresponding combination for the (ms) case, if the electric and magnetic moments are interchanged. The sign though is the same for combinations of pure electric or pure magnetic moments, but opposite for combinations of electric and magnetic moments.

The difference by two in the order of the highest computed order term between the power series for the Ω's and ΔE is due to the second derivatives that appear in Eq. (1).

Finally, in order to express ΔN also as a power series of v, we need to expand $dE_{\rm wave}/dt$ as a power series of v. As was explained in Sec. II A we cannot work out the perturbative analysis of gravitational-wave emission at a generic spacetime background; we can only obtain accurate expressions for $dE_{\rm wave}/dt$ up to v^4 after the leading order. However, we know the numerical factor of the higher order moment appearing at any higher than v^4 order. These higher order moments come from the power series expansion of ρ itself through Eq. (5) which describes the main contribution to energy radiation. All other contributions depend on lower moments at the same order of v. Therefore, in the following formulas for ρ and $dE_{\rm wave}/dt$, we write the power series coefficients explicitly up to the fourth order, while instead of giving the explicit form of all higher order coefficients, we give only the higher moment term that occurs at each order of the power expansion. More specifically, in order to compute the power expansion of $dE_{\rm wave}/dt$ we add up all three power series contributions of Eqs. (5)–(7). Thus, we yield

$$\rho = Mv^{-2}\left(1 + \sum_{n=2}^{\infty} \rho_n v^n\right), \tag{41}$$

and

$$\frac{dE_{\rm wave}}{dt} = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 v^{10}\left(1 + \sum_{n=2}^{\infty} W_n v^n\right), \tag{42}$$

where the ρ_n and W_n coefficients for the two electromagnetic cases are, respectively,

$$\begin{aligned}
&\rho_2^{(\rm es)} = -1 - \frac{E^2}{M^2}, \qquad \rho_3^{(\rm es)} = -\frac{2}{3}\frac{S_1}{M^2}, \qquad \rho_4^{(\rm es)} = -\frac{1}{2} - \frac{1}{2}\frac{M_2}{M^3} + \frac{1}{2}\frac{E^2}{M^2} - \frac{2}{9}\frac{E^4}{M^4},\\
&\rho_{4k+1}^{(\rm es)} = -(-1)^k\frac{2}{3}\frac{(2k-1)!!}{(2k-2)!!}\frac{H_{2k-1}E}{M^{2k+1}} + \text{LOM}, \qquad \rho_{4k+2}^{(\rm es)} = -(-1)^k\frac{(2k+2)}{3}\frac{(2k-1)!!}{(2k)!!}\frac{E_{2k}E}{M^{2k+2}} + \text{LOM},\\
&\rho_{4k+3}^{(\rm es)} = -(-1)^k\frac{2}{3}\frac{(2k+1)!!}{(2k)!!}\frac{S_{2k+1}}{M^{2k+2}} + \text{LOM}, \qquad \rho_{4k+4}^{(\rm es)} = -(-1)^k\frac{1}{3}\frac{(2k+3)!!}{(2k+2)!!}\frac{M_{2k+2}}{M^{2k+3}} + \text{LOM},
\end{aligned} \tag{43}$$

$$\begin{aligned}
&\rho_2^{(\rm ms)} = -1 - \frac{H^2}{M^2}, \qquad \rho_3^{(\rm ms)} = -\frac{2}{3}\frac{S_1}{M^2}, \qquad \rho_4^{(\rm ms)} = -\frac{1}{2} - \frac{1}{2}\frac{M_2}{M^3} + \frac{1}{2}\frac{H^2}{M^2} - \frac{2}{9}\frac{H^4}{M^4},\\
&\rho_{4k+1}^{(\rm ms)} = (-1)^k\frac{2}{3}\frac{(2k-1)!!}{(2k-2)!!}\frac{E_{2k-1}H}{M^{2k+1}} + \text{LOM}, \qquad \rho_{4k+2}^{(\rm ms)} = -(-1)^k\frac{(2k+2)}{3}\frac{(2k-1)!!}{(2k)!!}\frac{H_{2k}H}{M^{2k+2}} + \text{LOM},\\
&\rho_{4k+3}^{(\rm ms)} = -(-1)^k\frac{2}{3}\frac{(2k+1)!!}{(2k)!!}\frac{S_{2k+1}}{M^{2k+2}} + \text{LOM}, \qquad \rho_{4k+4}^{(\rm ms)} = -(-1)^k\frac{1}{3}\frac{(2k+3)!!}{(2k+2)!!}\frac{M_{2k+2}}{M^{2k+3}} + \text{LOM},
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
&W_2^{(\rm es)} = -\frac{1247}{336} - \frac{4}{3}\frac{E^2}{M^2}, \qquad W_3^{(\rm es)} = 4\pi - \frac{11}{4}\frac{S_1}{M^2}, \qquad W_4^{(\rm es)} = -\frac{44711}{9072} + \frac{1}{16}\frac{S_1^2}{M^4} - 2\frac{M_2}{M^3} + 6\frac{E^2}{M^2} - \frac{2}{9}\frac{E^4}{M^4},\\
&W_{4k+1}^{(\rm es)} = -(-1)^k\frac{8}{3}\frac{(2k-1)!!}{(2k-2)!!}\frac{H_{2k-1}E}{M^{2k+1}} + \text{LOM}, \qquad W_{2k+2}^{(\rm es)} = -4(-1)^k\frac{(2k+2)}{3}\frac{(2k-1)!!}{(2k)!!}\frac{E_{2k}E}{M^{2k+2}} + \text{LOM},\\
&W_{4k+3}^{(\rm es)} = -(-1)^k\frac{8}{3}\frac{(2k+1)!!}{(2k)!!}\frac{S_{2k+1}}{M^{2k+2}} + \text{LOM}, \qquad W_{4k+4}^{(\rm es)} = -4(-1)^{k)}\frac{1}{3}\frac{(2k+3)!!}{(2k+2)!!}\frac{M_{2k+2}}{M^{4k+5}} + \text{LOM},
\end{aligned} \tag{45}$$

$$W_2^{(\mathrm{ms})} = -\frac{1247}{336} - \frac{4}{3}\frac{H^2}{M^2}, \qquad W_3^{(\mathrm{ms})} = 4\pi - \frac{11}{4}\frac{S_1}{M^2}, \qquad W_4^{(\mathrm{ms})} = -\frac{44711}{9072} + \frac{1}{16}\frac{S_1^2}{M^4} - 2\frac{M_2}{M^3} + 6\frac{H^2}{M^2} - \frac{2}{9}\frac{H^4}{M^4},$$
$$W_{4k+1}^{(\mathrm{ms})} = (-1)^k \frac{8}{3}\frac{(2k-1)!!}{(2k-2)!!}\frac{E_{2k-1}H}{M^{2k+1}} + \mathrm{LOM}, \qquad W_{4k+2}^{(\mathrm{ms})} = -4(-1)^k \frac{(2k+2)}{3}\frac{(2k-1)!!}{(2k)!!}\frac{H_{2k}H}{M^{2k+2}} + \mathrm{LOM}, \tag{46}$$
$$W_{4k+3}^{(\mathrm{ms})} = -(-1)^k \frac{8}{3}\frac{(2k+1)!!}{(2k)!!}\frac{S_{2k+1}}{M^{2k+2}} + \mathrm{LOM}, \qquad W_{4k+4}^{(\mathrm{ms})} = -4(-1)^k \frac{1}{3}\frac{(2k+3)!!}{(2k+2)!!}\frac{M_{2k+2}}{M^{2k+3}} + \mathrm{LOM}.$$

In the expressions above the indices k run from one to infinity, and the term LOM is an abbreviation for lower order moments that appear at a specific order in the expansion. We note that in all these coefficients the same feature with respect to the corresponding terms in the two electromagnetic cases that was mentioned before arises.

By combining the power series for $\Delta E/\mu$ [Eq. (40)] with the one for dE_{wave}/dt [Eq. (42)] we obtain the power series expansion of ΔN:

$$\Delta N = \frac{5}{96\pi}\left(\frac{M}{\mu}\right)v^{-5}\left(1 + \sum_{n=2}^{\infty} N_n v^n\right), \tag{47}$$

where the N_n coefficients for the two electromagnetic cases are given by the following polynomials of the moments,

$$N_2^{(\mathrm{es})} = \frac{743}{336} + \frac{14}{3}\frac{E^2}{M^2}, \qquad N_3^{(\mathrm{es})} = -4\pi + \frac{113}{12}\frac{S_1}{M^2}, \qquad N_4^{(\mathrm{es})} = \frac{3058673}{1016064} - \frac{1}{16}\frac{S_1^2}{M^4} + 5\frac{M_2}{M^3} + \frac{12431}{504}\frac{E^2}{M^2} + \frac{179}{9}\frac{E^4}{M^4},$$
$$N_{4k+1}^{(\mathrm{es})} = (-1)^k \frac{(16k+20)}{3}\frac{(2k-1)!!}{(2k-2)!!}\frac{H_{2k-1}E}{M^{2k+1}} + \mathrm{LOM}, \qquad N_{4k+2}^{(\mathrm{es})} = (-1)^k \frac{(8k+6)(2k+2)}{3}\frac{(2k-1)!!}{(2k)!!}\frac{E_{2k}E}{M^{2k+2}} + \mathrm{LOM},$$
$$N_{4k+3}^{(\mathrm{es})} = (-1)^k \frac{(16k+28)}{3}\frac{(2k+1)!!}{(2k)!!}\frac{S_{2k+1}}{M^{2k+2}} + \mathrm{LOM}, \qquad N_{4k+4}^{(\mathrm{es})} = (-1)^k \frac{(8k+10)}{3}\frac{(2k+3)!!}{(2k+2)!!}\frac{M_{2k+2}}{M^{2k+3}} + \mathrm{LOM}, \tag{48}$$

$$N_2^{(\mathrm{ms})} = \frac{743}{336} + \frac{14}{3}\frac{H^2}{M^2}, \qquad N_3^{(\mathrm{ms})} = -4\pi + \frac{113}{12}\frac{S_1}{M^2}, \qquad N_4^{(\mathrm{ms})} = \frac{3058673}{1016064} - \frac{1}{16}\frac{S_1^2}{M^4} + 5\frac{M_2}{M^3} + \frac{12431}{504}\frac{H^2}{M^2} + \frac{179}{9}\frac{H^4}{M^4},$$
$$N_{4k+1}^{(\mathrm{ms})} = -(-1)^k \frac{(16k+20)}{3}\frac{(2k-1)!!}{(2k-2)!!}\frac{E_{2k-1}H}{M^{2k+1}} + \mathrm{LOM}, \qquad N_{4k+2}^{(\mathrm{ms})} = (-1)^k \frac{(8k+6)(2k+2)}{3}\frac{(2k-1)!!}{(2k)!!}\frac{H_{2k}H}{M^{2k+2}} + \mathrm{LOM},$$
$$N_{4k+3}^{(\mathrm{ms})} = (-1)^k \frac{(16k+28)}{3}\frac{(2k+1)!!}{(2k)!!}\frac{S_{2k+1}}{M^{2k+2}} + \mathrm{LOM}, \qquad N_{4k+4}^{(\mathrm{ms})} = (-1)^k \frac{(8k+10)}{3}\frac{(2k+3)!!}{(2k+2)!!}\frac{M_{2k+2}}{M^{2k+3}} + \mathrm{LOM}. \tag{49}$$

As in the other three measurable quantities, the power expansion of ΔN is such that at every order term a new moment, which was not present at any lower order term, occurs. This proves that all moments can in principle be unambiguously extracted from accurate measurements of ΔN.

IV. USING THE RESULTS IN GRAVITATIONAL-WAVE ANALYSIS

Although we have not quantitatively explored the implications of our results on the estimation of errors in determining the various moments from a gravitational-wave data analysis, as it has been done by Ryan in [6], we could make some general comments. Actually, the only difference of our results from the ones of [5] is that more moments are showing up at each coefficient in the power expansions of all observable quantities, and thus Ryan's estimates for each term apply equally well here.

As is shown in [6], the first generation of LIGO is not expected to be able to extract the first two moments (S_1 and M_2) with high accuracy (~ 0.05 for the former and ~ 0.5 for the latter one), by analyzing the phase of the waves. If we allow for electromagnetic fields as well, the corresponding monopole (which classically is expected to be very close to zero) will be measured with even higher accuracy than the other two mass moments, since the charge of the source (or the magnetic monopole in case of some exotic body) is present at even lower order, namely, in the v^2 term, while the electric dipole, or the magnetic dipole, that first shows up at the v^5 term will be measured with rather disappointing accuracy. On the other hand, analyzing the data of LISA leads to accuracies almost 2 orders of magnitude higher than the corresponding data for LIGO. Thus, it seems quite promising that LISA will give us the opportunity to measure the first few moments, including the electromagnetic dipole moments, quite accurately. Also, the fact that at every new order, in the power series of ΔN, a new moment appears is significant, since this means that in principle a unique set of moments arises from an accurate estimation of all power series terms. Actually, there are two possible sets of moments, one for each electromagnetic case, since we cannot *a priori* exclude one of them. We can only exclude one of the two sets on physical grounds, if only one of them leads to a physically reasonable classical object (for example, a highly magnetized compact object is physically preferable to a compact object with a huge electric dipole). If we manage to measure a few lower moments, we can check if they are interrelated as in a Kerr-Newman metric [20]. A positive outcome of such a test will be of support to the black-hole no-hair theorem in the case that the central object is a black hole. The case of observational violation of the black-hole no-hair theorem could either mean that the central object is not a black hole, or that the theorem does not hold. Of course to assume the latter an extra verification that the central object is indeed a black hole is necessary. Indications that the central massive compact object is not a black hole would imply the existence of an exotic object (e.g., soliton star, naked singularity, etc.). If the central body's mass is measured to be within the stellar limits (e.g., a massive neutron star) we could get valuable information about its electromagnetic field, like its magnetic dipole field.

While the phase of a gravitational wave is the quantity that can be most accurately measured, since a large number of cycles (a few thousand for LIGO and a few hundred thousand for LISA in case of binaries with high ratio of masses) is sweeping up the sensitive part of the detectors, the two precession frequencies Ω_ρ and Ω_z can in principle be measured if the detectors become more sensitive and templates that describe modulating waves are used [24]. If this ever becomes possible one could use any of them to test the no-hair theorem. This would demand no more than the four lower order terms, since according to this theorem all moments depend on only three quantities (mass, angular momentum, and total charge). Actually, from measurements of modulating frequencies we could not at first determine which frequency corresponds to each precession. However, the power expansions of the two frequencies begin at a different order, and thus we could discern them. Unfortunately, there are terms in these expansions that contain more than one first-occurring moment (for example, the first term of Ω_ρ, R_2, is a function of E or H and M). However, expansions of Ω_ρ and Ω_z, if used simultaneously, along with the intrinsic dependence of v on M, could finally lead to the full determination of the moments. Notice though that some corresponding terms in the series expansions of the two frequencies depend on the same set of multipole moments, like R_3 and Z_3. Assuming that the observed system is sufficiently well described by our model for the binary, this multipole information could serve to determine these moments with better precision.

The exploitation of our analysis presented above is mainly focused on measuring the various physical quantities through gravitational-wave analysis. Therefore, it is related mainly to binaries with massive black holes as a central object. Actually, our analysis applies to any binary system with one of the two bodies being much more massive than the other. For example, one could read the multipole moments of a compact object, like a neutron star, from the precession frequencies of a small object orbiting around the first one, or from the way the latter one loses its energy. However, such observations are rather hopeless, through gravitational-wave analysis, at least for the near future, since the strength of gravitational waves of low-mass bodies orbiting around neutron stars is rather prohibitive for corresponding observations. On the other hand, if we manage to obtain information about the precessing frequencies or the evolution of the orbits of such systems by other observational means, we could measure the multipole moments (mass moments, mass-current moments, and electromagnetic moments) of the compact central object and thus put strict restrictions on the models that are used to describe the interior of these objects.

Our analysis demonstrates that it will be possible to determine all types of multipole moments of the central object, from future gravitational-wave measurements. Thus, apart from spacetime geometry, we could also determine the central body's electromagnetic fields. Although the data of LISA should be suitable for extracting such information with high accuracy, the assumptions of circular and equatorial orbit are not that realistic. From this point of view we consider our work as a step towards a more detailed analysis with not so restrictive assumptions.

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