Hodograph: A useful geometrical tool for solving some difficult problems in dynamics

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The hodograph is very useful for solving complicated problems in dynamics. By simple geometrical arguments students can directly obtain the answer to problems that would otherwise be complicated exercises in algebra. Although beyond the level of undergraduates, we also use the hodograph to calculate by variational geometrical techniques, the well-known brachistochrone curve, thus illustrating this approach.
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I. INTRODUCTION

In 1847, Hamilton[1] invented the hodograph and used it to solve the famous problem that was first attacked with success by Newton; namely, to deduce the law of gravitation that makes the planets revolve in elliptical orbits around the Sun. Later, Maxwell[2] used the hodograph to introduce a short variation of Hamilton's solution. Feynman resurrected the geometrical technique of the hodograph to present an elementary way to solve Newton's problem.[3] Sadly the hodograph has almost entirely disappeared from most modern treatments of mechanics with the present emphasis being on analytical methods for deriving the orbits for the inverse square law. Also, the background of contemporary students in Euclidean geometry is not as strong as that of students a hundred years ago.

What is a hodograph? If the instantaneous velocity vectors of a moving particle are translated to the same initial point, the tip of the velocity vector traces a curve known as the hodograph. What makes the hodograph a useful tool is the fact that although Newton's law of motion is a second-order differential equation for the displacement

$$\frac{d^2\mathbf{r}}{dt^2}=\mathbf{F}/m, \tag{1}$$

it is a first-order differential equation for the velocity

$$\frac{d\mathbf{u}}{dt}=\mathbf{F}/m, \tag{2}$$

and thus the dynamics are more clearly reflected by the hodographic curve than by the trajectory itself (see Fig. 1).

The hodograph that corresponds to planetary motion can be shown to be a circle, and this property can be used to prove that the orbit follows from the inverse square force law (see, for example, Ref. 2). In this paper, we will use the hodograph for a much simpler force: the one produced by a uniform field, such as gravity near the Earth's surface. Because $d\mathbf{u}/dt=\mathbf{g}=\text{constant}$, the velocity vector of a particle (for example, a projectile) that is moving freely in the uniform gravitational field traces a vertical line on which it moves at a constant rate. In velocity space, the initial and final states of the projectile can be pictured as a triangle, one side of which is vertical and linearly proportional to the time of flight, while the other two sides have magnitudes equal to the initial and final velocities, with these velocities related to each other by energy conservation. This triangle is the main tool that will be used in Sec. II to demonstrate the beauty and power of the hodograph technique. In Sec. III we use the hodograph to solve the brachistochrone problem. Although we will use the classical variational technique to solve the problem, the use of the hodograph will make the analysis far easier than the usual analysis, and the cycloidal curve will arise much more naturally.

II. THE HODOGRAPH OF A PROJECTILE

As was outlined in Sec. I, the tip of the velocity of a particle that is moving in a uniform force field traces a straight line parallel to the direction of the field. Without loss of generality, we will assume that the field is that of uniform gravity near the Earth's surface. The equation of motion, $d\mathbf{u}/dt=\mathbf{g}$, where $\mathbf{u}$ is the velocity of the particle and $\mathbf{g}$ is the constant acceleration of free fall, can easily be integrated to produce the well-known relation between velocities at different times for uniform acceleration:

$$\mathbf{u}_2=\mathbf{u}_1+\mathbf{g}(t_2-t_1). \tag{3}$$

We now discuss the physical content hidden in the geometrical picture formed by the initial and final velocity vectors (see, for example, the triangle of Fig. 2). The height of the triangle ($u_0 \cos\theta$) perpendicular to its vertical side is easily identified as the constant horizontal velocity (in general, the velocity component that is perpendicular to the direction of the field). The vertical side, $g(t_2-t_1)$, of the triangle is proportional to the time of flight. Therefore, the area, $\frac{1}{2}g(t_2-t_1)u_0\cos\theta$, of this triangle is proportional to the horizontal distance, $u_0\cos\theta(t_2-t_1)$, traveled by the particle during its flight. This is the main idea that will be further explored in Sec. II A.

A. The maximum range of a projectile

First consider a particle thrown with initial velocity u_0 from the edge of a cliff that is a height H above a plane valley. We want to determine the shot angle θ that will make the projectile land the farthest from the cliff. According to our previous discussion, we must maximize the area of the triangle ABC of Fig. 2 to maximize the horizontal distance. Note that the two sides of this triangle have fixed magnitudes; u_0 is the initial speed, and u_f is the final landing velocity at the valley, which by conservation of energy is $u_f=\sqrt{u_0^2+2gH}$. This triangle assumes its maximum area

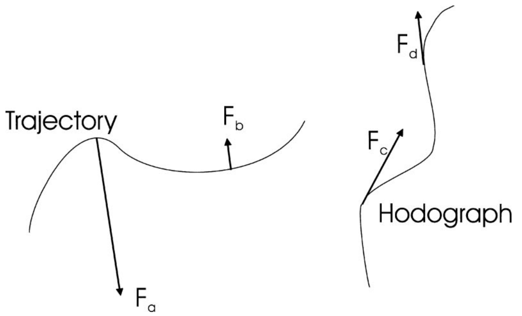

Fig. 1. While the force vector points along the instantaneous center of curvature of a particle's trajectory, it is simply tangential to the hodographic curve. Therefore, the connection between the hodograph and the force law is much more direct.

when the two sides are perpendicular to each other. Then, from the properties of similar triangles, it is easy to see that θ is equal to the angle $\hat{C}$, that is,

$$\theta = \tan^{-1}\left(\frac{u_0}{\sqrt{u_0^2 + 2gH}}\right). \tag{4}$$

Equation (4) is valid even for negative values of H (such as when a particle is thrown from the bottom of a well) as long as the trajectory does not intersect the topography of the ground. The maximum horizontal range achieved when a projectile is shot at this angle is then easy to calculate. The area of the velocity triangle $\mathcal{E}_{ABC}$ is $g/2$ times the horizontal distance. Thus

$$R_{\max} = \frac{2\mathcal{E}_{ABC}}{g} = \frac{u_0\sqrt{u_0^2 + 2gH}}{g}. \tag{5}$$

For $H = 0$ we obtain the well-known value u_0^2/g that corresponds to $\theta = 45°$ inclination angle [see Eq. (4)].

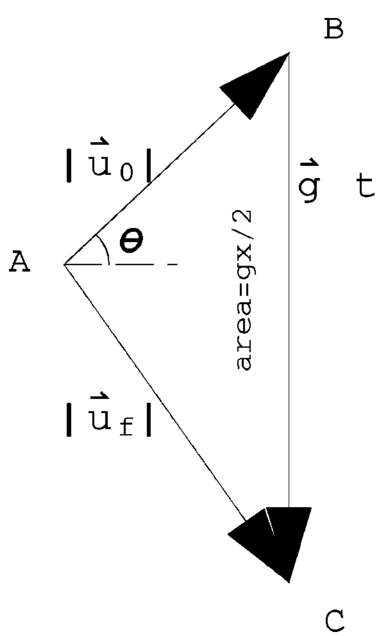

Fig. 2. This triangle is formed by the initial and final velocities of flight. The vertical side of triangle ABC is the hodograph of motion, while the area of the triangle is proportional to the horizontal distance traveled. The angle θ is the initial shot angle that has to be optimized.

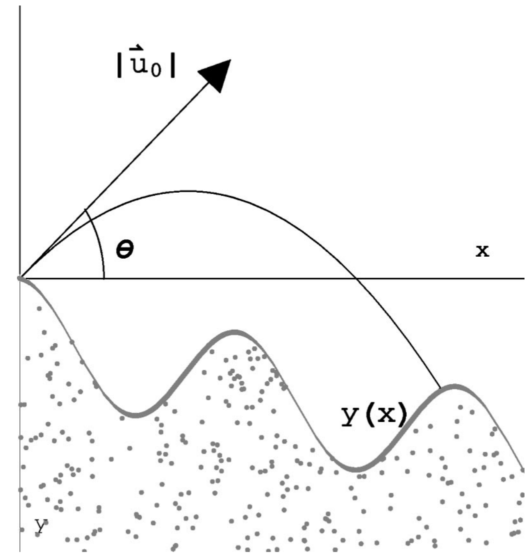

Fig. 3. A projectile shot at angle θ from the top of a hill, the topography of which is described by the function $y(x)$.

We next investigate a more general projectile problem. We will assume that the particle is shot from a point on the ground, the topography of which is described by the monotonic function $y(x)$, where x is the horizontal distance and y is directed down (see Fig. 3). Initially the particle is located at $(x=0, y=0)$. A trajectory that ends on the ground at $(x_f, y_f = y(x_f))$ corresponds to a velocity triangle, the two sides of which have magnitudes $AB = u_0$, and $AC = \sqrt{u_0^2 + 2gy_f}$, respectively (see Fig. 4). If we keep the angle ϕ between initial and final velocity fixed, then the intersection of the curves

$$f_1(y) = \sin\phi \frac{u_0\sqrt{u_0^2 + 2gy}}{g} \tag{6}$$

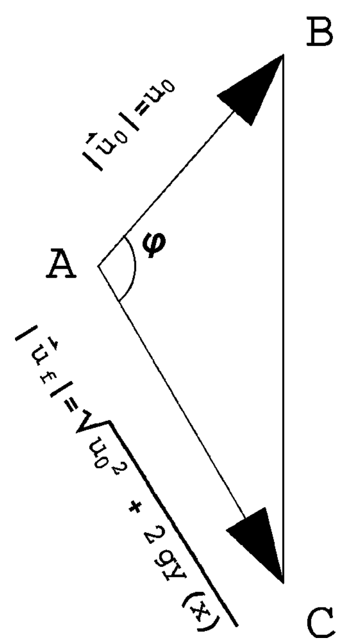

Fig. 4. The velocity triangle for a projectile that is shot from the top of a hill.

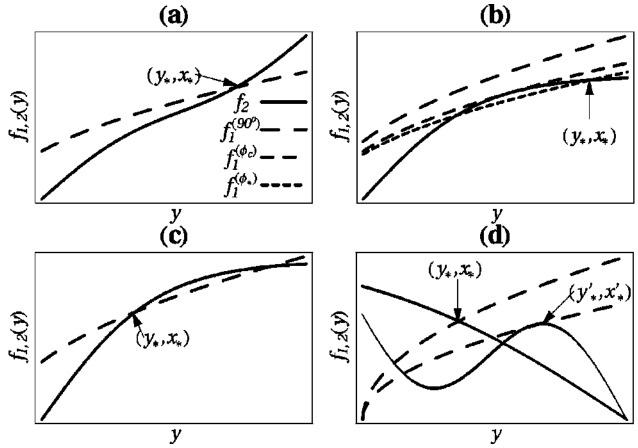

Fig. 5. The graphical solution of the maximum range for the three cases (a, b, and c) are shown in the corresponding diagrams (a), (b), and (c). In case (b), the solution is not the one that corresponds to the critical $\sin\phi = s_c$ value, but rather, as discussed in the main text, the one with even lower $\sin\phi$ value ($s_\star$). Diagram (d) shows the graphical solution when the projectile is shot from the deepest point. The black lines correspond to the simple case of continuously rising ground, while the gray lines correspond to more complicated ground morphologies, for example, with cavity formations.

($2/g$ times the area of the velocity triangle), and

$$f_2(y) = x(y) \tag{7}$$

(the inverse of $y(x)$ of the ground morphology), gives the coordinates of the ground point (if it exists) where the particle will fall, given the angle between the initial and the final velocity of the trajectory. It is easy to see why this is so. Because $f_1(y)$ equals the horizontal distance traveled by the particle (as was explained in the previous section), the intersection point of the curves f_1 and f_2 corresponds to the point where the projectile trajectory meets the ground.

We will examine the three possible cases that exhaust all possible downhill ground morphologies for which the initial point is the highest one: (a) The graph of f_2 has only one intersection point with f_1 regardless of the value of ϕ [see Fig. 5(a)]; (b) $f_2(y)$ does not increase as fast as $f_1(y)$; consequently, there is no intersection point for $\sin\phi > s_c$ [see Fig. 5(b)]; (c) there are multiple solutions of the equation $f_1(y) = f_2(y)$ for a range of ϕ, including the extreme case $\phi = 90°$ [see Fig. 5(c)].

In case (a) we can easily find the maximum horizontal distance, because $f_1(y)$ and $f_2(y)$ are both monotonically increasing functions, and there is only one intersection point; the higher the value of $\sin\phi$, the higher the value of x at the intersection point. Therefore, we achieve the longest range when $\phi = 90°$, just as in the simple case analyzed previously The optimal inclination angle θ can be estimated, given the graphical solution for $\phi = 90°$:

$$\theta = \tan^{-1}\left(\frac{u_0}{\sqrt{u_0^2 + 2gy_\star}}\right), \tag{8}$$

where $y_\star$ is the y coordinate of the graphical solution. We can simplify the calculations that follow by introducing the characteristic length α:

$$\alpha \equiv \frac{u_0^2}{2g}. \tag{9}$$

Then, we can express the optimal value of θ that is given in Eq. (8) as

$$\tan\theta = \frac{1}{\sqrt{1 + y_\star/\alpha}}. \tag{10}$$

Also, because the graphical solution assumes that $f_1^{(90°)}(y) = f_2(y)$ (where the superscript $^{(90°)}$ refers to the angle between the initial and final velocities), that is, $u_0\sqrt{u_0^2 + 2gy_\star}/g = x_\star$,

$$\sqrt{1 + \frac{y_\star}{\alpha}} = \frac{x_\star}{2\alpha}, \tag{11}$$

and thus $\tan\theta = 2\alpha/x_\star$, which can be transformed after some algebra to

$$\tan 2\theta = x_\star / y_\star. \tag{12}$$

But $x_\star/y_\star$ is the cotangent of the inclination angle of the line connecting the initial and final points. This form of the solution gives a straightforward answer to the case where the ground has constant slope b: the particle should be shot at an angle $\theta = 45° - \omega/2$.

In case (b) there is no graphical solution for $\sin\phi > s_c$, and the solution that corresponds to $\sin\phi = s_c$ [where the two graphs $f_1(y)$ and $f_2(y)$ are tangent to each other] seems to give the required answer. However, this assumption is not true. The graph of $f_1(y)$ for slightly lower values of $\sin\phi$ intersects the graph of $f_2(y)$ at even higher x. The apparently paradoxical fact that the plot of $f_1(y)$ goes "under" the ground [there is a region of y where $f_1(y) < f_2(y)$] does not mean that the trajectory penetrates the ground. Remember that $f_1(y)$ does not describe an actual trajectory, because $(f_1(y), y)$ is the locus of points of different trajectories, where the angle between the initial and final velocities is held fixed, while for an actual trajectory this angle varies from point to point. As long as the angle ϕ_2 (the angle between the final velocity and the horizon) is larger than the inclination of the ground, the intersection of the plots of f_1 and f_2 corresponds to arrival at this point from above the ground, while if the angle ϕ_2 is smaller than the inclination of the ground, the intersection point corresponds to arrival from beneath the ground, and thus, it has no physical meaning. For example, at the critical value of ϕ ($\sin\phi_c = s_c$), where the two curves just touch each other, it is easy to show (see Ref. 4) that the slope of the ground is not equal to ϕ_2. If $\phi_c < \pi/2$, then the ground at the intersection point is steeper than the trajectory of the projectile, while, if $\phi_c > \pi/2$, then the trajectory is steeper. Therefore, this graphical solution point cannot be reached with the oblique angle solution of $\sin\phi_c = s_c$, but it can be reached with the corresponding obtuse angle solution. In the latter case, even higher values of ϕ will yield even longer range. The longest range will be obtained at an obtuse angle $\phi = \phi_\star$ [see Fig. 5(b)], that at the intersection point the slope of the ground equals the inclination of the final velocity, ϕ_2.

Finally, in case (c), where there are multiple solutions for $\phi = 90°$, the longest range $x_\star$ that the projectile can reach is the first (the one with the smallest $y_\star$ value) graphical solu-

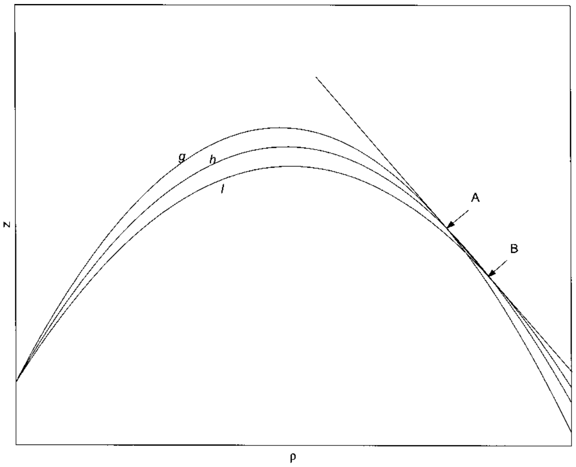

Fig. 6. Three consecutive parabolas (g, h, and l) that correspond to trajectories with the same initial velocity, but with slightly different shot angles. Intersection points A and B of g and h, and h and l, respectively, define a line that leaves all three parabolas at the same half-plane. The locus of all such intersection points between neighboring parabolas forms the boundary of the safe region.

tion of $f_1^{(90°)}(y)=f_2(y)$, because the projectile cannot go any further than the curve $f_1^{(90°)}(y)$. Thus, no other graphical solution can be achieved, because otherwise the projectile would penetrate the ground [see Fig. 5(c)]. Again, the optimal inclination angle can be calculated exactly as in case (a).

For completeness, we describe the case where the initial point is now the lowest point on the ground. Again, we can resort to the graphical solution by extending the graph $f_1(y)$ to $y<0$. Depending on the ground morphology, we can find the maximum $x_\star$ by trying various values of ϕ. For ground that is constantly uphill, it is obvious that the maximum range will be accomplished for $\phi=90°$ (see the black curves in Fig. 5(d)). If there are cavity-like formations on the ground, the maximum horizontal range (deep in the cavity) can be achieved at $\phi<90°$ [see the gray curves in Fig. 5(d)].

B. The safe region around an explosion

We now apply the geometrical method to solve another projectile problem. For a given maximum initial velocity u_0 of the fragments of an explosion, how far from the explosion should one stand to be safe? The classical way to solve such a problem is by determining, for a given z value (vertical distance above the explosion), the maximum horizontal distance from the explosion, ρ, so that the initial and final point can be connected by a parabolic projectile orbit with initial velocity u_0. Although, such a solution is neither difficult nor especially lengthy, it is not direct, because, we should first optimize the angle for a given ρ and then use that angle as a function of ρ in the orbital equation to yield the safe region $z(\rho)$. What we are really trying to find is the envelope of all free fall parabolas, emanating from the location where the explosion takes place, with velocity u_0 at various angles. The envelope of all such parabolas with a common initial point is the locus of all intersection points between coplanar parabolas with slightly different initial inclination angles (see Fig. 6). The argument goes as follows: Imagine three such successive parabolas g, h, and l. The points A and B, where g and h, and h and l intersect each other, respectively, define a line that leaves all three parabolas at the same half-plane. The intersection of all such half-planes formed by consecutive parabolas is the region of points that can be reached by some fragment. Next, we will calculate the locus of all such intersection points at a given vertical plane, and then we will obtain the entire safe region by revolving this envelope around the vertical axis that passes through the explosion point.

Let us draw the velocity triangle for two projectile trajectories, slightly different with respect to the initial inclination angle. The intersection point of these two parabolas will have the same (ρ,z) coordinates, and thus both triangles will have equal velocity sides and equal corresponding areas. But two triangles with the same pair of sides and different opening angles between the corresponding pairs have the same area only if these opening angles are supplementary. On the other hand, because the values of ϕ of the two triangles are infinitesimally different (see Fig. 4), the triangles will be almost orthogonal. This observation leads naturally to the desired envelope shape; that is, the locus of the points with coordinates ($\rho=2/g$ times the area of each triangle, z). Because the triangles are orthogonal

$$\rho=\frac{1}{g}u_0\sqrt{u_0^2-2gz}, \tag{13}$$

and thus the envelope is given by

$$z=\frac{u_0^2}{2g}-\frac{g}{2u_0^2}\rho^2=\alpha-\frac{\rho^2}{4\alpha}. \tag{14}$$

If we identify ρ with the distance from the vertical axis z, and use the axial symmetry of the problem, we recognize in Eq. (14) the analytical description of the parabolic safe "cup" that surrounds the explosion. The extra information we directly obtain with this method is that at each point of the envelope, the corresponding parabolic orbit that passes from this point has an initial direction that is perpendicular to the tangent of the envelope at that point.

III. CALCULATING THE BRACHISTOCHRONE CURVE VIA THE HODOGRAPH TECHNIQUE

Next, we use the hodograph to calculate the curve that connects two points at the same height on which a body that slides freely under a uniform gravitational field goes from one point to the other in the shortest time. Instead of directly trying to find the curve that minimizes the time, we instead determine the hodograph curve that corresponds to minimum travel time. We first point out that there is a one-to-one correspondence between a real curve in space and the corresponding hodograph, because at each point (x,y) of the curve, the corresponding velocity is $\sqrt{-2gy}$, which is tangential to the curve (see Fig. 7). Thus, if we determine the hodograph, we can go back and calculate the actual curve.

The body is assumed to start from rest, and by conservation of energy, it will end its trip with zero velocity. Therefore, the hodograph will be a closed curve described by the polar graph $u(\theta)$, where θ is the inclination angle of the instantaneous velocity with respect to the horizontal line. For $\theta=0$, the velocity has its maximum value $\sqrt{-2gy_{\min}}$ (if the curve has a unique minimum). If we consider an infinitesimal part ds of the curve where the body passes by with velocity $u(\theta)$, the Cartesian components of velocity are

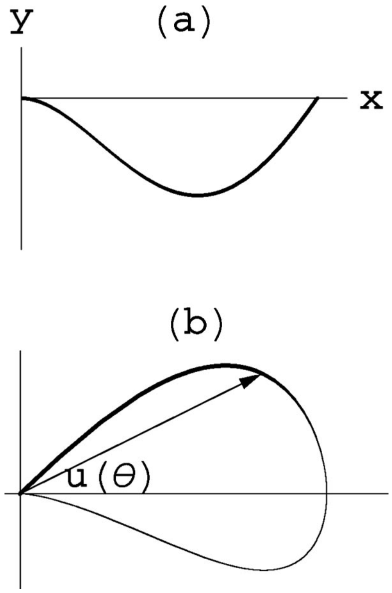

Fig. 7. For each curve that connects the initial and final points, there is a distinct hodograph, because at every point of the curve, the velocity of the sliding body is determined; the magnitude of the velocity is derived from the principle of conservation of energy, while its direction is tangential to the curve at that point.

$$u \cos \theta = \frac{dx}{dt}, \tag{15a}$$

$$u \sin \theta = \frac{dy}{dt}. \tag{15b}$$

On the other hand, from conservation of energy

$$y = -\frac{u^2}{2g}. \tag{16}$$

Thus,

$$dt = -\frac{du}{g \sin \theta}, \tag{17}$$

and

$$dx = -\frac{u du}{g} \cot \theta. \tag{18}$$

Hence, we are looking for that closed curve $u(\theta)$ that minimizes $\int dt$, while $\int dx$ is held fixed. By introducing a Lagrange multiplier λ, we have to minimize the following functional:

$$A[\theta(u);\lambda] = -\int \frac{du}{g \sin \theta} - \lambda \left[-\int \frac{u du}{g} \cot \theta - L \right], \tag{19}$$

where L is the horizontal distance between the departure and arrival points, and the integration is calculated along the closed hodograph curve. Assume now that we deform the space curve slightly so that the slope of the curve at a fixed y is not θ, but $\theta + \epsilon \eta$, where ϵ is a very small positive number, and η is an arbitrary continuous function of position that is zero at the end points. Minimizing[5] the functional A means that we are trying to find a function $\theta(y)$, or equivalently $\theta(u)$,[6] such that the above deformation results in a second-order variation of A with respect to ϵ. The constraint of fixed horizontal distance between the end points is provided by

$$\frac{dA}{d\lambda} = 0. \tag{20}$$

The required hodograph curve, then, arises by expanding A with respect to ϵ and setting the first-order term equal to zero. That is,

$$\int \eta \frac{du}{g \sin^2 \theta} (\cos \theta - \lambda u) = 0. \tag{21}$$

To make this expression hold for any function η, it is clear that the rest of the integrand must be identically zero. Thus, $u = (1/\lambda) \cos \theta$, with $\theta \in [-\pi/2, \pi/2]$.[7] If we introduce this function in Eq. (20), we obtain the value of λ:

$$\lambda = \sqrt{\frac{\pi}{2gL}}. \tag{22}$$

Thus, the desired hodographic curve is

$$u(\theta) = \sqrt{\frac{2gL}{\pi}} \cos \theta. \tag{23}$$

Equation (23) is the polar equation of a circle, with the origin at the left-most point of the circle. It is not difficult to find the shape of the actual curve that leads to such a hodograph. If we had shifted this circular hodograph horizontally by one radius (by subtracting a horizontal velocity equal to the radius of the circle), we would obtain the characteristic hodograph of uniform circular motion (a circular hodograph with the origin at its center). Thus the motion that minimizes the travel time is uniform circular motion plus a rectilinear uniform motion with a velocity equal to that of circular motion. This path is a cycloidal curve, that is, the path traced out by a point on a vertical circular ring that rolls on a horizontal surface. The minimum travel time can be calculated easily once the hodograph is known. We integrate Eq. (17) to obtain

$$t_{\text{tot}} = \sqrt{\frac{2\pi L}{g}}. \tag{24}$$

This time could also be calculated by dividing the distance L traveled by the circular ring by its rolling velocity, that is, by the radius r of the circular hodograph $(r = 1/2\lambda = \sqrt{gL/2\pi})$.

IV. SUGGESTED PROBLEM

Consider an electron beam consisting of electrons that move with initial velocity u_0. A system of two charged parallel plates is used to deflect the beam. The uniform electric field between the two plates is $\mathcal{E}$ and the distance between the two plates is d. Use the hodograph technique to find the angle at which the plates should be placed with respect to the initial electron beam so that the beam deflection is a maximum. Consider both cases: (a) the beam passes from one side of the plates to the other side, (b) the beam is reflected back to the side that is initially moving.

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[1]William Rowan Hamilton, ''The Hodograph, or a new method of expressing in symbolical language the Newtonian law of attraction,'' Proc. R. Ir. Acad. **3**, 344–353 (1847).

[2]James Clerk Maxwell, *Matter and Motion* (Dover, New York, 1991).

[3]David and Judith Goodstein, *Feynman's Lost Lecture* (W. W. Norton, New York, 1996).

[4]If $\sin\phi = s_c$, so that the two graphs just touch each other [see Fig. 5(b)], the slope of the ground at the intersection point is equal to the inverse of the slope of f_1, because $[dy_{\text{ground}}/dx]_0 = [df_2/dy]_0^{-1} = [df_1/dy]_0^{-1} = [(d/dy)s_c 2a\sqrt{1+y/a}]_0^{-1} = (\sqrt{1+y_0/a})/s_c$. If we call ϕ_1 and ϕ_2 the angles formed by the initial and final velocities, respectively, with the horizon we have that $\phi_1+\phi_2=\phi_c$, and $\cos\phi_1=\sqrt{1+y_0/a}\cos\phi_2$, where the last relation comes from the geometry of Fig. 4. With a little algebra we obtain $\tan\phi_2=\sqrt{(1+y_0/a)}/s_c-\cot\phi_c$. Therefore if ϕ_c is the oblique angle that satisfies $\sin\phi_c=s_c$, the slope of the projectile trajectory is smaller than the slope of the ground at the landing point, which is impossible (because this would mean that the projectile arrives at this point from inside the ground). Thus, only the corresponding obtuse-angle solution can lead the projectile to this point. The same reasoning holds for values of ϕ such that $\sin\phi$ is slightly less than s_c. Both intersection points then can be reached by the obtuse-angle solution, and the nearest point can be reached by the oblique-angle solution only if $dy_{\text{ground}}/dx \leqslant \tan\phi_2$.

[5]The stationary point we calculated here corresponds to a minimum, because an infinitely deep well would lead to infinite time travel.

[6]Because all expressions are written as functions of u, we try to optimize the inverse function of $u(\theta)$; namely, $\theta(u)$.

[7]The end points of θ are $\theta_{\text{in}}=-\pi/2$, and $\theta_{\text{fin}}=\pi/2$, because these values lead to zero velocities when substituted in $u=(1/\lambda)\cos\theta$, the hodographic curve.

THE CROCKER CRACKER

Even before the sixty-inch design—dubbed the ''Crocker Cracker''—was completed, Lawrence was thinking of one ''ten times greater,'' a truly huge cyclotron for nuclear physics or, as Isidor Isaac Rabi of Columbia called it, ''the beam to end all beams.'' For Lawrence, money, not technology, was the chief obstacle. In a radio broadcast, he announced he was considering constructing a cyclotron ''to weigh 2,000 tons and to produce 100 million-volt particles . . . It would require more than half a million dollars.'' With the active encouragement of Loomis and other big-thinking admirers, it would increase steadily in size and cost over the next year. ''He was building a cyclotron as big as money would permit him,'' said Loomis, adding that ''we got up to 210 inches'' before it was finally cut back to 184 inches. ''The idea would go up and up. He did very courageous things. Most people would not want to make such big calculations, but he was so confident.''

Jennett Conant, *Tuxedo Park* (Simon & Schuster, New York, NY, 2002), p. 139.