On the 2nd Kahn-Kalai conjecture and Bayesian inference connections

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University of Athens seminar

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- **Erdős-Rényi model on** n **vertices,** $p = p_n$**:** $\mathbf{G} \sim G(n, p)$

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Huge Literature...

A natural approach

First moment method

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- H K_4+ *antenna*: $\mathbb{E}\mathcal{Z}_H \sim n^5p^7, p_1 \sim 1/n^{5/7}$ **but...** $p_c(H) \sim 1/n^{2/3}$.

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Issue: $H' = K_4$ is a subgraph of $H=K_4+$ *antenna.*

$$\mathbb{P}_p(\mathcal{Z}_H \geq 1) \leq \mathbb{P}_p(\mathcal{Z}_{H'} \geq 1) \leq \mathbb{E}\mathcal{Z}_{H'}.$$

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Refined first moment threshold

Using subgraphs H' of H,

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Perfect matching, Hamiltonian cycle, $p_E(H) \sim 1/n, p_c(H) \sim \log n/n$ *(beyond 2nd MM)*.

A bit more literature

Any non-trivial monotone property $\mathcal{F}$ on random set $\mathbf{V} \sim \mathrm{Bern}(p)^{\otimes N}$.

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- (Park, Pham '22): Proof of the Kahn-Kalai conjecture!

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(3) **New proof of spread lemma** using *Bayesian inference tools.*

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- log-**necessary:** Perfect matching $p_{\tilde{E}}(H) \sim 1/n$, $p_c(H) \sim \log n/n$.

The spread lemma and proof

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Spread measure

If $G_1, \ldots, G_M$ are *subgraphs of* K_n, the *uniform measure* $\mathbf{A} \sim \pi$ on G_i is R-**spread** if *for all* $S \subseteq K_n$ $\pi(S \subseteq \mathbf{A}) \leq R^{-e(S)}$.

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