Asymptotic Theory for Estimator of the Hüsler-Reiss Distribution via Block Maxima Method

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Richard Smith

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Motivating Example

- Trying to measure extreme meteorological events (ex. temperature or rainfall) over some spatial area.
- Goal is to model the annual maxima at multiple sites, capturing dependence structure.
- We assume these underlying processes are Gaussian.

(Max) Domains of Attraction

- $X_1,...,X_m \sim F$, i.i.d.
- $M_m := \max\{X_1, ..., X_m\}$. F is in the (max) domain of attraction of H iff, for $a_m > 0, b_m$ as $m \to \infty$:

$$\mathbb{P}\left(\frac{M_m - b_m}{a_m} \leq x\right) = F^m(a_m x + b_m) \to H(x).$$

- In the univariate case, H must be a Generalized Extreme Value Distribution.

$$H(x) = \begin{cases} \exp\{-(1 + \zeta\frac{x-\mu}{\psi})_+^{-1/\zeta}\} & \zeta \neq 0 \\ \exp\{-\exp\{-\frac{x-\mu}{\psi}\}\} & \zeta = 0 \end{cases}$$

Max-Stable Processes

- Extension of max-stability from finite dimensional distributions to processes.
- A process Y is considered max-stable iff for any finite set of points $\{x_i\}_{i=1}^{d}$ in the domain of this process, the law of $m\{Z(x_1), ..., Z(x_d)\}$ is the same as $\{\max_{j\in[m]}(Z_j(x_1)), ..., \max_{j\in[m]}(Z(x_d))\}$.
- General construction: $\{R_i\}$ are points of a Poisson process on $\mathbb{R}^+$ with intensity $s^{-2}ds$. Let W^+ be a strictly positive process with expectation 1 at the origin.

$$\bigvee_{i=1}^{\infty} R_i W_i^+(t) \tag{1}$$

Brown-Resnick Process

- A class of max-stable models
- Let U_i be points of a Poisson point process with intensity $e^{-y}dy$, and $W_i(t)$ is some stationary Gaussian process with variance $\sigma^2(t)$,

$$\bigvee_{i=1}^{\infty} U_i + W_i(t) - \sigma(t)^2/2 \tag{2}$$

- For any two points, the joint law of a Brown-Resnick process is Hüsler-Reiss.

Hüsler-Reiss Distribution

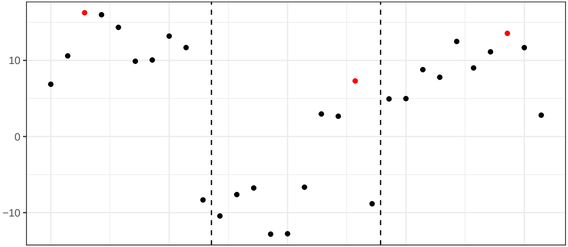
- (X_i, Y_i) distributed by 0-mean, unit-variance bivariate normal distribution with correlation parameter ρ_m.
- b_m by the relationship $b_m = m\phi(b_m)$, where ϕ is the standard normal density function.
- The Hüsler-Reiss Distribution arises as the limiting distribution of $b_m \max\{(X_1, Y_1), ..., (X_m, Y_m)\} - b_m^2$

$$H_\lambda(x, y) := \exp\left\{-e^{-x}\Phi(\lambda + \tfrac{y-x}{2\lambda}) - e^{-y}\Phi(\lambda + \tfrac{x-y}{2\lambda})\right\}$$

- When correlation is constant, H_λ is a product of independent Gumbel distributions ($\lambda = \infty$)[5]
- The single parameter arises from the limit $\ln(m)(1 - \rho_m) \to \lambda^2$

Block Maxima Method

- Given a sample $x_1, ..., x_n$ how do we estimate the distribution of the maxima of these random variables?
  - Split sample of size n into k groups of size m.
  - Take the maximum for each of the k blocks.
  - Proceed with a sample of k maxima.
- We will use Maximum Likelihood Estimator



Problem Set-Up

- $(X_i, Y_i) \sim \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & \rho_m\\ \rho_m & 1\end{pmatrix}\right)$
- We want to estimate the limiting parameter λ of the Hüsler-Reiss distribution
- Treat the scaled maximum of each block as if it were exactly distributed according to a Hüsler-Reiss distribution, performing maximum likelihood estimation with the *misspecified* Hüsler-Reiss density.

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- **How do we decide the values of m and k for optimal inference?**

Main Result

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Let (X_i, Y_i) be i.i.d. standard bivariate normal random variables with correlation ρ_m such that $\ln(m)(1-\rho_m) \to \lambda_0^2 \in (0, \infty)$. Define $\hat{\lambda}_n$ as the maximum likelihood estimator using the misspecified likelihood. Where k_n, the number of blocks in the block maxima method, and m_n, the block size, both converge to infinity with sample size, we have

$$\sqrt{k_n}\left(\lambda_0 - \hat{\lambda}_n\right) \xrightarrow{d} N\left(I_\lambda^{-1} A, I_\lambda^{-1}\right)$$

where I_λ is the Fisher Information Matrix of the Hüsler-Reiss distribution and A is a depends on k_n, m_n and λ_0.

Preliminary Results

- Denote $h_\lambda(x, y)$ and $\phi_{\rho_m}(x, y)$ to be the relevant densities. $\phi_{\rho_m}(x, y) \to h_\lambda(x, y)$.

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- $\mathbb{E}_{H_\lambda}\left(\frac{\partial^2}{\partial^2 \lambda} L(X, Y, \lambda_0)\right) := -I_\lambda$ is finite, negative for all values of $\lambda_0 \in (0, \infty)$.

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  - Dominated Convergence Theorem

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  - Split quantity into 3 terms:
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- A may be infinite, depending on balance between k and m.
  - Most terms converge at a rate of $\frac{1}{b_m^2} \approx \frac{1}{2\ln(m)}$
  - Other terms depend on the convergence rate of $\lambda_0 - b_m\sqrt{\frac{1-\rho_m}{1+\rho_m}}$

Convergence of Maximizers

- By breaking up the above equation we can define the two random processes

$$M_n(h) = \tilde{L}(h) - \tilde{L}(0)$$
$$M(h) = h\mathcal{N}(A, I_\lambda) - \frac{1}{2}h^2 I_\lambda$$

- Argmax Theorem is used to show the maximizer of M_n converges in distribution to the maximizer of M

Argmax Corollary [6]

Suppose $M_n \xrightarrow{d} M$ in $l^\infty(K)$ for every compact subset K of $\mathbb{R}^k$ and M has continuous sample paths that have unique maxima $\hat{h}$. If the domain of M_n converges to that of M, $M_n(\hat{h}) \geq M_n(H_n) - op(1)$, and $\hat{h}_n$ is uniformly tight, then $\hat{h}_n \xrightarrow{d} \hat{h}$.

Interpreting the Result

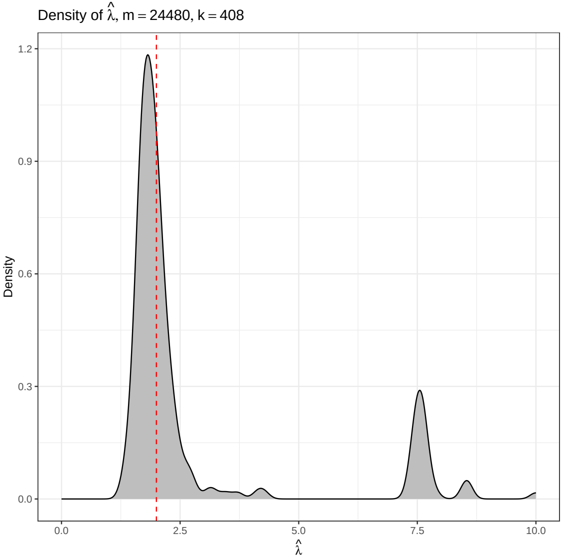
- Since convergence to the Hüsler-Reiss distribution is only asymptotic, do we introduce bias by assuming our sample from a Hüsler-Reiss distribution?
  - Yes, but it can be managed.
  - Non-trivial bias exists, but can be handled either by picking a suboptimal bias-variance trade-off, or estimating the theoretical bias.
- How do we decide the values of m and k for optimal inference?
  - Bias term $A := \lim_{n\to\infty} \sqrt{k_n}\mathbb{E}_{\phi_m}\left(\frac{\partial}{\partial\lambda}L(X, Y, \lambda_0)\right)$ shows the bias-variance trade-off.
  - The optimal value of k_n depends on the behavior of ρ_m, but is no greater than $C\ln(n)$.

Finding a value for A

- $A := \lim_{n \to \infty} \sqrt{k_n} \mathbb{E}_{\phi_m} \left(\frac{\partial}{\partial \lambda} L(X, Y, \lambda_0) \right)$
- Since $\mathbb{E}_{\phi_m}(\frac{\partial}{\partial \lambda} L(X, Y, \lambda)) \to \mathbb{E}_{H_\lambda}(\frac{\partial}{\partial \lambda} L(X, Y, \lambda)) = 0$, we can use a Taylor Series expansion for ϕ_m, cancelling any terms which are non-zero in the limit.

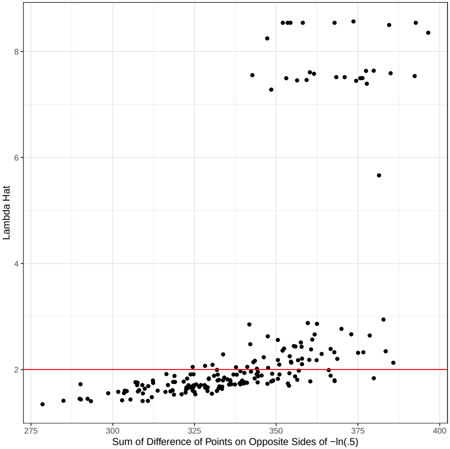
$$\Phi_{\rho_m}\left(u_m(x), u_m(y)\right)^{m-2} \left[\frac{m-1}{m} e^{-y-x-\frac{y^2+x^2}{2b_m^2}} \Phi\left(\frac{u_m(x) - \rho_m u_m(y)}{\sqrt{1-\rho_m^2}} \right) \Phi\left(\frac{u_m(y) - \rho_m u_m(x)}{\sqrt{1-\rho_m^2}} \right) \right.$$
$$\left. + \Phi_{\rho_m}\left(u_m(x), u_m(y)\right) e^{-x} e^{-\frac{x^2}{2b_m^2}} \phi\left(\frac{u_m(y) - \rho_m u_m(x)}{\sqrt{1-\rho_m^2}} \right) \frac{1}{b_m \sqrt{1-\rho_m^2}} \right]$$

Simulation Results



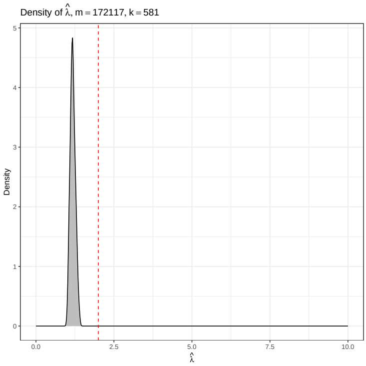
Explaining the Second Mode

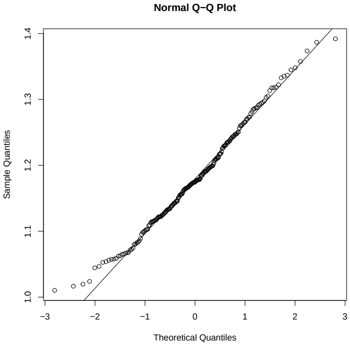
- Taking the limit of log-likelihood as $\lambda \to \infty$ has the same sign as $e^{-x}(1 - e^{-y}) + e^{-y}(1 - e^{-x}) - \frac{1}{2}$
  - Equivalent to $x, y < -\ln(.5)$ or $x, y > -\ln(.5)$.



Second Simulation

- ρ increased from .668 to .865





1. Empirical variance: .006464. Theoretical variance: .006061

Limitations

- Restricted to bivariate sites
  - Hüsler-Reiss distribution has multivariate extensions
- Correlation depending on m seems artificial

Composite Likelihood

- Given a family $\{f(\psi, y); y \in \mathcal{Y} \subset \mathbb{R}^k, \psi \in \Psi \subset \mathbb{R}^q\}$ and a set of events $\{\mathcal{I}_k : k \in K \subset \mathbb{N}\}$, and non-negative weights w_i,

$$l_C(\psi, y) := \sum_{k \in K} w_k f(\psi, \{y_i : y_i \in I_k\}) \quad (3)$$

- Let $\{\mathcal{I}_k\}$ be every bivariate observation from our data $\{x_i\}_1^n \in \mathbb{R}^J$

$$l_C(x, \psi) = \frac{2}{J(J-1)} \sum_{i=1}^{n} \sum_{j=1}^{J-1} \sum_{h=j+1}^{J} f((x_{n,j}, x_{n,h}), \psi) \quad (4)$$

- Use composite bivariate composite likelihood to estimate Brown-Resnick Processes

Spatial Scaling

- From [3]:

$$\eta_m(t) = \bigvee_{i=1}^{m} b_m(X(s_m * t) - b_m) \tag{5}$$

- Spatial scaling allows us to control ρ_m.
- Paired with infill-statistics to ensure our samples are sufficiently dense.

Future Work

- Combine composite likelihood and spatial scaling to estimate Brown-Resnick processes
  - Extending the misspecification from finite dimensional distributions to processes
- Increase number of sites to higher dimensions
- New distributions

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