High-dimensional latent Gaussian count time series

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Outline

1. Motivation for (vector-valued) count time series models
2. Model
3. Concentration inequalities for autocovariance matrix estimates
4. Sparse estimation for latent VAR processes
5. Conclusions

Examples of correlated counts observed at different time points:

- Number of patients with different but related symptoms

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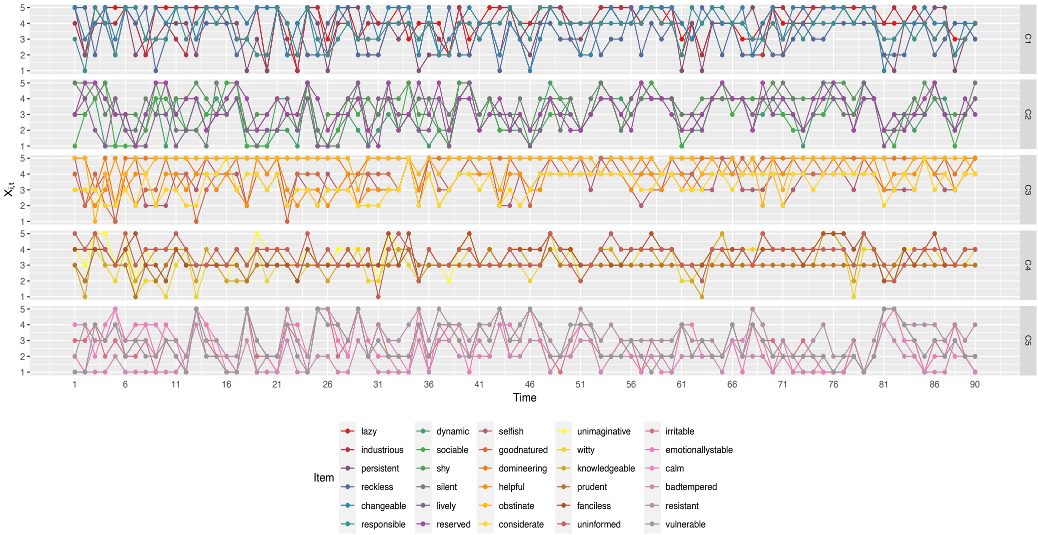
Literature:

- (Multivariate) INAR, INGARCH

Survey: Discussion of (dis)advantages for several classes of count time series; see Davis et al. (2021).


C1
C2
C3
C4
C5
$X_{i,t}$
Time
1
6
11
16
21
26
31
36
41
46
51
56
61
66
71
76
81
86
90
5
4
3
2
1
Item
lazy
industrious
persistent
reckless
changeable
responsible
dynamic
sociable
shy
silent
lively
reserved
selfish
goodnatured
domineering
helpful
obstinate
considerate
unimaginative
witty
knowledgeable
prudent
fanciless
uninformed
irritable
emotionallystable
calm
badtempered
resistant
vulnerable

Model Copula transformation and latent Gaussian variable

- Latent: $Z_t = (Z_{1,t}, \ldots, Z_{d,t})'$, $t \in \mathbb{Z}$, is a d-dimensional stationary Gaussian series with zero mean and unit variance: $\mathrm{E}[Z_{i,t}] = 0, \mathrm{E}[Z_{i,t}^2] = 1$.

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- Observed: $X_t = (X_{1,t}, \dots, X_{d,t})'$, $t \in \mathbb{Z}$, is a d-dimensional stationary count time series ($X_{i,t} \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$) with marginal CDF $F_i(x) = \mathrm{P}[X_{i,t} \leq x]$.

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- We model $\{X_t\}$ as

$$X_t = (X_{1,t}, \dots, X_{d,t})' = (G_1(Z_{1,t}), \dots, G_d(Z_{d,t}))' = G(Z_t),$$

with

$$G_i(z_i) = F_i^{-1}(\Phi(z_i)), \quad G(z) = (G_1(z_1), \dots, G_d(z_d))', \quad z \in \mathbb{R}^d.$$

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- By construction $\{X_t\}$ has marginal CDF F_i. F_i depends on a parameter vector $\theta_i \in \mathbb{R}^{K_i}$.

Example with $d = 1$ and Bernoulli marginals. $X_t \sim \text{Bernoulli}(p)$, $Z_t = \phi Z_{t-1} + \varepsilon_t$.

$$G(z) = F^{-1}(\Phi(z)) = \begin{cases} 1, & z \geq \Phi^{-1}(1-p), \\ 0, & z < \Phi^{-1}(1-p). \end{cases}$$

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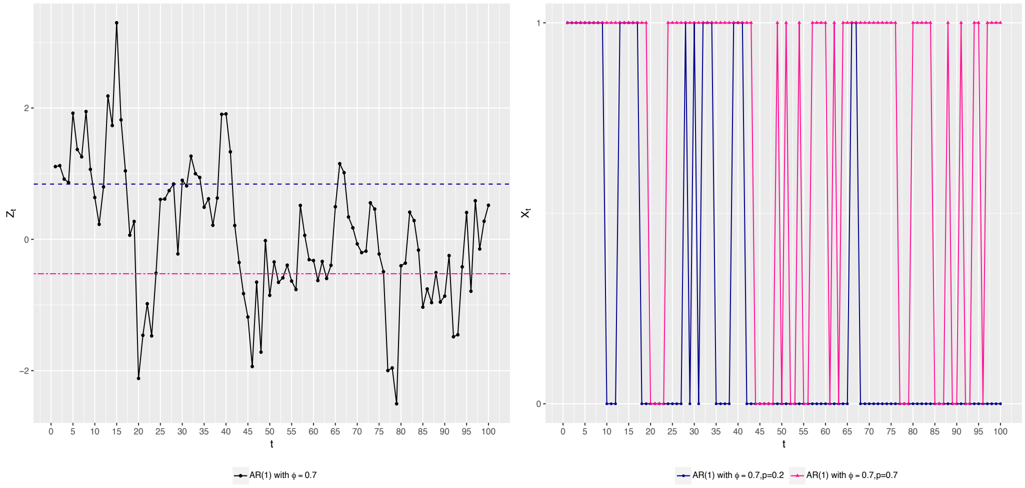

Model Autocovariance matrices and their relationships

Recall that $G_i(z_i) = F_i^{-1}(\Phi(z_i))$.

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- Expand G_i using Hermite polynomials

$$G_i(z) = \sum_{k=0}^{\infty} \frac{c_{i,k}}{k!} H_k(z), \quad c_{i,k} = \mathrm{E}(G_i(Z_{i,0})H_k(Z_{i,0}));$$

with the kth Hermite polynomial defined as

$$H_k(z) = (-1)^k e^{\frac{z^2}{2}} \frac{\partial^k}{\partial z^k} e^{\frac{-z^2}{2}}.$$

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Examples: $H_0(z) = 1, H_1(z) = z, H_2(z) = z^2 - 1, H_3(z) = z^3 - 3z, H_4(z) = z^4 - 6z^2 + 3, \dots$

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- Jia et al. (2021):

$$c_{i,k} = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{-Q_{i,n}^2/2} H_{k-1}(Q_{i,n})$$

with $Q_{i,n} = \Phi^{-1}(C_{i,n})$ and $C_{i,n} = \mathrm{P}[X_{i,t} \leq n]$. $C_{i,n}$ depends on parameter vector $\theta_i \in \mathbb{R}^{K_i}$.

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- The autocovariances of $\{Z_t\}$ (latent) can be associated to $\{X_t\}$ (observed) through

$$\Gamma_X(h) := \mathrm{E}[X_{t+h}X_t'] - \mathrm{E}[X_{t+h}]\,\mathrm{E}[X_t'] = \left(\sum_{k=1}^{\infty} \frac{c_{i,k}c_{j,k}}{k!} R_{Z,ij}(h)^k \right)_{i,j=1,\dots,d}, R_Z(h) = \mathrm{E}[Z_{t+h}Z_t'].$$

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- More compactly

$$\Gamma_X(h) = \ell(\Gamma_Z(h)), \;\; \ell(u) = (\ell_{ij}(u))_{i,j=1,\dots,d} \quad \text{with} \quad \ell_{ij}(u) = \sum_{k=1}^{\infty} \frac{c_{i,k}c_{j,k}}{k!} u^k.$$

- $c_{i,k}$ depends on $\theta_i \rightarrow \ell_{ij}$ depends on θ_i, θ_j.

Example

Example with $d = 2$ and Bernoulli marginals.

- $Z_t = (Z_{1,t}, Z_{2,t})'$, $t \in \mathbb{Z}$, stationary Gaussian series with $\mathrm{E}[Z_{i,t}] = 0, \mathrm{E}[Z_{i,t}^2] = 1$ and lag-h autocovariance matrix

$$\Gamma_Z(h) = \begin{pmatrix} \rho_{1,1}(h) & \rho_{1,2}(h) \\ \rho_{2,1}(h) & \rho_{2,2}(h) \end{pmatrix}.$$

- $X_{i,t} \sim \text{Bernoulli}(p)$.
- Then,

$$G_i(z) = F_i^{-1}(\Phi(z)) = \begin{cases} 1, & z \geq \Phi^{-1}(1-p), \\ 0, & z < \Phi^{-1}(1-p). \end{cases}$$

Suppose $p = \frac{1}{2}$ such that $\Phi^{-1}(1-p) = 0$

- Then,

$$\Gamma_X(h) = \frac{1}{2\pi} \begin{pmatrix} \arcsin(\rho_{1,1}(h)) & \arcsin(\rho_{1,2}(h)) \\ \arcsin(\rho_{2,1}(h)) & \arcsin(\rho_{2,2}(h)) \end{pmatrix}.$$

Recall that $\ell(u) = (\ell_{ij}(u))_{i,j=1,\dots,d}$ with $\ell_{ij}(u) = \sum_{k=1}^{\infty} \frac{c_{i,k}c_{j,k}}{k!} u^k$.

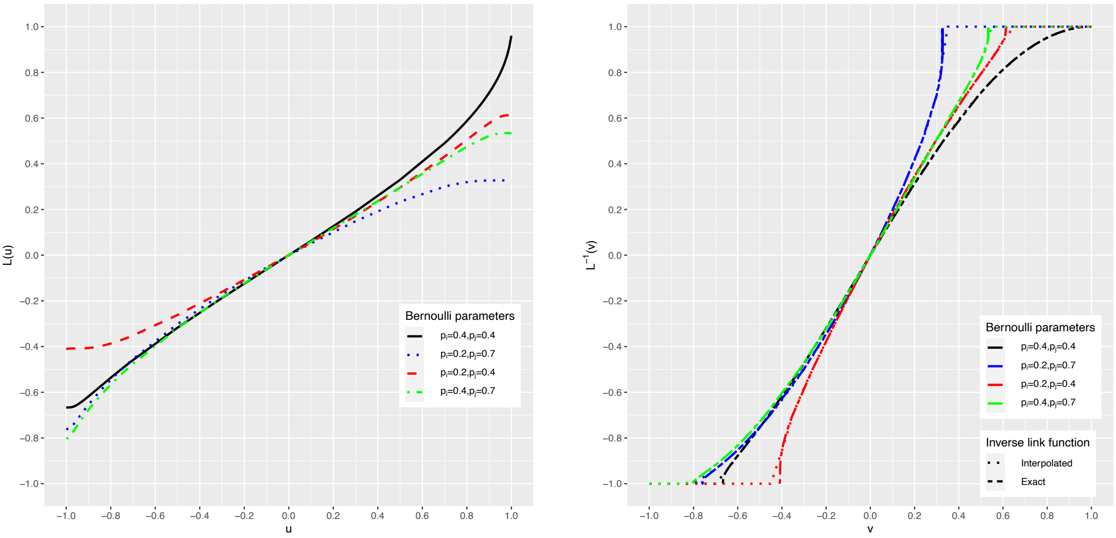
Proposition (Jia et al. (2021))

For $u \in (-1,1)$, the link function ℓ satisfies

$$\ell'_{ij}(u) = \frac{1}{2\pi\sqrt{1-u^2}} \sum_{n_0,n_1=0}^{\infty} \exp\left(-\frac{1}{2(1-u^2)}(Q^2_{i,n_0} + Q^2_{j,n_1} - 2uQ_{i,n_0}Q_{j,n_1})\right)$$

with $Q_{i,n} = \Phi^{-1}(C_{i,n})$ and $C_{i,n} = \mathrm{P}[X_{i,t} \leq n]$.

There is no explicit representation of ℓ.



L(u)
u
Bernoulli parameters
p_i=0.4,p_j=0.4
p_i=0.2,p_j=0.7
p_i=0.2,p_j=0.4
p_i=0.4,p_j=0.7
$L^{-1}(v)$
v
Bernoulli parameters
p_i=0.4,p_j=0.4
p_i=0.2,p_j=0.7
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Inverse link function
Interpolated
Exact

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- Establish concentration bounds for the differences between the estimated and true latent Gaussian autocovariances, in terms of those for the observed count series and the estimated marginal parameters. (Main result I)
- Impose parametric model on latent process (e.g. VAR).
- Utilize Main result I to find high probability bound on sparse estimators for transition matrices of VAR. (Main result II)

Literature

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Our contributions: Theoretical guarantees for consistent estimation of a latent parametric model in a possibly high-dimensional regime.

Recall that $\ell(u) = (\ell_{ij}(u))_{i,j=1,\dots,d}$ with $\ell_{ij}(u) = \sum_{k=1}^{\infty} \frac{c_{i,k} c_{j,k}}{k!} u^k$.

- The function ℓ_{ij} depends on the marginal CDF parameters θ_i, θ_j only, so that for all lags h,

$$\Gamma_X(h) = \ell(\Gamma_Z(h)).$$

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- If $\widehat{\theta}_i$ is an estimator of θ_i and $\widehat{\ell}$ one of ℓ, one estimator of $\widehat{\Gamma}_Z$ is

$$\widehat{\Gamma}_Z(h) = \widehat{\ell}^{-1}(\widehat{\Gamma}_X(h)),$$

where $\widehat{\Gamma}_X(h)$ is a standard ACVF estimator of $\Gamma_X(h)$ based on $X_1, \ldots, X_T$.

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  where $\widehat{\Gamma}_X(h)$ is a standard ACVF estimator of $\Gamma_X(h)$ based on $X_1, \dots, X_T$.
- Assuming that the observations have a zero mean, the autocovariance matrices $\mathbf{\Gamma}_X = (\Gamma_X(r-s))_{r,s=1,\dots,L}$ can be estimated as $\widehat{\mathbf{\Gamma}}_X = \frac{1}{N}\mathcal{X}_X'\mathcal{X}_X$ with $N = T - L$ and

$$\mathcal{X}_X = \begin{pmatrix} X_L' & \dots & X_1' \\ \vdots & \ddots & \vdots \\ X_{T-1}' & \dots & X_{T-L}' \end{pmatrix}.$$

  With a slight abuse of notation, we write both $\mathbf{\Gamma}_X = \ell(\mathbf{\Gamma}_Z)$ and $\Gamma_X(h) = \ell(\Gamma_Z(h))$.

Main result I

Recall that $\widehat{\Gamma}_Z(h) = \widehat{\ell}^{-1}(\widehat{\Gamma}_X(h))$ and set $\mathbf{\Gamma}_X = (\Gamma_X(r-s))_{r,s=1,\dots,L}$

Proposition (D., Lund, Pipiras)

Under mild moment conditions *on* $\{X_t\}$, *we have, for* $\delta, \varepsilon > 0$,

$$\mathrm{P}\left[\|\widehat{\mathbf{\Gamma}}_Z - \mathbf{\Gamma}_Z\|_s > Q(\mathbf{\Gamma}_Z)\delta\right] \lesssim \mathrm{P}[\|\widehat{\mathbf{\Gamma}}_X - \mathbf{\Gamma}_X\|_s > \delta] + \mathrm{P}[\|\widehat{\mathbf{\Gamma}}_X - \mathbf{\Gamma}_X\|_s^2 > \delta] + \mathrm{P}[\|\widehat{\theta} - \theta\|_{\max} > \delta \wedge \varepsilon] + \mathrm{P}[\|\widehat{\theta} - \theta\|_{\max}^2 > \delta]$$

with $Q(\mathbf{\Gamma}_Z) := Q(\mathbf{\Gamma}_Z, \varepsilon, \delta)$.

The constant $Q(\mathbf{\Gamma}_Z)$ depends on

$$\mu_i^{(k)}(u) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \exp\left(-\frac{1}{2u} Q_{i,n}^2\right) |Q_{i,n}|^k \left(\|\nabla_{\theta_i} Q_{i,n}\|_1\right)^b, \quad b \in \{0,1\}.$$

Sparse estimation for latent VAR processes

$$Z_t = \sum_{u=1}^{p} \Phi_u Z_{t-u} + \varepsilon_t, \quad t \in \mathbb{Z},$$

for some $\Phi_u \in \mathbb{R}^{d \times d}$ and white noise series $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ characterized by

$$\mathrm{E}[\varepsilon_t] = 0, \quad \mathrm{E}[\varepsilon_t \varepsilon_t'] = \Sigma_\varepsilon, \quad \mathrm{E}[\varepsilon_s \varepsilon_t'] = 0 \quad \text{for } s \neq t.$$

The VAR(p) model can be written in a linear models form as

$$\begin{pmatrix} Z'_{p+1} \\ \vdots \\ Z'_T \end{pmatrix} = \begin{pmatrix} Z'_p & \dots & Z'_1 \\ \vdots & \ddots & \vdots \\ Z'_{T-1} & \dots & Z'_{T-p} \end{pmatrix} \begin{pmatrix} \Phi'_1 \\ \vdots \\ \Phi'_p \end{pmatrix} + \begin{pmatrix} \varepsilon'_{p+1} \\ \vdots \\ \varepsilon'_T \end{pmatrix} \quad \text{or} \quad \mathcal{Y}_Z = \mathcal{X}_Z B_0 + \mathcal{E}.$$

A vectorized version:

$$\begin{aligned} \mathrm{vec}(\mathcal{Y}_Z) &= \mathrm{vec}(\mathcal{X}_Z B_0) + \mathrm{vec}(\mathcal{E}) \\ &= (I_d \otimes \mathcal{X}_Z)\,\mathrm{vec}(B_0) + \mathrm{vec}(\mathcal{E}), \\ Y &= Z\beta_0 + E, \end{aligned}$$

Estimation

Recall that $\text{vec}(\mathcal{Y}_Z) = \text{vec}(\mathcal{X}_Z B_0) + \text{vec}(\mathcal{E})$ and $Y = Z\beta_0 + E$

The transition matrices can be estimated through

$$\widehat{\beta} = \underset{\beta \in \mathbb{R}^q}{\arg\min} \Big(-2\beta'\widehat{\gamma} + \beta'\widehat{\Gamma}\beta + \lambda_N \|\beta\|_1 \Big).$$

- observed Basu and Michailidis (2015)

$$\widehat{\gamma} = \text{vec}(\widehat{\gamma}_Z) = \text{vec}(\mathcal{X}_Z' \mathcal{Y}_Z),$$
$$\widehat{\Gamma} = I_d \otimes \widehat{\mathbf{\Gamma}}_Z = I_d \otimes \mathcal{X}_Z' \mathcal{X}_Z / N.$$

- unobserved D., Lund, Pipiras (2023)

$$\widehat{\gamma} = \text{vec}(\widehat{\gamma}_Z) = \text{vec}(\widehat{\ell}^{-1}(\widehat{\gamma}_X)),$$
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General path to consistency

Observed Basu and Michailidis (2015)

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Restricted Eigenvalue: A symmetric matrix $\widehat{\Gamma} \in \mathbb{R}^{q \times q}$ satisfies the restricted eigenvalue condition with curvature $\alpha > 0$ and tolerance $\tau > 0$ if

$$x'\widehat{\Gamma}x \geq \alpha \|x\|^2 - \tau \|x\|_1^2 \quad \text{for all} \quad x \in \mathbb{R}^q.$$

We write $\widehat{\Gamma} \sim RE(\alpha, \tau)$ for short.

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We write $\widehat{\Gamma} \sim RE(\alpha, \tau)$ for short.
Deviation bound: There exists a deterministic function $Q(\beta_0)$ such that

$$\|\widehat{\gamma} - \widehat{\Gamma}\beta_0\|_{\max} \leq Q(\beta_0)\sqrt{\frac{\log(q)}{N}}, \quad N = T - p, \quad q = d^2 p.$$

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Restricted Eigenvalue: A symmetric matrix $\widehat{\Gamma} \in \mathbb{R}^{q \times q}$ satisfies the restricted eigenvalue condition with curvature $\alpha > 0$ and tolerance $\tau > 0$ if

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We write $\widehat{\Gamma} \sim RE(\alpha, \tau)$ for short.
Deviation bound: There exists a deterministic function $Q(\beta_0)$ such that

$$\|\widehat{\gamma} - \widehat{\Gamma}\beta_0\|_{\max} \leq Q(\beta_0)\sqrt{\frac{\log(q)}{N}}, \quad N = T - p, \quad q = d^2 p.$$

Both properties can be reduced to finding bounds on:

$$\mathrm{P}[|v'(\widehat{\mathbf{\Gamma}}_Z - \mathbf{\Gamma}_Z)v| > \delta]$$

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Unobserved D., Lund, Pipiras (2023)

$$\widehat{\beta} = \underset{\beta \in \mathbb{R}^q}{\arg\min} \Big(-2\beta'\widehat{\gamma} + \beta'\widehat{\Gamma}\beta + \lambda_N \|\beta\|_1 \Big).$$

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Apply Main result I:

$$\mathsf{P}\left[\|\widehat{\mathbf{\Gamma}}_Z - \mathbf{\Gamma}_Z\|_s > Q(\mathbf{\Gamma}_Z)\delta\right] \lesssim \mathsf{P}[\|\widehat{\mathbf{\Gamma}}_X - \mathbf{\Gamma}_X\|_s > \delta] + \mathsf{P}[\|\widehat{\mathbf{\Gamma}}_X - \mathbf{\Gamma}_X\|_s^2 > \delta]$$
$$+ \mathsf{P}[\|\widehat{\theta} - \theta\|_{\max} > \delta \wedge \varepsilon] + \mathsf{P}[\|\widehat{\theta} - \theta\|_{\max}^2 > \delta].$$

Main result II

Recall that $\widehat{\beta} = \arg\min_{\beta \in \mathbb{R}^q} \left(-2\beta'\widehat{\gamma} + \beta'\widehat{\Gamma}\beta + \lambda_N \|\beta\|_1 \right)$.

There exist finite positive constants c_1 and c_2 such that for any $v \in \mathcal{K}(2s)$,

$$\mathrm{P}[|v'(\widehat{\mathbf{\Gamma}}_X - \mathbf{\Gamma}_X)v| > \delta] \leq c_1 \exp\left(-c_2 \frac{N\delta^2}{s^2}\right), \quad N = T - p.$$

There exist finite positive constants c_1 and c_2 such that

$$\mathrm{P}[\|\widehat{\theta} - \theta\|_{\max} > \varepsilon] \leq c_1 dK \exp\left(-c_2 T\varepsilon^2\right).$$

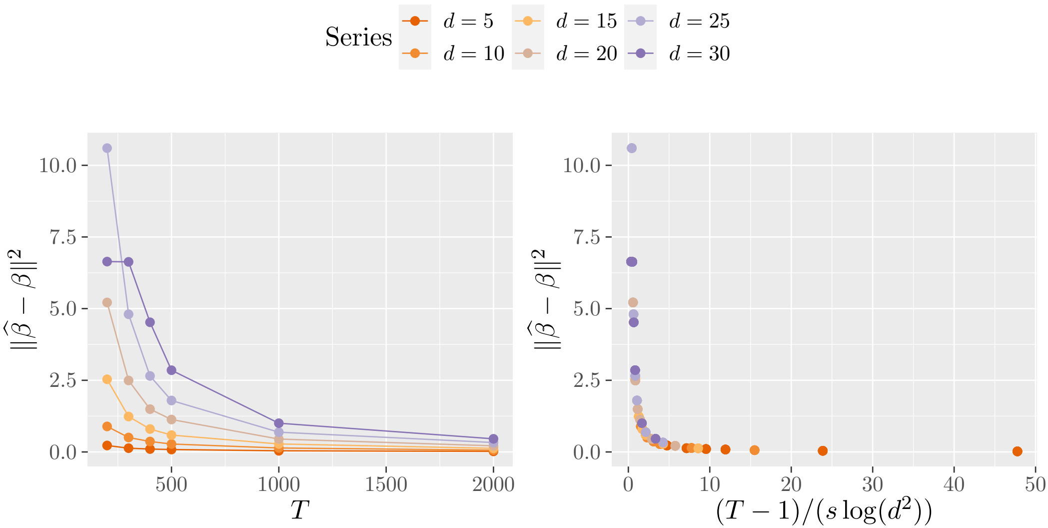
Proposition (D., Lund, Pipiras)

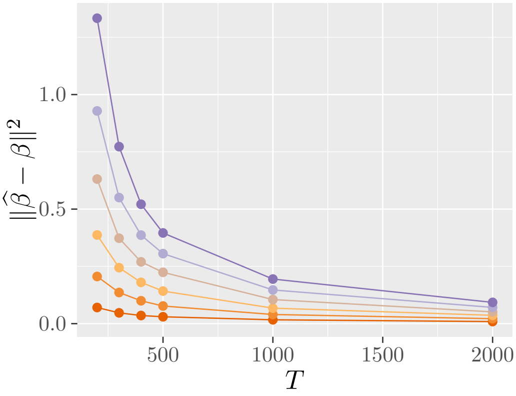
Then, with high probability, for any $\lambda_N \geq 4Q(\beta_0)\sqrt{\frac{\log(q)}{N}}$,

$$\|\widehat{\beta} - \beta_0\| \leq 64s\frac{\lambda_N}{\alpha}.$$

Same convergence rate as in Basu and Michailidis (2015)!

▸ More


Series
$d = 5$
$d = 10$
$d = 15$
$d = 20$
$d = 25$
$d = 30$
10.0
7.5
5.0
2.5
0.0
$\|\widehat{\beta} - \beta\|^2$
500
1000
1500
2000
T
0
10
20
30
40
50
$(T-1)/(s\log(d^2))$



1.0
0.5
0.0
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500
1000
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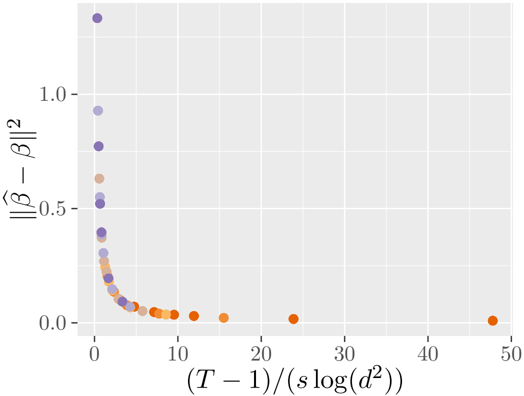
2000
T



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