Opinion Dynamics on Directed Inhomogeneous Networks

Panagiotis Andreou

University of North Carolina at Chapel Hill

pandreou@email.unc.edu

Joint work with Prof. Mariana Olvera-Cravioto

Seminar of Statistics & Operations Research

Department of Mathematics, NKUA

October 20, 2023

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Motivation for Opinion Dynamics

1. Political Science:
    - Understand polarization in modern societies
    - Influence of the media in opinion shaping
    - Debunk myths about political personas

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1. Political Science:
    - Understand polarization in modern societies
    - Influence of the media in opinion shaping
    - Debunk myths about political personas
2. Probability Theory:
    - Stochastic Processes on Networks
    - Influence maximization in Social Networks
    - Community detection and clustering

Goals

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1. Study the behavior of the system under varying **density** regimes and check for phase transitions.
2. Understand how the opinion process is affected by the passing of **time** and the change of the **network size**.
3. Study the **typical** stationary opinion on an inhomogeneous network.

Mathematical Tools

1. Random Graphs
2. Mean-field approximation
3. Local Weak Convergence
4. Stochastic fixed-point equations

Opinion Models on Fixed Graphs

❶ DeGroot model

- The first opinion model to formally study consensus.
- The simplest form of linear opinion updating:

$$\mathbf{R}^{(t)} = W\mathbf{R}^{(t-1)},$$

for a stochastic matrix W.

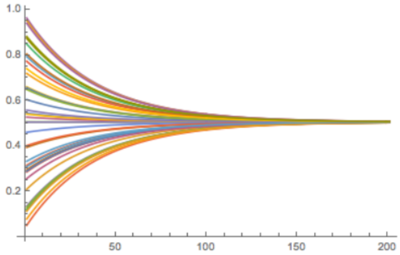

Figure: Evolution of the DeGroot model towards consensus (Source: Noorazar '20)

❷ Friedkin - Johnsen model

- Extension of the DeGroot model:

$$\mathbf{R}^{(t)} = DW\mathbf{R}^{(t-1)} + (I - D)\mathbf{R}^{(0)}$$

- Allows for stubborn agents via the matrix $I - D$.

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❸ Bounded confidence models

- Agents interact only when their opinions are close:

$$\begin{cases} R_i^{(t)} = R_i^{(t-1)} + \mu(R_j^{(t-1)} - R_i^{(t-1)}) \\ R_j^{(t)} = R_j^{(t-1)} + \mu(R_i^{(t-1)} - R_j^{(t-1)}) \end{cases}$$

when $|R_i^{(t-1)} - R_j^{(t-1)}| \leq \epsilon$, for a specified confidence radius ϵ.

- Model selective exposure

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- In practice, we have a specified social network G.
- *Idea:* think of G as a realization of a random graph model.
- *Insight:* Even though the math of random graphs is harder, this idea allows us to talk about the *typical* stationary opinion and get way more general results that don't depend on the specific G.

Random Graphs

- Graphs where each edge is present with some probability.
- Useful for modeling first-order properties:
  1. Degree distribution
  2. Connectivity
  3. Community structure
  4. Average distances (small-world phenomenon)

Classification of Random Graphs

1. Static:
   - Snapshots of large networks
   - $G(V_n, E_n)$ and $G(V_{n+1}, E_{n+1})$ can be quite different
   - *Examples:* Erdös-Rényi, Stochastic Block Model, Configuration Model

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Our opinion process is evolving on a **static** random graph.

Erdös - Rényi (static)

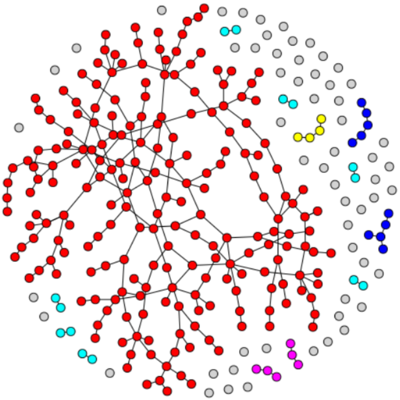

Figure: Different colors for different connected components (source: Fluid Limits and Random Graphs)

Stochastic Block Model (static)

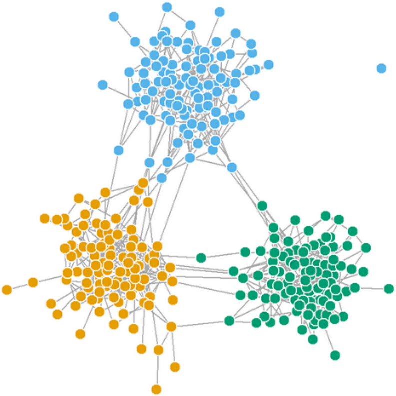

Figure: SBM with 3 communities
(source: Mathematics sin Fronteras)

Preferential Attachment (dynamic)

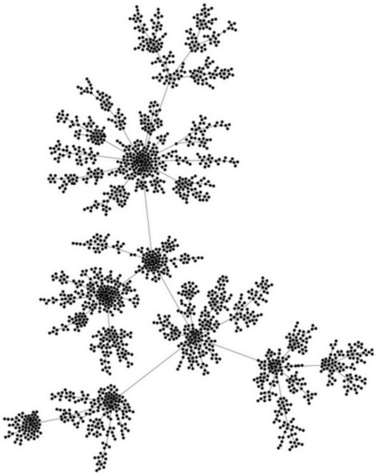

Figure: PA model - “the rich get richer” (source: ResearchGate)

dSBM

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- Each vertex $i \in V_n$ has a **community label** $J_i \in [K]$.
- Two nodes $i, j \in V_n$ are connected with an edge with probability

$$p_{ij}^{(n)} = \frac{\kappa(J_i, J_j)\theta_n}{n} \wedge 1,$$

where $\kappa \in \mathbb{R}_+^{K \times K}$ and θ_n is a **density** parameter.

Density regimes

- The expected degree of a vertex is of order θ_n.
- We call the graph **sparse** if $\theta_n = O(1)$.
- We call the graph **semi-sparse** if $\theta_n \to \infty$ and $\theta_n = O(\log n)$.
- We call the graph **dense** if $\frac{\theta_n}{\log n} \to \infty$ as $n \to \infty$.
- Our work covers the entire spectrum of sequences satisfying $\theta_n \to \infty$ as $n \to \infty$.

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- $R_i^{(k)} \in [-1, 1]$: opinion of individual i at time k.
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- $C_{ij} \in [0, c]$: the weight that i puts in j's opinion.

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- $C_{ij} \in [0, c]$: the weight that i puts in j's opinion.
- At each time step $k \geq 1$, individual i updates their opinion according to

$$R_i^{(k)} = \sum_{j=1}^{n} C_{ij} R_j^{(k-1)} + W_i^{(k)} + (1 - c - d) R_i^{(k-1)},$$

where $0 \leq c + d \leq 1$.

The Weights

- Define the weight C_{ij} that i puts on j's opinion as

$$C_{ij} = \frac{cB_{ij}1(j \to i)}{\sum_{r=1}^{n} B_{ir}1(r \to i)} 1(D_i^- > 0, i \neq j),$$

where $D_i^- := \sum_{r=1}^{n} 1(r \to i)$ is the in-degree of i.

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- This is key in our analysis, as it creates a **contraction**.
- Assume that the external media signals $\{W_i^{(k)} : k \geq 0, i \in V_n\}$ are independent.

Simulations

- dSBM with 2 communities
- Expected degree $\sim \log n$
- Run the Markov chain until stationarity (roughly 500 iterations)

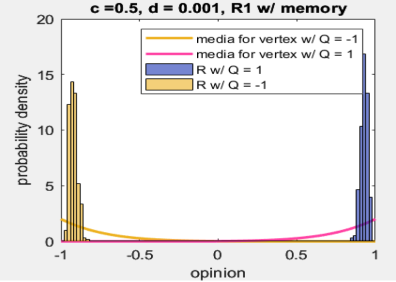

Figure: Polarization

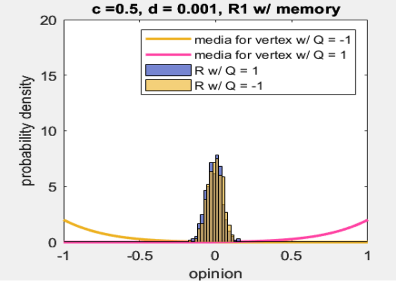

Figure: Consensus

Mean Field Theory

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- *Idea:* replace all the interactions in a complex system by an average interaction.
- *Intuition:* the presence of many particles should reduce the effect of each particle on the entire system.
- *Practicality:* reduce the initial high-dimensional problem of a stochastic process on a network to one of much lower dimension.

Notation

- Define $\mathscr{A}_n := \{J_i : i \in V_n\}$, $\mathscr{F}_n := \sigma(\mathscr{A}_n)$, $\mathbb{E}_n[\cdot] := E[\cdot|\mathscr{F}_n]$.
- $\pi_r^{(n)} := \frac{1}{n}\sum_{i=1}^{n} 1(J_i = r)$, the proportion of vertices having community $r \in [K]$.
- *Assumption:* $\pi_r^{(n)} \xrightarrow{P} \pi_r$, where $\pi_1 + \cdots + \pi_K = 1$.
- Define the matrix $M \in [0,1]^{K\times K}$ by

$$m_{rs} = \frac{c\,\pi_s \beta_{r,s}\kappa(s,r)}{\pi_1\beta_{r1}\kappa(1,r) + \cdots + \pi_K\beta_{rK}\kappa(K,r)},$$

  where $\beta_{r,s} = E[B_{ij}|J_i = r, J_j = s]$.
- Let $a_{l,s} = \binom{s}{l}(1-c-d)^{s-l}$, for $0 \le l \le s$.

The mean-field limit

- Define the process $\boldsymbol{\mathcal{R}}^{(k)} = (\mathcal{R}_1^{(k)}, \dots, \mathcal{R}_n^{(k)})'$ according to

$$\mathcal{R}_i^{(0)} = R_i^{(0)}$$

and for $k \geq 1$,

$$\mathcal{R}_i^{(k)} = \sum_{t=0}^{k-1} (1-c-d)^t W_i^{(k-t)} + 1(k \geq 2) \sum_{t=1}^{k-1} \sum_{s=1}^{t} a_{s,t} (M^s \mathbf{w})_{J_i} + \sum_{s=1}^{k} a_{s,k} (M^s \mathbf{r}_0)_{J_i} + (1-c-d)^k R_i^{(0)},$$

$i \in [n]$, where $w(l) = E[W_i^{(0)} | J_i = l]$ and $r_0(l) = E[R_i^{(0)} | J_i = l]$, $l \in [K]$.

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- *Key-observation*: The components of $\boldsymbol{\mathcal{R}}^{(k)}$ are **independent**.

Main Theorem

Theorem (A., Olvera-Cravioto '23)

Suppose $\theta_n / \log n \to \infty$ *as* $n \to \infty$. *Then,*

$$\sup_{k \geq 0} \mathbb{E}_n \left[\| \mathbf{R}^{(k)} - \mathcal{R}^{(k)} \|_\infty \right] = O\left(\sqrt{\frac{\log n}{\theta_n}} + \max_{1 \leq r,s \leq K} \left| \frac{\pi_s^{(n)} \pi_r - \pi_s \pi_r^{(n)}}{\pi_r^{(n)} \pi_s} \right| \right).$$

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Moreover, for any sequence θ_n *satisfying* $\theta_n \to \infty$ *as* $n \to \infty$,

$$\sup_{k\geq 0} \frac{1}{n} \mathbb{E}_n \left[\|\mathbf{R}^{(k)} - \boldsymbol{\mathcal{R}}^{(k)}\|_1 \right] \xrightarrow{P} 0.$$

Remarks

- Since $\|\mathbf{x}\|_1 \leq n\|\mathbf{x}\|_\infty$ for any $\mathbf{x} \in \mathbb{R}^n$, Theorem 1 shows that the approximation is stronger when $\theta_n/\log n \to \infty$, and it gradually weakens as the rate at which θ_n grows drops below the *critical* rate $\log n$.

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- *Intuition*: the average number of neighbors that any vertex has grows with θ_n. The larger the number of neighbors, the more their aggregate contributions behave as the average opinion.
- The weakest result is valid for any $\theta_n \to \infty$, regardless of how slow the growth is.
- Since the components of $\{\mathcal{R}^{(k)} : k \geq 1\}$ are independent of each other, Theorem 1 yields that the trajectories of the process $\{\mathbf{R}^{(k)} : k \geq 0\}$ are asymptotically independent, i.e., the system exhibits *propagation of chaos*.

Proof steps

1. First, write the opinion recursion in matrix form:

$$\mathbf{R}^{(k)} = A\mathbf{R}^{(k-1)} + \mathbf{W}^{(k)},$$

where $A_{ij} = C_{ij}1(i \neq j) + (1 - c - d)1(i = j)$.

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2. Iterate the recursion:

$$\mathbf{R}^{(k)} = \sum_{t=0}^{k-1} A^t \mathbf{W}^{(k-t)} + A^k \mathbf{R}^{(0)}.$$

3. Note that

$$A^t = (C + (1 - c - d)I)^t = \sum_{s=0}^{t} \binom{t}{s} (1 - c - d)^{t-s} C^s = \sum_{s=0}^{t} a_{s,t} C^s.$$

3 Note that

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4 Thus, the recursion becomes

$$\mathbf{R}^{(k)} = \sum_{t=0}^{k-1} \sum_{s=0}^{t} a_{s,t} C^s \mathbf{W}^{(k-t)} + \sum_{s=0}^{k} a_{s,k} C^s \mathbf{R}^{(0)}.$$

3. Define the *approximate* mean $\tilde{M}$ of the matrix C:

$$\tilde{M}_{ij} = \frac{c\beta_{J_i,J_j}\kappa(J_j, J_i)}{n\left(\beta_{J_i,1}\pi_1^{(n)}\kappa(1, J_i) + \cdots + \beta_{J_i,K}\pi_K^{(n)}\kappa(K, J_i)\right)} 1(i \neq j).$$

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4. Approximate meaning that

$$\tilde{M}_{ij} = \frac{\mathbb{E}_n[cB_{ij}1(j \to i)]}{\mathbb{E}_n\left[\sum_{i=1}^{n} B_{ir}1(r \to i)\right]} \approx \mathbb{E}_n[C_{ij}].$$

❻ *Key idea:* Define the **intermediate** process

$$\tilde{\mathbf{R}}^{(k)} = \sum_{t=0}^{k-1} (1-c-d)^t \mathbf{W}^{(k-t)} + 1(k \geq 2) \sum_{t=1}^{k-1} \sum_{s=1}^{t} a_{s,t} \tilde{M}^s \bar{\mathbf{w}} + \sum_{s=1}^{k} a_{s,k} \tilde{M}^s \bar{\mathbf{r}}_0 + (1-c-d)^k \mathbf{R}^{(0)}, \quad k \geq 1, \qquad \tilde{\mathbf{R}}^{(0)} = \mathbf{R}^{(0)},$$

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❼ *Intuition:* $\tilde{\mathbf{R}}^{(k)}$ replaces all neighbor contributions with their approximate means, i.e., every term of the form $C^s\mathbf{X}$ with $s \geq 1$ and $\mathbf{X}$ a random vector is replaced with $\tilde{M}^s\mathbb{E}_n[\mathbf{X}]$. That's the essence of mean-field approximation!

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8. *Key fact:* The components of $\tilde{\mathbf{R}}^{(k)}$ are independent, since the only randomness comes from the media signals.

7. *Goal:* bound $\mathbb{E}_n \left[\|\mathbf{R}^{(k)} - \tilde{\mathbf{R}}^{(k)}\|_p \right]$ and $\mathbb{E}_n \left[\|\tilde{\mathbf{R}}^{(k)} - \boldsymbol{\mathcal{R}}^{(k)}\|_\infty \right]$, for $p \geq 1$.

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❽ Bound these terms for different ranges of θ_n:

- If $\theta_n / \log n \to \infty$, use concentration inequalities.
- If $\theta_n / \log n \to 0$, use local weak convergence.

Local Weak Convergence

- *Idea:* if the graph is sparse enough, then cycles take long to form, so it **locally** looks like a tree.

Local Weak Convergence

- *Idea:* if the graph is sparse enough, then cycles take long to form, so it **locally** looks like a tree.
- Pick vertex I_n uniformly at random and explore its inbound neighborhood.

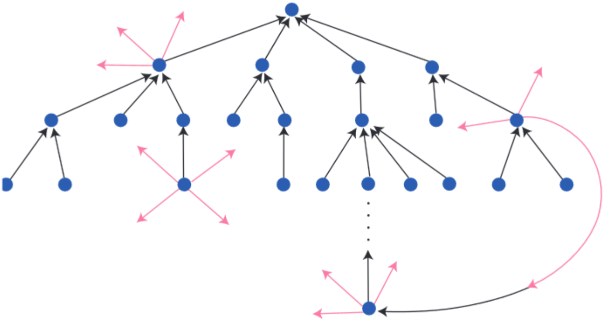

Figure: Graph created by M. Olvera-Cravioto.

Local Weak Limit

- In a general random graph, we can't compute $E[C^s\mathbf{X}]$ for $s \geq 2$:

$$(C^s\mathbf{X})_i = \sum_{j_1,\dots,j_s} C_{ij_1} C_{j_1 j_2} \cdots C_{j_{s-1}j_s} X_{j_s},$$

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- The local weak limit of our K-community dSBM is a K-type Galton-Watson process.

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  since the existence of the edge (j_1, j_2) is no longer Bernoulli.

- Random graphs are hard because they contain cycles!

- The local weak limit of our K-community dSBM is a K-type Galton-Watson process.

- *Key-insight:* The **conditional independence** of branching processes balances the lack of enough averaging in the subcritical regimes.

Typical Opinion

Theorem (Fraiman, Lin, Olvera-Cravioto '22)

Suppose G is locally finite and $d > 0$. Then, there exists a random vector $\mathbf{R}$ such that $\mathbf{R}^{(k)} \Rightarrow \mathbf{R}$ as $k \to \infty$. This is the stationary distribution of the Markov Chain.

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- *Typical opinion:* R_{I_n}, for I_n uniformly chosen vertex.
- The typical opinion reflects the average behavior on the graph.

Theorem (A., Olvera-Cravioto '23)

Fix $k \geq 0$ and define the random variables $(\{\mathcal{R}_\emptyset^{(k)} : k \geq 0\}, \mathcal{J}_\emptyset)$ according to:

$$P\left((\mathcal{R}_\emptyset^{(0)}, \mathcal{R}_\emptyset^{(1)}, \ldots, \mathcal{R}_\emptyset^{(k)}) \in A \middle| \mathcal{J}_\emptyset = s \right) = \mathbb{P}_n \left((\mathcal{R}_i^{(0)}, \mathcal{R}_i^{(1)}, \ldots, \mathcal{R}_i^{(k)}) \in A \middle| J_i = s \right)$$

and $P(\mathcal{J}_\emptyset = s) = \pi_s, 1 \leq s \leq K$. Then, $\forall 1 \leq r \leq K$, $\forall f \in C_b([-1,1]^{k+1})$, we have

$$\frac{1}{n} \sum_{i=1}^{n} f(R_i^{(0)}, R_i^{(1)}, \ldots, R_i^{(k)}) 1(J_i = r) \xrightarrow[n\to\infty]{P} E\left[f(\mathcal{R}_\emptyset^{(0)}, \mathcal{R}_\emptyset^{(1)}, \ldots, \mathcal{R}_\emptyset^{(k)}) 1(\mathcal{J}_\emptyset = r) \right].$$

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Furthermore, when $\theta_n / \log n \to \infty$, we have for any arbitrary collection of vertices $\{i_1, \ldots, i_m\} \subseteq V_n$ having community labels $\{r_1, \ldots, r_m\}$, $m \geq 1$,

$$E\left[\prod_{j=1}^{m} f_j(R_{i_j}^{(0)}, \ldots, R_{i_j}^{(k)}) \right] \xrightarrow{n\to\infty} \prod_{j=1}^{m} E\left[f_j(\mathcal{R}_\emptyset^{(0)}, \ldots, \mathcal{R}_\emptyset^{(k)}) \middle| \mathcal{J}_\emptyset = r_j \right],$$

for any set of continuous bounded functions $\{f_1, \ldots, f_m\}$ on $[-1,1]^{k+1}$, $k \geq 0$.

Time and Network Size

Theorem (A., Olvera-Cravioto '23)

There exists a random variable $\mathcal{R}_\emptyset$ such that $R_{I_n} \Rightarrow \mathcal{R}_\emptyset$ as $n \to \infty$, and $\mathcal{R}_\emptyset^{(k)} \Rightarrow \mathcal{R}_\emptyset$ as $k \to \infty$. Hence, the following diagram commutes.

$$\begin{array}{ccc} R_{I_n}^{(k)} & \xrightarrow{k\to\infty} & R_{I_n} \\ \Big\downarrow{\scriptstyle n\to\infty} & & \Big\downarrow{\scriptstyle n\to\infty} \\ \mathcal{R}_\emptyset^{(k)} & \xrightarrow{k\to\infty} & \mathcal{R}_\emptyset \end{array}$$

Key takeaways

- Mathematical insights:
    1. When the graph is sufficiently dense, the high number of neighbors allows us to use Chernoff bounds.
    2. As the graph gets sparser, concentration inequalities are not useful, so we need to get independence from somewhere else. That's what branching processes do.
    3. Random graphs are hard because they contain cycles. When possible, couple them with trees.

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- Mathematical insights:
  1. When the graph is sufficiently dense, the high number of neighbors allows us to use Chernoff bounds.
  2. As the graph gets sparser, concentration inequalities are not useful, so we need to get independence from somewhere else. That's what branching processes do.
  3. Random graphs are hard because they contain cycles. When possible, couple them with trees.

- Practical implications:
  1. When the network is sparse, individual opinions matter significantly.
  2. As the network gets denser, individuals essentially don't interact but rather update based on the "average" opinion.

Future directions

- Replace linear opinion recursion by general **contraction**, namely

$$R_i^{(k)} = \Phi\left(W_i^{(k)}, \{R_j^{(k-1)}\}_{j \in V_n}, J_i\right),$$

where Φ is a ℓ-Lipschitz function with $\ell < 1$.

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- Extend the analysis to **multiopinion** dynamics, i.e., assume individuals are interacting on more than one topics. Try to explain mathematically the political scientists' hypothesis that "political personas are not real".

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