Generalized B*-algebras: A short introduction and overview

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Topological algebras

A *topological algebra* is an algebra which is also a Hausdorff topological vector space in which multiplication is separately continuous.

A *locally convex $*$-algebra* is a topological algebra with continuous involution, which is also a locally convex space.

GB*-algebras (*G. R. Allan, 1967*)

Let $A[\tau]$ be a locally convex $*$-algebra.

Denote by $\mathcal{B}^*$ the family of all subsets B of A such that:

(i) B is absolutely convex,
$B^2 \subseteq B$, $B^* = B$ and $1 \in B$,

(ii) B is bounded and closed.

A unital pseudo-complete locally convex $*$-algebra A is called a *GB^*-algebra* if there is a C*-algebra $A_b[\|\cdot\|]$ contained in A such that

(i) $(1+x^*x)^{-1} \in A_b$ for all $x \in A$,

(ii) the unit ball of A_b is the greatest member of $\mathcal{B}^*$ w.r.t set inclusion.

Proposition. If $A[\tau]$ is a GB*-algebra, then $\tau|_{A_b} \preceq \|\cdot\|$ on A_b.

Examples of GB*-algebras

(1) *Pro-C*-algebras* : Complete locally convex $*$-algebras $A[\tau]$ for which τ is defined by a family of C*-seminorms. (*A. Inoue, 1971*)

(2) $C(X)$, with a topology weaker than the compact-open topology, where X is a completely regular Hausdorff space.

(3) $L^\omega([0,1]) = \cap_{1\leq p<\infty} L^p([0,1])$. This is a Fréchet GB*-algebra which is not a pro-C*-algebra.

(4) Every unital closed $*$-subalgebra of a GB*-algebra is a GB*-algebra.

(5) Suppose that $A[\tau]$ is a locally convex $*$-algebra, where τ is defined by a directed family $\Gamma = \{(p_\lambda)_{\lambda \in \Lambda}\}$ of seminorms with the properties that for each $\lambda \in \Lambda$, there exists $\lambda' \in \Lambda$ such that for all $x, y \in \mathcal{A}$,

$$p_\lambda(xy) \leq p_{\lambda'}(x)p_{\lambda'}(y),$$

$$p_\lambda(x^*) \leq p_{\lambda'}(x),$$

$$p_\lambda(x)^2 \leq p_{\lambda'}(x^*x).$$

We call $A[\tau]$ above a *C^*-like locally convex $*$-algebra* if it is complete and
$A_b := \{x \in A : \sup p_\lambda(x) < \infty\}$
is dense in A.

(6) Let M be a von Neumann algebra on a Hilbert space H with a faithful semifinite normal trace ϕ. The $*$-algebra $\widetilde{M}$ of all ϕ-measurable operators affiliated with M is a GB*-algebra, in the topology of convergence in measure τ_{cm}, over the closed unit ball of M.

The GB*-algebra $\widetilde{M}[\tau_{cm}]$ is, however, not necessarily locally convex.

Remark. If $M \cong L_\infty([0,1])$, then $\widetilde{M} \cong L_0([0,1])$.

(7) Let M be a von Neumann algebra with a faithful *finite* normal trace ϕ, and let $\widetilde{M}$ denote the $*$-algebra of all ϕ-measurable operators affiliated with M. For every $1 \leq p < \infty$, let
$L^p(M, \phi) = \{x \in \widetilde{M} : \phi(|x|^p)^{\frac{1}{p}} < \infty\}$.

The $*$-algebra
$L^\omega(M, \phi) = \cap_{1 \leq p < \infty} L^p(M, \phi)$ is a C*-like locally convex $*$-algebra w.r.t the family of C*-like seminorms $\| \cdot \|_p = \phi(1)^{-\frac{1}{p}} \phi(| \cdot |^p)^{\frac{1}{p}}$.

Representation theorem for commutative GB*-algebras

(*G. R. Allan, 1967*)

Theorem. If $A[\tau]$ is a commutative GB*-algebra, then the Gelfand map $x \mapsto \hat{x}$ $(x \in A_b)$ of A_b extends to an algebra $*$-isomorphism of A onto a $*$-algebra of functions on the maximal ideal space of A_b.

Representation theorem for non-commutative GB*-algebras
(*P. G. Dixon, 1970*)

Let $A[\tau]$ be a GB*-algebra. Then there exists a faithful $*$-representation π of A onto a $*$-algebra of closed linear operators on a Hilbert space, with common dense domain. The elements of B_0 correspond to the closed unit ball of $\pi(A)$.

Derivations of pro-C*-algebras

(*R. Becker, 1992*)

A *derivation* is a linear map $\delta : A \to A$ with

$$\delta(xy) = x\delta(y) + \delta(x)y,$$

for every $x, y \in A$.

All derivations of pro-C*-algebras are continuous.

The zero derivation is the only derivation of a commutative pro-C*-algebra.

A *-derivation of a pro-C*-algebra $A[\tau]$ is the generator of a one-parameter group of *-automorphisms of A.

Derivations of GB*-algebras

(*M. Weigt, I. Zarakas, 2013*)

Theorem. The zero derivation is the only derivation of a complete commutative GB*-algebra having jointly continuous multiplication.

Corollary. If $A[\tau]$ is a commutative Fréchet GB*-algebra, then the zero derivation is the only derivation of A.

Question. What about the non-commutative case?

Is every derivation of a Fréchet GB*-algebra continuous?

A GB*-algebra A is said to be *nuclear* if A_b is a nuclear C*-algebra.

Theorem. Every derivation of a smooth Fréchet nuclear GB*-algebra is continuous (*M. Weigt, I. Zarakas, 2015*).

Derivations of Fréchet GB*-algebras A for which A_b is a W*-algebra

(*M. Weigt, I. Zarakas, 2013*)

Theorem.

Every derivation of a Fréchet GB*-algebra $A[\tau]$, with A_b a W*-algebra, is inner.

Unbounded derivations of GB*-algebras
(*M. Weigt, I. Zarakas, 2017*)

(1) An investigation of the closure of the domain of an unbounded derivation of a GB*-algebra, under the holomorphic functional calculus.

(2) A generalization of the Lumer-Phillips theorem in semigroup theory on Banach spaces, to complete locally spaces.

GB*-algebras and Gelfand-Mazur algebras

A *Gelfand-Mazur algebra* is a unital topological algebra $A[\tau]$ such that $A/M \cong \mathbb{C}$ up to topological algebra isomorphism, for every closed two-sided ideal M of A which is maximal as a left or right ideal of A.

Theorem. (*M. Weigt, I. Zarakas, 2021*)
Every GB*-algebra is a Gelfand-Mazur algebra.

Tensor products of GB*-algebras
(*M. Fragoulopoulou, A. Inoue and M. Weigt, 2014*)

Convention. From here on, unless stated otherwise, all locally convex $*$-algebras are assumed to have *jointly continuous* multiplication.

With this convention in place, if A_1 and A_2 are locally convex $*$-algebras, then $A_1 \underset{\tau}{\hat{\otimes}} A_2$ will be a locally convex $*$-algebra, and not only a locally convex space.

The completion of a C*-algebra under a locally convex $*$-topology (*F. Bagarello, C. Trapani, A. Inoue, M. Fragoulopoulou, 2006*)

Theorem. Let $A[\|\cdot\|]$ be a C*-algebra equipped with a topology τ such that $A[\tau]$ is a locally convex $*$-algebra with jointly continuous multiplication, and $\tau \preceq \|\cdot\|$. Let $\widetilde{A}[\tau]$ denote the τ-completion of A. Then $\widetilde{A}[\tau]$ is a GB*-algebra over the τ-closure of the closed unit ball of A.

Example. Let $A[\tau]$ be a complete GB*-algebra, and let (p_λ) denote a family of $*$-seminorms on A defining the topology τ. If X is a locally compact Hausdorff space, let $C(X, A)$ denote the locally convex $*$-algebra of all continuous A-valued functions on X, with $q_{K,\lambda}(f) := \sup_{x \in K} p_\lambda(f(x))$ a family of $*$-seminorms defining the topology on $C(X, A)$, where K is a compact subset of X. Then $C(X, A) = C(X) \hat{\underset{\epsilon}{\otimes}} A$ is a GB*-algebra.

If A is a C*-like locally convex $*$-algebra, then so is $C(X, A)$.

Main Result. Let $A_1[\tau_1]$ and $A_2[\tau_2]$ be GB*-algebras, with B_0^1, B_0^2 the maximal members of $\mathcal{B}_1^*$, $\mathcal{B}_2^*$ respectively. Let τ be a $*$-admissible topology on $A_1 \otimes A_2$. The following are equivalent.

(1) $A_1 \hat{\underset{\tau}{\otimes}} A_2$ is a GB*-algebra.

(2) $A_1[B_0^1] \hat{\underset{\|\cdot\|}{\otimes}} A_2[B_0^2]$ is a C*-algebra contained in $A_1 \hat{\underset{\tau}{\otimes}} A_2$ for some C*-crossnorm $\| \cdot \|$.

Tensor Product GB*-algebra

When the equivalent conditions of the Main Result hold, then we call $A_1 \hat{\underset{\tau}{\otimes}} A_2$ the *tensor product GB*-algebra* of the GB*-algebras $A_1[\tau_1]$ and $A_2[\tau_2]$.

Example. If $A[\tau]$ is a commutative C*-like locally convex $*$-algebra, and $M_n(A)$ denotes the $*$-algebra of all $n \times n$ matrices over A, then $M_n(A) = A_b \hat{\underset{\epsilon}{\otimes}} M_n(\mathbb{C})$ is a C*-like locally convex $*$-algebra, and hence a tensor product GB*-algebra.

Example. If $A[\tau]$ is any C*-like locally convex $*$-algebra, then $L^{\omega}([0,1])\hat{\underset{\epsilon}{\otimes}}A$ is a tensor product GB*-algebra.

Applications to Mathematical Physics (*Weigt, 2018-*)

(1) Applications of GB*-algebras to quantum entanglement, including further results on pure states of GB*-algebras.

(2) Applications of GB*-algebras to quantum irreversible dynamics, and is still ongoing.

(3) Applications of unbounded operator algebras to relativistic quantum field theory (local nets of unbounded operator algebras, and Wightman theory) and is work in progress.