Quantitative ergodic theory

Romain Tessera

CNRS, Université Paris Cité et Sorbonne Université

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- *Rotations*: $\mathbb{Z} \curvearrowright (S^1, \lambda)$ generated by an irrational rotation,
- *Bernoulli shift*: $\Lambda \curvearrowright \{0,1\}^\Lambda$.

Context

- Ergodic theory,
- Representation theory,
- Operator algebras,
- Percolation theory (probabilities),
- Lattices in Lie groups...

Isomorphism

Definition

Two pmp actions $\Lambda \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are **isomorphic**, if there exist an isomorphism of measure spaces $\Psi : (X, \mu) \to (Y, \nu)$, and a group isomorphism: $\theta : \Lambda \to \Gamma$ such that

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- $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}$ and $\mathbb{Z} \curvearrowright \{0,1,2\}^{\mathbb{Z}}$ are *not* isomorphic (Kolmogorov-Sinai);

Orbit equivalence

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Two pmp actions $\Lambda \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are **orbit equivalent** (OE), if there exists an isomorphism of measure spaces $\Psi : (X, \mu) \to (Y, \nu)$ such that for a.e. $x \in X$,

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Theorem (Dye)

Any two ergodic pmp actions of $\mathbb{Z}$ are OE.

Amenable groups

Definition

A countable group Λ is **amenable** if it admits a sequence of "almost-invariant finite subsets" $A_n \subset \Lambda$, i.e. such that for all $\lambda \in \Lambda$,

$$\frac{|A_n\lambda \vartriangle A_n|}{|A_n|} \to 0.$$

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- free groups F_k on $k \geq 2$ generators **are not** amenable.

A famous theorem of Ornstein-Weiss

Theorem (Ornstein-Weiss 80)

Let Λ and Γ be two (infinite) countable amenable groups. Then any pmp ergodic actions $\Lambda \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are OE.

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Things are very different for **non-amenable** groups. For instance

Theorem (Gaboriau 00)

If F_k and $F_{k'}$ have OE pmp actions, then $k = k'$.

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Problem: Quantify how "distorted" are these bijections "in average".

Word metric on a group

Definition (Word distance)

Let Λ be a group generated by a finite subset S. The word length on Λ associated to S is defined as

$$|g|_S = \min\{n \in \mathbb{N} \mid g = s_1^{\pm 1} \dots s_n^{\pm 1}; s_i \in S\}.$$

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is **integrable** (similarly for β).

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$$(L^2 - OE) \Rightarrow (L^1 - OE) \Rightarrow (L^{1/2} - OE) \Rightarrow (\log(t) - OE)...$$

No quantitative version of OW's theorem

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

For all Λ amenable, and all increasing unbounded φ, there exists another (explicit) amenable group Γ such that no pmp action of Γ is φ-OE to a pmp action of Λ.

Quantify amenability: Følner profile

Definition

Let Λ be a group generated by a finite subset S. Define its Følner function

$$F\o l(n) = \min\left\{|A| \mid \frac{|As \vartriangle A|}{|A|} \leq 1/n,\ \forall s \in S\right\}$$

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- *If Γ has exponential growth and if Γ and $\mathbb{Z}$ are are φ-OE, then $\varphi(n) \lesssim \log n$.*

What about a converse?

The previous result is optimal in a number of situation. For instance

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Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

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- **The** 4**-odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$
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Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

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These actions **preserve the product measure** on $\{0,1\}^{\mathbb{N}}$ and $\{0,1,2,3\}^{\mathbb{N}}$.

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Two sequences belong to the **same orbit** if and only if they differ by at most finitely many coordinates.

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

The actions of $\mathbb{Z}$ and $\mathbb{Z}^2$:

- We let $\mathbb{Z}$ acts on the 4-odometer: $\{0,1,2,3\}^{\mathbb{N}}$

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The orbit equivalence: $F : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1,2,3\}^{\mathbb{N}}$ is defined

$$F(x,y) = x + 2y.$$

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Example: if $x = (0,1,1,\ldots)$, $y = (1,0,1,\ldots)$, then

$$F(x,y) = (0+2, 1+0, 1+2, \ldots) = (2,1,3,\ldots).$$