Normalised solutions for beginners

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The problem

Let $N \geq 2$, $2 < p < \frac{2N}{N-2}$ with $p \neq 2 + \frac{4}{N}$, $\rho > 0$, and consider the problem

$$-\Delta u + \lambda u = |u|^{p-2} u \quad \text{in } \mathbb{R}^N, \tag{1}$$

where $\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$, paired with the constraint

$$\int_{\mathbb{R}^N} u^2 \, \mathrm{d}x = \rho^2, \tag{2}$$

with $\lambda \in \mathbb{R}$ to be determined.

If (λ, u) solves (1)–(2), we call it a normalised solution.

The problem (1)–(2) appears when looking for solutions to

$$\begin{cases} \mathrm{i}\frac{\partial \Phi}{\partial t} - \Delta \Phi = |\Phi|^{p-2}\Phi & \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} |\Phi(x,t)|^2 \, \mathrm{d}x = \rho^2 & \text{(conserved in time)} \end{cases}$$

as standing waves, i.e.,

$$\Phi(x,t) = e^{-\mathrm{i}\lambda t} u(x).$$

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The settings

Let us define

$$H^1(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) \;\middle|\; \frac{\partial u}{\partial x_j} \in L^2(\mathbb{R}^N)\, \forall j = 1, \dots, N \right\}.$$

What is $\frac{\partial u}{\partial x_j}$ for $u \in L^2(\mathbb{R}^N)$? It is defined via

$$\int_{\mathbb{R}^N} \frac{\partial u}{\partial x_j} \varphi \, \mathrm{d}x = - \int_{\mathbb{R}^N} u \frac{\partial \varphi}{\partial x_j} \, \mathrm{d}x \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N).$$

One can prove $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for every $2 \le q \le \frac{2N}{N-2}$ ($2 \le q < \infty$ if $N = 2$).

Take $v \in H^1(\mathbb{R}^N)$. If we multiply (1) by v and integrate by parts, we get

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v + \lambda uv \, \mathrm{d}x = \int_{\mathbb{R}^N} |u|^{p-2} uv \, \mathrm{d}x. \tag{3}$$

We say that $u \in H^1(\mathbb{R}^N)$ is a weak solution to (1) iff (3) holds for every $v \in H^1(\mathbb{R}^N)$.

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If $I\colon H^1(\mathbb{R}^N) \to \mathbb{R}$ and $u, v \in H^1(\mathbb{R}^N)$, we denote

$$I'(u)v := \lim_{t \to 0} \frac{I(u+tv) - I(u)}{t}.$$

Then $I'(u) \in \left(H^1(\mathbb{R}^N)\right)'$.

We write $I \in \mathcal{C}^1\left(H^1(\mathbb{R}^N)\right)$ iff $u \mapsto I'(u)$ is continuous.

We say that $u \in H^1(\mathbb{R}^N)$ is a critical point for $I \in \mathcal{C}^1\left(H^1(\mathbb{R}^N)\right)$ iff $I'(u) = 0$, i.e., $I'(u)v = 0$ for every $v \in H^1(\mathbb{R}^N)$.

Recall: If $A \subset H^1(\mathbb{R}^N)$ is open and $u \in A$, then $I|_A'(u) = 0 \Leftrightarrow I'(u) = 0$.

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Let $\mathcal{Q} := \{ u \in H^1(\mathbb{R}^N) \mid G(u) = 0 \} \neq \emptyset$ for some $G \in \mathcal{C}^1\big(H^1(\mathbb{R}^N)\big)$ such that $G'(u) \not\equiv 0$ for every $u \in \mathcal{Q}$.

If $u \in \mathcal{Q}$, then

$$I|'_{\mathcal{Q}}(u) = 0$$

if and only if

$$I'(u)v = 0 \text{ for every } v \in \mathcal{T}_u\mathcal{Q}$$

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$$I'(u) = \lambda G'(u) \text{ for some } \lambda \in \mathbb{R}.$$

Such λ is called a Lagrange multiplier.

If, moreover, $I(u) = \min \{ I(v) \mid v \in H^1(\mathbb{R}^N) \text{ and } G(v) \le 0 \}$, then $\lambda \le 0$. Finally, $\lambda = 0$ if $G(u) < 0$.
Equivalently, $I'(u) + \lambda G'(u) = 0$ and $\lambda \ge 0$ (or $\lambda > 0$).

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Let $J\colon H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$J(u) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 - \frac{1}{p}|u|^p \,\mathrm{d}x = \int_{\mathbb{R}^N} \frac{1}{2}\sum_{j=1}^N \left|\frac{\partial u}{\partial x_j}\right|^2 - \frac{1}{p}|u|^p \,\mathrm{d}x$$

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$$\mathcal{S} := \left\{ u \in H^1(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} u^2 \,\mathrm{d}x = \rho^2 \right\}.$$

If u is a critical point of $J|_{\mathcal{S}}$, i.e.,

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$H^1_{\text{rad}}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N)$ for every $2 < q < \frac{2N}{N-2}$, hence $u_n \in \mathcal{S}$, $u_n \rightharpoonup u$ in $H^1_{\text{rad}}(\mathbb{R}^N) \not\Rightarrow u \in \mathcal{S}$. For this reason we consider

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$H^1_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N)$ for every $2 < q < \frac{2N}{N-2}$, hence $u_n \in \mathcal{S}$, $u_n \rightharpoonup u$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N) \not\Rightarrow u \in \mathcal{S}$. For this reason we consider

$$\mathcal{D} := \left\{ u \in H^1(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} u^2 \,\mathrm{d}x \le \rho^2 \right\}.$$

Clearly $u_n \in \mathcal{D}$, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N) \Rightarrow u \in \mathcal{D}$.

The proof ($2 < p < 2 + 4/N$)

Theorem

There exists a solution (λ, u) to (1)–(2) such that $J(u) < 0$ and $\lambda > 0$.

Gagliardo–Nirenberg inequality: for every $2 < q < \frac{2N}{N-2}$ there exists $C_{q,N} > 0$ such that for every $u \in H^1(\mathbb{R}^N)$ there holds

$$|u|_q \le C_{q,N} |\nabla u|_2^{\delta_q} |u|_2^{1-\delta_q},$$

with $\delta_q = N\left(\frac{1}{2} - \frac{1}{q}\right) \in]0,1[$. In particular, $q\delta_q < 2 \Leftrightarrow q < 2 + \frac{4}{N}$ (resp. '$=$', '$>$'). We work in $H^1_{\text{rad}}(\mathbb{R}^N)$. The norm in $H^1(\mathbb{R}^N)$ is given by

$$\|u\|^2 := |u|_2^2 + |\nabla u|_2^2.$$

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Lemma

$J|_{\mathcal{D}}$ is coercive (and bounded below).

Proof.

If $u \in \mathcal{D}$, then from the Gagliardo–Nirenberg inequality

$$J(u) = \frac{1}{2}|\nabla u|_2^2 - \frac{1}{p}|u|_p^p \geq \frac{1}{2}|\nabla u|_2^2 - \frac{C_{N,p}^p}{p}|\nabla u|_2^{p\delta_p}|u|_2^{p(1-\delta_p)}$$
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Lemma

$\inf_{\mathcal{D}} J < 0$.

For $u \in H^1(\mathbb{R}^N)$ and $s > 0$ define $s \star u(x) := s^{N/2} u(sx)$. Notice that $|s \star u|_2 = |u|_2$, hence $u \in \mathcal{D} \Rightarrow s \star u \in \mathcal{D}$.

Proof.

Fix $u \in \mathcal{D} \setminus \{0\}$. If $0 < s \ll 1$, then

$$J(s \star u) = \frac{s^2}{2} |\nabla u|_2^2 - \frac{s^{N(p/2-1)}}{p} |u|_p^p < 0$$

because $N(p/2 - 1) = p\delta_p < 2$. □

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$\inf_{\mathcal{D}} J$ is attained (i.e., there exists $u \in \mathcal{D}$ such that $J(u) = \inf_{\mathcal{D}} J$).

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Let $u_n \in \mathcal{D}$ such that $J(u_n) \to \inf_{\mathcal{D}} J$. Since $J|_{\mathcal{D}}$ is coercive, u_n is bounded, therefore there exists $u \in \mathcal{D}$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ and $u_n \to u$ in $L^p(\mathbb{R}^N)$ (up to a subsequence). Then

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Proof of the main Theorem.

Let $u \in \mathcal{D}$ such that $J(u) = \min_{\mathcal{D}} J < 0$. Then u is a critical point of $J|_{\mathcal{D}}$, i.e., there exists $\lambda \geq 0$ such that

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v + \lambda uv \, \mathrm{d}x = \int_{\mathbb{R}^N} |u|^{p-2} uv \, \mathrm{d}x$$

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Fix $u \in \mathcal{D} \setminus \{0\}$. Then (recall $p\delta_p > 2$)

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If (λ, u) solves (1), then (Nehari)

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Moreover, one can prove that (Pohožaev)

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$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = N\left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} |u|^p \, dx. \tag{4}$$

Define

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and note that for every $u \in \mathcal{M}$

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$$r := r(u) := \sqrt{N \left(\frac{1}{2} - \frac{1}{p} \right) \frac{|\nabla u|_2^2}{|u|_p^p}}.$$

Theorem

There exists a solution (λ, u) *to* (1)–(2) *such that* $J(u) > 0$ *and* $\lambda > 0$.

Lemma

$$\inf_{u \in \mathcal{M} \cap \mathcal{D}} |\nabla u|_2 > 0.$$

Proof.

$|\nabla u|_2^2 = C|u|_p^p \leq C'|\nabla u|_2^{p\delta_p}|u|_2^{p(1-\delta_p)} \leq C'\rho^{p(1-\delta_p)}|\nabla u|_2^{p\delta_p}$, $p\delta_p > 2$. □

Lemma

J *is coercive on* $\mathcal{M} \cap \mathcal{D}$ *and* $m := \inf_{\mathcal{M} \cap \mathcal{D}} J > 0$.

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Since $J(u) = C_0|u|_p^p$ if $u \in \mathcal{M} \cap \mathcal{D}$, $\frac{1}{2}|\nabla u|_2^2 = \frac{1}{p}|u|_p^p + J(u) = CJ(u)$. □

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m is attained.

Proof.

Let $u_n \in \mathcal{M} \cap \mathcal{D}$ such that $J(u_n) \to m$. Then it is bounded, hence there exist $u \in \mathcal{D}$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ and $u_n \to u$ in $L^p(\mathbb{R}^N)$ (up to a subsequence).

Next, $0 < m = \lim_n C_0 |u_n|_p^p = C_0 |u|_p^p$, therefore $u \neq 0$ and we can consider $u_r \in \mathcal{M}$.

Moreover, $r^2 = r(u)^2 = N\left(\frac{1}{2} - \frac{1}{p}\right)\frac{|u|_p^p}{|\nabla u|_2^2} \geq \lim_n N\left(\frac{1}{2} - \frac{1}{p}\right)\frac{|u_n|_p^p}{|\nabla u_n|_2^2} = 1$, thus $u_r \in \mathcal{D}$.

Finally, $m \leq J(u_r) = C_0 |u_r|_p^p = C_0 r^{-N} |u|_p^p \leq C_0 |u|_p^p = \lim_n C_0 |u_n|_p^p$
$= \lim_n J(u_n) = m$, therefore $r = 1$ and $J(u) = m$. □

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Proof of the Theorem.

Let $u \in \mathcal{M} \cap \mathcal{D}$ such that $J(u) = m > 0$. There exist $\lambda \geq 0$ and $\sigma \in \mathbb{R}$ such that $-(1+2\sigma)\Delta u + \lambda u = \left(1 + \sigma N \frac{p-2}{2}\right) |u|^{p-2}u$. Then the Nehari identity $(1+2\sigma)|\nabla u|_2^2 + \lambda |u|_2^2 = \left(1 + \sigma N \frac{p-2}{2}\right) |u|_p^p$ and the Pohožaev identity $(1+2\sigma)\frac{N-2}{2N}|\nabla u|_2^2 + \frac{\lambda}{2}|u|_2^2 = \left(1 + \sigma N \frac{p-2}{2}\right) \frac{1}{p}|u|_p^p$ hold. From the two identities we obtain $(1+2\sigma)|\nabla u|_2^2 = N\left(1 + \sigma N \frac{p-2}{2}\right)\left(\frac{1}{2} - \frac{1}{p}\right)|u|_p^p$, and from $u \in \mathcal{M}$ we obtain $\sigma\left(\frac{1}{2} - \frac{1}{p}\right)\left(N(p-2) - 4\right)|u|_p^p = 0$, whence $\sigma = 0$ and $-\Delta u + \lambda u = |u|^{p-2}u$. If $\lambda = 0$, then the Nehari and Pohožaev identities read $\frac{1}{p}|u|_p^p = \frac{N-2}{2N}|\nabla u|_2^2 = \frac{N-2}{2N}|u|_p^p$, so $\lambda > 0$ and $u \in \mathcal{S}$. □

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