Numerical methods for large-scale differential matrix equations in control theory

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Outline

- Differential Riccati equations
- The LQR finite-Horizon problem
- Krylov-subspace methods
- Krylov projection + BDF (Backward Differentiation Formulae) for DRE's
- The Hamiltonian-Exponential approach
- Lyapunov differential matrix equations
- Krylov-BDF and Krylov-exponential approaches
- Some numerical experiments.

Solving large scale differential matrix equations

- Symmetric Differential Riccati equations (continuous case)

$$\dot{X}(t) = A(t)^T X(t) + X(t)A(t) - X(t)B(t)B(t)^T X(t) + C(t)^T C(t).$$

- Differential Lyapunov matrix equations

$$\dot{X}(t) = A(t)X(t) + X(t)A(t)^T + E(t)$$

Application: Control theory, Model reductions, Linear time varying dynamical systems, ...

PART I: Differential Riccati Equations (DREs)

In thist talk, we consider the continuous-time differential Riccati equation (DRE in short) on the time interval $[0, T_f]$ of the form

$$\begin{cases} \dot{X}(t) = A^T X(t) + X(t) A - X(t) B B^T X(t) + C^T C \\ X(0) = X_0, \ t \in [0 \ T_f] \end{cases} \quad (1)$$

where X_0 is some given $n \times n$ matrix, $A \in \mathbb{R}^{n \times n}$ is assumed to be large, sparse and nonsingular, $B \in \mathbb{R}^{n \times s}$ and $C \in \mathbb{R}^{s \times n}$.
The matrices B and C are assumed to have full rank with $s \ll n$.
A, B, and C are time-independent.

The Finite-Horizon LQR problem

The Linear Quadratic Regulator (LQR) problem is a well known design technique in the theory of optimal control. It can be described as follows. For each initial state x_0, find the optimal cost $J(x_0, \hat{u})$ such that:

$$J(x_0, \hat{u}) = \inf_{u} \left\{ \int_0^{T_f} \left(y(t)^T y(t) + u(t)^T u(t) \right) dt \right\},$$

under the dynamic constrains

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A\, x(t) + B\, u(t), \quad x(0) = x_0. \\ y(t) & = & C\, x(t) \end{array} \right.$$

$x(t)$ is the state vector of dimension n, $u(t) \in \mathbb{R}^p$ the control vector and $y(t)$ the output vector of length s.

Assuming that the pair (A, B) is stabilizable (*i.e.* there exists a matrix S such that $A - B\,S$ is stable) and the pair (C, A) is detectable (i.e., (A^T, C^T) stabilizable): the optimal input control $\hat{u}(t)$ is given by

$$\hat{u}(t) = -B^T P(t)\, x(t),$$

where $P(t) \in \mathbb{R}^{n\times n}$ is the unique solution to the following differential Riccati equation

$$\dot{P} + A^T P + PA - PBB^T P + C^T C = 0; \quad P(T_f) = 0.$$

The optimal state trajectory satisfies $\dot{\hat{x}}(t) = (A - BB^T P(t))\hat{x}(t)$ and could be expressed as

$$\hat{x}(t) = e^{tA}\hat{x}_0 + \int_0^t e^{(t-\tau)A} B u(\tau) d\tau.$$

The optimal cost is given by the following quadratic function of the initial state x_0:

$$J(x_0, \hat{u}) = x_0^T P(0) x_0. \tag{2}$$

We notice that if we set $X(t) = P(T_f - t)$ and $X_0 = 0$, then

$$P(t) = X(T_f - t).$$

Then we recover the solution X to the differential Riccati equation (1) and

$$J(x_0, \hat{u}) = x_0^T X(T_f) x_0.$$

These results are summarized in the following theorem:

Theorem

Assume that the pair (A, B) *is stabilizable and the pair* (A, C) *is detectable. Then, the differential algebraic Riccati equation* (1) *has a unique positive solution* X *on* $[0,\ T_f]$ *and for any initial state* x_0*, the optimal cost of* $J(x_0, u)$ *is given by*

$$J(x_0, \hat{u}) = x_0^T X(T_f) x_0,$$

where the optimal control is given by

$$\hat{u}(t) = -B^T X(T_f - t)\hat{x}(t),$$

and the optimal trajectory is determined by

$$\dot{\hat{x}}(t) = (A - BB^T X(T_f - t))\hat{x}(t), \text{ with } \hat{x}(0) = x_0.$$

The infinite horizon LQR problem

The optimal cost is given by

$$J(x_0, u_\infty) = \inf_u \left\{ \int_0^\infty \left(y(t)^T y(t) + u(t)^T u(t) \right) dt \right\} = x_0^T X_\infty x_0,$$

where X_∞ is the unique postitive and stabilizing solution of the algebraic Riccati equation

$$A^T X_\infty + X_\infty A - X_\infty B B^T X_\infty + C^T C = 0,$$

and the optimal feedback is given by $u_\infty = -B^T X_\infty \hat{x}(t)$.

The discrete case

For the discrete case, LQR finite-horizon problem is described as follows

$$J(x_0, u) = \inf_u \left\{ \sum_0^N \left(y_k^T\, y_k + u_k^T R\, u_k \right) \right\},$$

under the discrete-dynamic constrains

$$\left\{ \begin{array}{lcl} x_{k+1} & = & A\, x_k + B\, u_k. \\ y_k & = & C\, x_k, \end{array} \right. \tag{3}$$

The optimal control is given by

$$u_k = -F_k u_k, \text{ where } F_k = (R + B^T Z_{k+1} B)^{-1} B^T Z_{k+1} A,$$

and Z_{k+1} is computed by solving a discrete-time algebraic Riccati equation

Z_{k+1} is computed by solving the following discrete-time algebraic Riccati equation

$$Z_k = A^T Z_{k+1} A - A^T Z_{k+1} B (R + B^T B)^{-1} B^T Z_{k+1} A + C^T C.$$

When N tends to infinity, we obtain the infinite-horizon discrete-time LQR, $Z_\infty = \lim_{k \to \infty} Z_k$ is the unique positive definite solution to the discrete time algebraic Riccati equation (DARE)

$$Z_\infty = A^T Z_\infty A - A^T Z_\infty B \left(R + B^T Z_\infty B \right)^{-1} B^T Z_\infty A + C^T C.$$

Low rank solutions for matrix Riccati equations

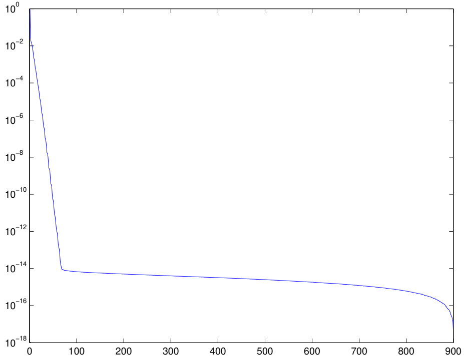

FIGURE – A=fdm900. Singular values of the exact stabilizing solution

The singular values **decay rapidaly to zero** and this suggests to seek for methods producing approximate solutions having a **low rank**.

Existing methods for large algebraic and differential cases

- Algebraic Riccati equations
  - `Block Arnoldi` : [Jaimoukha(98); J. (03); Simoncini & Szyld (12,16)]
  - `Newton-LR-ADI`:[Benner & al. 98, 05, 14]
  - `Global Arnoldi` : [J. (05)]
  - `Extended Block Arnoldi`: [Heyouni & J. (08)]
  - `EBA for Nonsymmetric CAREs` : [Bentbib, J. & Sadek (15)]
- **Differential Riccati equations**
  - `LR-ADI-Based methods`: [Benner Mena & Stillfjord (15)]
  - `EBA for Diff. Riccati equations`: [Hached & J. (16)].
  - `Expo-Krylov for Diff. Lyapunov equations`: [Hached & J. (17)]
  - `The expo-Hamiltonian approach`: [J. (17)].
  - `Rational-Krylov LQR for Large-scale DAREs`: [J. (23)].

The extended block Arnoldi algorithm

The extended block-Krylov subspace $\mathcal{K}_m(A, V)$ of $\mathbb{R}^n$ which is considered here is:

$$\mathcal{K}_m(A, V) = Range([V, A^{-1}\,V, A\,V, A^{-2}\,V, A^2\,V, \dots, A^{-m}\,V, A^{m-1}\,V]).$$

Notice that the subspace $\mathcal{K}_m(A, V)$ is a sum of two block Krylov subspaces

$$\mathcal{K}_m(A, V) = \mathbb{K}_m(A, V) + \mathbb{K}_m(A^{-1}, A^{-1}V)$$

where $\mathbb{K}_m(A, V) = Range([V, A\,V, \dots, A^{m-1}\,V])$ and $V \in \mathbb{R}^{n\times s}$.

The Extended Block Arnoldi (EBA) builds an orthonormal basis $\mathcal{V}_m$ of the Krylov subspace $\mathrm{Range}(V, A\,V, \ldots, A^{m-1}\,V, A^{-1}\,V, \ldots, (A^{-1})^m\,V)$ and a block upper Hessenberg matrix H_m whose non zeros blocks are the $H_{i,j}$.

Let $\mathcal{T}_m = \mathcal{V}_m^T\,A\,\mathcal{V}_m$ and set $\bar{\mathcal{T}}_m = \mathcal{V}_{m+1}^T\,F\,\mathcal{V}_m$, then we have

$$\begin{aligned} A\,\mathcal{V}_m &= \mathcal{V}_{m+1}\,\bar{\mathcal{T}}_m, && (4)\\ &= \mathcal{V}_m\,\mathcal{T}_m + V_{m+1}\,T_{m+1,m}\,E_m^T. && (5)\end{aligned}$$

$E_m = [O_{2s\times 2(m-1)s}, I_{2s}]^T$ is the matrix of the last $2s$ columns of the $2ms \times 2ms$ identity matrix I_{2ms}.

The BDF method for solving DREs

The BDF approximation X_{k+1} of $X(t_{k+1})$ is given by the implicit relation

$$X_{k+1} = \sum_{i=0}^{p-1} \alpha_i X_{k-i} + h\beta\mathcal{F}(X_{k+1}),$$

where $h = t_{k+1} - t_k$ is the step size, α_i and β_i are the coefficients of the BDF method as listed in Table 1 and $\mathcal{F}(X)$ is given by

$$\mathcal{F}(X) = A^T X + X A - X B B^T X + C^T C.$$

p	β	α_0	α_1	α_2
1	1	1		
2	2/3	4/3	-1/3	
3	6/11	18/11	-9/11	2/11

TABLE – Coefficients of the p-step BDF method with $p \leq 3$.

Then, X_{k+1} solves the following matrix equation

$$-X_{k+1}+h\beta(C^TC+A^TX_{k+1}+X_{k+1}A)-X_{k+1}BB^TX_{k+1}+\sum_{i=0}^{p-1}\alpha_iX_{k-i}=0,$$

which can be written as the following continuous-time algebraic Riccati equation

$$\mathcal{A}^T\,X_{k+1}+\mathcal{A}\,X_{k+1}-X_{k+1}\,\mathcal{B}\,\mathcal{B}^T\,X_{k+1}+\mathcal{C}_{k+1}^T\mathcal{C}_{k+1}=0,$$

assuming that at each timestep, X_k can be factorized as a low rank product $X_k \approx Z_kZ_k^T$, $Z_k \in \mathbb{R}^{n\times m_k}$, with $m_k \ll n$, the coefficients matrices are given by

$$\mathcal{A}=h\beta A-\frac{1}{2}I,\ \ \mathcal{B}=\sqrt{h\beta}B;\mathcal{C}_{k+1}=[\sqrt{h\beta}C,\sqrt{\alpha_0}Z_k^T,\ldots,\sqrt{\alpha_{p-1}}Z_{k+1-p}^T]^T.$$

First approach: BDF+Newton+EBA method

After using the BDF integration formulae, we solve the obtained Algebraic Riccati equation and we can use:

- Direct solvers: `care` with matlab for moderate sizes ($n < 1000$)
- For large problems: Inexact Newton-Kleinman's method combined with an iterative method (EBA or LR-ADI) for the numerical resolution of large-scale Lyapunov equations.
- Using EBA for large-scale algebraic Riccati equations [Heyouni & J. 09]

When using the Kleinman-Newton method, we obtain the following algorithm

We define a sequence of approximates to X_{k+1} as follows:

- Set $X_{k+1}^0 = X_k$
- Build the sequence $\left(X_{k+1}^l\right)_{l\in\mathbb{N}}$ defined by

$$X_{k+1}^{l+1} = X_{k+1}^l - \mathcal{F}_{X_{k+1}^l}(F(X_{k+1}^l))$$

where the Fréchet derivative $\mathcal{F}$ of F at X_{k+1}^l is given by

$$\mathcal{F}_{X_{k+1}^l}(H) = (\mathcal{A} - \mathcal{B}\,\mathcal{B}^T X_{k+1}^l)^T H + H(\mathcal{A} - \mathcal{B}\,\mathcal{B}^T X_{k+1}^l)$$

X_{k+1}^{l+1} is the solution to the Lyapunov equation

$$(\mathcal{A}-\mathcal{B}\,\mathcal{B}^T X_{k+1}^l)^T X + X(\mathcal{A}-\mathcal{B}\,\mathcal{B}^T X_{k+1}^l) + X_{k+1}^l \mathcal{B}\mathcal{B}^T X_{k+1}^l + \mathcal{C}_{k+1}^T\mathcal{C}_{k+1} = 0.$$

and we apply Extended-Block-Arnoldi (EBA) to the last Lyapunov matrix equation.

The second approach: Projecting and solving the low dimensional DRE

Apply the Extended Block Arnoldi (EBA) algorithm to the pair (A^T, C^T) and get the orthonormal matrix $\mathcal{V}_m$ and the block-Hessenberg matrix $\mathcal{T}_m = \mathcal{V}_m^T A^T \mathcal{V}_m$.
Let $X_m(t)$ be the desired low-rank approximate solution to (1) given as

$$X_m(t) = \mathcal{V}_m Y_m(t) \mathcal{V}_m^T, \tag{6}$$

and satisfying the Petrov-Galerkin orthogonality condition

$$\mathcal{V}_m^T R_m(t) \mathcal{V}_m = 0, \tag{7}$$

where $R_m(t)$ is the residual

$$R_m(t) = \dot{X} - A^T X_m(t) - X_m(t) A + X_m(t) B B^T X_m(t) - C^T C$$

Then, from (6) and (7), we obtain the low dimensional differential Riccati equation

$$\dot{Y}_m - \mathcal{T}_m Y_m - Y_m \mathcal{T}_m^T + Y_m B_m B_m^T Y_m - C_m^T C_m = 0. \quad (8)$$

- We use the BDF method for this **low-dimensional differential Riccati equation**.
- At each time-step of the BDF process, we get a low-order algebraic Riccati equation that is solved by a direct method (`care`).
- To stop the iterations without computing the intermediade approximations, we can use the result of the following theorem

Theorem

Let $X_m = \mathcal{V}_m Y_m \mathcal{V}_m^T$ *be the approximation obtained at step* m *by the Extended Block Arnoldi-BDF(p) method and* Y_m *solves the low-dimensional differential Riccati equation (8), Then the residual* R_m *satisfies*

$$\| R_m \| = \| T_{m+1,m} \hat{Y}_m \|,$$

where $\hat{Y}_m$ *is the* $2s \times 2ms$ *matrix corresponding to the last* $2s$ *rows of* Y_m.

- The approximate solution is not required at each iteration
- We compute it only when convergence is achieved and in a **factored form** (save storage and useful for applications) as follows:

Consider the singular value decomposition of the matrix

$$Y_m(t) = U \Sigma U^T$$

where Σ is the diagonal matrix of the singular values of Y_m sorted in decreasing order.
Let U_l be the $2m \times l$ matrix of the first l columns of U corresponding to the l singular values of magnitude greater than some tolerance *dtol*. We obtain the truncated singular value decomposition

$$Y_m(t) \approx U_l \Sigma_l U_l^T$$

where $\Sigma_l = \mathrm{diag}[\sigma_1, \ldots, \sigma_l]$. Setting $Z_m(t) = \mathcal{V}_k U_l \Sigma_l^{1/2}$, it follows that

$$X_m(t) \approx Z_m(t) Z_m(t)^T.$$

The following result shows that the approximation X_m is an exact solution of a perturbed differential Riccati equation.

Theorem

Let X_m be the approximate solution given by (6). Then we have

$$\dot{X}_m(t) = (A - F_m)^T X_m + X_m (A - F_m) - X_m B B^T X_m + C^T C.$$

where $F_m = V_m T_{m+1,m}^T V_{m+1}^T$.

For the error, we have

Theorem

Let X be a solution of (1) and let X_m be the approximate solution obtained at step m. The error $E_m = X - X_m$ satisfies the following differential Riccati equation

$$\dot{E}_m(t) = (A^T - XBB^T)E_m + E_m(A - BB^T X) + E_m BB^T E_m + R_m,$$

The projected Finite-horizon LQR problem

Consider the LTI system

$$\begin{cases} \dot{x}(t) &= A\,x(t) + B\,u(t), \quad x(0) = x_0. \\ y(t) &= C\,x(t) \end{cases}$$

and the projected low order dynamical system (onto the extended block Krylov subspace $\mathcal{K}_m(A^T, C^T)$):

$$\begin{cases} \dot{\tilde{x}}_m(t) &= \mathcal{T}_m\,\tilde{x}_m(t) + B_m\,\tilde{u}_m(t), \quad \tilde{x}_m(0) = x_{m,0}. \\ \tilde{y}_m(t) &= C_m\,\tilde{x}_m(t) \end{cases} \tag{9}$$

where $B_m = \mathcal{V}_m^T B$, $C_m^T = \mathcal{V}_m^T C^T = \mathcal{E}_1\,\Lambda_{1,1}$ and $x_{m,0} = \mathcal{V}_m^T x_0$.
For small values of the iteration number m,
$x_m(t) = \mathcal{V}_m \tilde{x}_m(t)$ is an approximation of the original large state $x(t)$.

Theorem

Assume at step m that $(\mathcal{T}_m, B_m)$ *is stabilizable and that* $(C_m, \mathcal{T}_m)$ *is detectable and consider the low dimension LQR problem with finite time-horizon:*
Minimize

$$J_m(x_{m,0}, \tilde{u}_m) = \int_0^{T_f} \left(\tilde{y}_m(t)^T \, \tilde{y}_m(t) + \tilde{u}_m(t)^T \, \tilde{u}_m(t) \right) \, dt, \qquad (10)$$

under the dynamic constrains (9). Then, the unique optimal feedback $\tilde{u}_{*,m}$ *minimizing the cost function (10) is given by*

$$\tilde{u}_{*,m}(t) = -B_m \widetilde{Y}_m(t) \hat{x}_m(t),$$

where $\widetilde{Y}_m(t) = Y_m(T_f - t)$ *and* Y_m *is the unique stabilizing solution of the corresponding low-dimensional differential Riccati equation..*

The optimal projected state satisfies $\dot{\tilde{x}}_m(t) = (A - BB^T \widetilde{Y}_m(t))\tilde{x}_m(t)$ which can also be expressed as

$$\tilde{x}_m(t) = e^{tA}\tilde{x}_{m,}0 + \int_0^t e^{(t-\tau)\mathcal{T}_m} B_m \tilde{u}_{*,m}(\tau) d\tau,$$

and the optimal cost is given by the following quadratic function of the initial state x_0

$$\begin{aligned} J_m(x_{m,0}, \hat{u}_m) &= x_{m,0}^T \widetilde{Y}_m(0) x_{m,0} = x_0 \mathcal{V}_m \widetilde{Y}_m(0) \mathcal{V}_m^T x_0 && (11) \\ &= x_0 P_m(0) x_0 = x_0^T X_m(T_f) x_0, && (12) \end{aligned}$$

This shows clearly that the reduced optimal cost is an approximation of the initial optimal cost.

It ca be shown that $X(t)$ converges to a symmetric and positive matrix X_∞ as $t \to \infty$:

$$X_\infty = \lim_{t\to\infty} X(t),$$

satisfying the following algebraic Riccati equation

$$A^T X_\infty + X_\infty A - X_\infty BB^T X_\infty + C^T C = 0.$$

and X_∞ is the only positive and stabilizing solution to this algebraic Riccati equation.

An Exponential-Hamiltonian approach (in progress)

Let H be the Hamiltonian matrix given by

$$H = \begin{pmatrix} A & -BB^T \\ -C^T C & -A^T \end{pmatrix}. \tag{13}$$

and consider the associated differential Riccati equation:

$$\dot{P}(t) + A^T P(t) + P(t)A - P(t)BB^T P(t) + C(t)^T C(t) = 0; \; P(T_f) = 0.$$

It can be shown that if we consider the matrix partition of e^{tH} as

$$e^{tH} = \begin{pmatrix} \Phi_1(t) & \Phi_2(t) \\ \Phi_3(t) & \Phi_4(t) \end{pmatrix},$$

the differential Riccati solution $P(t)$ can be expressed as follows

$$P(t) = \Phi_3(t - T_f)\Phi_1(t - T_f)^{-1},$$

Consider now the orthonormal matrix $\mathcal{V}_m = [V_1, \ldots, V_m]$ obtained by applying the extended block Arnoldi algorithm to the pair (A^T, C^T) Define the new matrix $\mathbb{V}_m$ by

$$\mathbb{V}_m = \begin{pmatrix} \mathcal{V}_m & 0 \\ 0 & \mathcal{V}_m \end{pmatrix}.$$

Then we have the following projected Hamiltonian matrix

$$H_m = \mathbb{V}_m^T H \mathbb{V}_m = \begin{pmatrix} \mathcal{T}_m^T & -B_m B_m^T \\ -C_m^T C_m & \mathcal{T}_m \end{pmatrix}.$$

The projected Hamiltonian matrix H_m is associated to the low dimensional differential Riccati equation

$$\dot{Y} + \mathcal{T}_m Y + Y \mathcal{T}_m^T - Y B_m B_m^T + C_m^T C_m = 0, \;\; Y(T_f) = 0.$$

This differential Riccati equation can also be computed as

$$Y_m(t) = \Phi_{3,m}(t - T_f)\Phi_{1,m}(t - T_f)^{-1},$$

where

$$e^{tH_m} = \begin{pmatrix} \Phi_{1,m}(t) & \Phi_{2,m}(t) \\ \Phi_{3,m}(t) & \Phi_{4,m}(t) \end{pmatrix},$$

and the approximate solution $P_m(t)$ is given as

$$P_m(t) = \mathcal{V}_m Y_m(t) \mathcal{V}_m^T.$$

For this approach, no need of an intergation method such as BDF.
Upper bounds for the error,..., is in progress.

Part II: Large-scale differential Lyapunov matrix equations

We consider the Lyapunov differential matrix equation (DRE in short) of the form

$$\dot{X}(t) = A(t)\,X(t) + X(t)\,A^T(t) + B(t)B(t)^T \tag{14}$$

and $X(t_0) = X_0, \;\; t \in [t_0, \; T_f],$
The problem (14) is equivalent to the following

$$\dot{x}(t) = \mathcal{A}(t)x(t) + b(t), \tag{15}$$

where $\mathcal{A} = I \otimes A(t) + A(t) \otimes I$, $x(t) = vec(X(t))$.
Then one can use an integration method to solve (15) . However, solving numerically the problem (15) is not recommended for large problems.

Theorem

The unique solution of the general Lyapunov differencial equation

$$\dot{X}(t) = A(t)\,X + X\,A(t)^T + M(t); \;\; X(0) = X_0$$

is defined by

$$X(t) = \Phi_A(t, t_0) X_0 \Phi_A^T(t, t_0) + \int_{t_0}^{t} \Phi_A(t,\tau) M(\tau) \Phi_A^T(t,\tau) d\tau.$$

where the transition matrix $\Phi_A(t, t_0)$ *is the unique solution of the problem*

$$\dot{\Phi}_A(t, t_0) = A(t)\Phi_A(t, t_0), \;\; \Phi_A(t_0, t_0) = I.$$

Futhermore, if A is constant, then

$$X(t) = e^{(t-t_0)A} X_0 e^{(t-t_0)A^T} + \int_{t_0}^{t} e^{(t-\tau)A} M(\tau) e^{(t-\tau)A^T} d\tau, \;\; t \in \mathbb{R}.$$

Projecting onto a Krylov subspace

We first apply the extended block Arnoldi algorithm to the pair (A, B) to get the matrices $\mathcal{V}_m$ and $\bar{\mathcal{T}}_m$; A is assumed to be time-invariant!!
Let $X_m(t)$ be the desired approximate solution given as

$$X_m(t) = \mathcal{V}_m Y_m(t) \mathcal{V}_m^T,$$

satisfying the Petrov-Galerkin orthogonality condition

$$\mathcal{V}_m^T R_m(t) \mathcal{V}_m = 0,$$

We obtain the low dimensional differential Lyapunov equation

$$\dot{Y}_m(t) - \mathcal{T}_m Y_m - Y_m \mathcal{T}_m^T - B_m B_m^T = 0,$$

solved by the BDF integration method.

Theorem

The error $E_m = X - X_m$ *satisfies the following equation*

$$\dot{E}_m(t) = AE_m(t) + E_m(t)A^T + R_m(t), \qquad (16)$$

where $R_m(t)$ *is the residual given by*
$R_m(t) = \dot{X}_m(t) - A\,X_m(t) - X_m(t)\,A^T - BB^T.$

Theorem

Assume that A *is stable matrix and* $X(0) = X_m(0)$. *Then we have the following upper bound*

$$\| \, E_m(t) \, \| \leq (t - t_0) \;\| \, R_m \, \|_\infty \; e^{2\mu_2(A)t},$$

where $\mu_2(A) = \dfrac{1}{2}\lambda_{max}(A + A^T)$ *is the 2-logarithmic norm*

A second approach: using an approximation of the matrix exponential

We give a second approach for computing approximate solutions to large-scale Lyapunov differential equations. Coming to the expression of the exact solution as

$$X(t) = e^{(t-t_0)A} X_0 e^{(t-t_0)A^T} + \int_{t_0}^{t} e^{(t-\tau)A} BB^T e^{(t-\tau)A^T} \, d\tau, \;\; t \in \mathbb{R}. \quad (17)$$

We can see that one possibility for approximating $X(t)$, is to approximate the term $e^{(t-\tau)A}B$ and then use a quadrature method to compute the desired approximate solution.

Let $\mathcal{V}_m = [V_1, \ldots, V_m]$ be the matrix whose block columns are obtained by the extended block Arnoldi algorithm. Then, an approximation to $Z = e^{(t-\tau)A}\,B$ can be obtained as

$$Z_m(t) = \mathcal{V}_m e^{(t-\tau)\mathcal{T}_m}\,\mathcal{V}_m^T B$$

where $\mathcal{T}_m = \mathcal{V}_m A \mathcal{V}_m^T$ and $\mathcal{V}_m^T\,B = \mathcal{E}_1\,\Lambda_{1,1}$(QR-decomposition). Therefore, the term appearing in the integral expression can be approximated as

$$e^{(t-\tau)A} B B^T e^{(t-\tau)A^T} \approx Z_m(t) Z_m(t)^T.$$

If for simplicity we set $X_0 = 0$, an approximation to the solution of the differential Lyapunov equation can be expressed as

$$X_m(t) = \mathcal{V}_m G_m(t) (\mathcal{V}_m)^T,$$

where

$$G_m(t) = \int_{t_0}^{t} \widetilde{G}_m(\tau) \widetilde{G}_m^T(\tau) d\tau, \tag{18}$$

and $\widetilde{G}_m(\tau) = e^{(t-\tau)\mathcal{T}_m} \mathcal{E}_1 \Lambda_{1,1}$.

- Notice that we have

$$\mathcal{V}_m^T R_m(t) (\mathcal{V}_m) = 0$$

- For the practical computation of $e^{(t-\tau)\mathcal{T}_m}$ we can use the 'scaling and squaring method', by Higham `expm`.
- $G_m(t)$ can be computed using a quafrature formulae.

We can also state the following result.

Theorem

Let $G_m(t)$ as defined by (18), then it satisfies the following low-order differential Lyapunov matrix equation

$$\dot{G}_m(t) = \mathcal{T}_m G_m(t) + G_m(t)\mathcal{T}_m + \widetilde{C}_m \widetilde{C}_m^T$$

- $G_m(t)$ could be computed as a solution of a differential Lyapunov matrix equation (which is also the first approach)
- OR by a matrix exponential approximation + a quadrature formulation.
- The two approches are theoreticaly equivalent.

Balanced truncation for linear time-varying dynamical systems

Consider the linear-time varying (LTV) dynamical systems

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A(t)x(t) + B(t)u(t),\ x(0) = 0, \\ y(t) & = & C(t)x(t), \end{array} \right. \tag{19}$$

The LTV dynamical system (19) can also be denoted as

$$\Sigma(t) \equiv \left[\begin{array}{c|c} A(t) & B(t) \\ \hline C(t) & 0 \end{array} \right].$$

The reduced order LTV dynamical system can be stated as follows

$$\Sigma_m \begin{cases} \dot{x}_m(t) = A_m(t)x_m(t) + B_m(t)u(t) \\ \\ y_m(t) = C_m(t)x_m(t) \end{cases}$$

where $x_m \in \mathbb{R}^m$, $y_m \in \mathbb{R}^s$, $A_m \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times p}$ and $C_m \in \mathbb{R}^{s\times m}$ with $m \ll n$.

Problem: : How to choose the reduced order model?; A_m ...

The reduced order dynamical system should be constructed such that

- the norm $\|y - y_m\|$ should be computable and close to zero.
- Some properties of the original system such as stability should be preserved.
- The method should be efficient for large problems.

Methods: two classes

- Balanced truncation (solve large diff Lyapunov or diff Riccati equations)
- Moment matching (use Krylov based methods: Arnoldi-type, Lanczos-type,...)

One of the well known methods for contructing such reduced-order dynamical systems is the balanced truncation method for LTV systems.

This method requires the LTV controllability and observability Gramians $P(t)$ and $Q(t)$ defined as the solutions of the differential Lyapunov matrix equations

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)^T + B(t)B(t)^T,\ P(0) = 0,$$

and

$$\dot{Q}(t) = A^T(t)P(t) + P(t)A(t) + C(t)^T C(t),\ Q(T_f) = 0.$$

Let us see now how to construct the low order model. Consider the Cholesky decompositions of the Gramians $P(t)$ and $Q(t)$:

$$P(t) = L_c L_c^T, \quad Q(t) = L_o(t) L_o(t)^T,$$

and the singular value decomposition of $L_c(t)^T L_o(t)$ as

$$L_c(t)^T L_o(t) = \mathcal{Z}(t)\Sigma(t)\mathcal{Y}(t)^T.$$

We set

$$V_m(t) = L_o(t)\mathcal{Y}_m(t)\Sigma_m(t)^{-1/2} \text{ and } W_m(t) = L_c(t)\mathcal{Z}_m(t)\Sigma_m(t)^{-1/2},$$

where $\Sigma_m(t) = diag(\sigma_1(t), \ldots, \sigma_m(t))$; $\mathcal{Z}_m(t)$ and $\mathcal{Y}_m(t)$ correspond to the leading m columns of the matrices $\mathcal{Z}(t)$ and $\mathcal{Y}(t)$.

The matrices A_m, B_m and C_m of the reduced LTV system are such that

$$W_m(t)^T V_m(t) A_m(t) = V_m(t)^T A(t) W_m(t) - V_m(t)^T \dot{W}_m(t),$$

and

$$B_m(t) = V_m(t)^T B(t),\ C_m(t) = C(t) W_m(t).$$

The state $x(t)$ is approximated by $\tilde{x}(t) = V_m(t) x_m(t)$.

For large problems, instead of using the Cholesky factors, it is more convenient to use the low-rank decompositions

$$P(t) \approx Z_{m,p}(t)Z_{m,p}^T(t), \quad Q(t) \approx Z_{m,q}(t)Z_{m,q}^T(t).$$

- Another possibility is to use directly the matrix-basis of the extended block Arnoldi to construct the reduced order model (without solving differential Lyapunov equations)
- We have no formulation for the upper bounds of the errors!!!
- Using trnasfer functions!!

Numerical Examples

We reported the results given by the following approaches :

- EBA-PDF: EBA+ BDF to the projected problem.
- EBA-EXP: EBA-EXP + quadrature formulae, for diff. Lyapunov Eqs.
- BDF-Newton= BDF + Newton+ (EBA or LR-ADI) for Lyapunov equations
- LRSOS: Low-rank second-order splitting.

Example 1. The matrix A was obtained from the 5-point discretization of the operators

$$L_A = \Delta u - f_1(x,y)\frac{\partial u}{\partial x} + f_2(x,y)\frac{\partial u}{\partial y} + g_1(x,y),$$

on the unit square $[0,1] \times [0,1]$ with homogeneous Dirichlet boundary conditions. The number of inner grid points in each direction is n_0 and the dimension of the matrix A was $n = n_0^2$.

We set $f_1(x,y) = 10xy$, $f_2(x,y) = e^{x^2 y}$, $f_3(x,y) = 100y$, $f_4(x,y) = x^2 y$, $g_1(x,y) = 20xy$ and $g_2(x,y) = x\,y$.

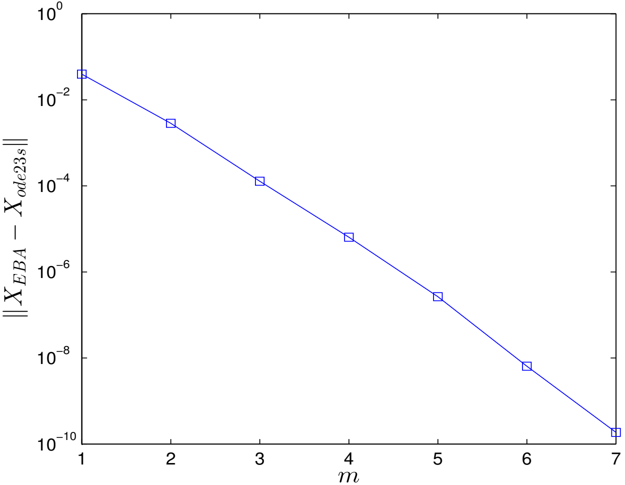

FIGURE – Norms of the errors $\|X_{EBA}(T_f) - X_{ode23s}(T_f)\|_F$.

size(A)	ode23s	EBA-BDF	BDF-Newton-EBA	BDF-Newton-Lyap
49	39.3	0.87	2.41	0.77
100	1470	1.20	3.31	4
900	$--$	3.07	186	964

TABLE – Execution times for ode23s, EBA+BDF(2), BDF(2)+Newton, BDF(2)+Newton-Lyap

Example 2. In this example, we considered the same problem.We reported the times, residual norms and number of the Extended Arnoldi iterations for various sizes of A.

size(A)	Execution time (s)	Residual norms	Iterations (m)
6400	86.7	2.31×10^{-9}	19
10000	125.4	3.07×10^{-9}	21

TABLE – Times (s), residual norms and the number of Extended-Arnoldi iterations (m) for the EBA-BDF(2) method

The next figure shows the norm of the residual R_m versus the number m of Extended Block Arnoldi iterations for the $n = 6400$ case.

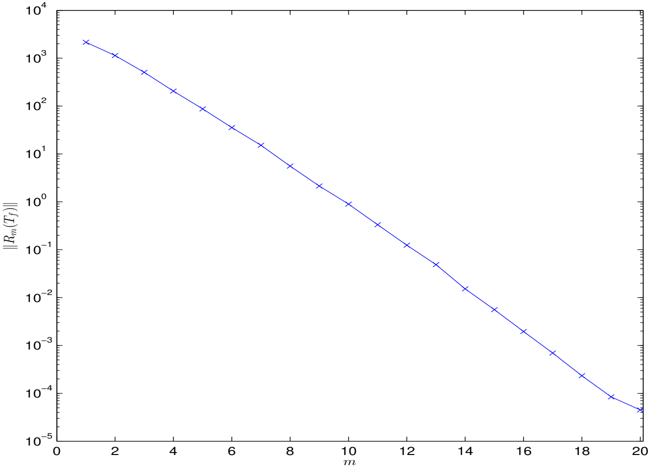

FIGURE – Residual norms $\|R_m(Tf)\|$ versus number of Extended-Arnoldi iterations (m)

Example 3 This example comes from the autonomous linear-quadratic optimal control problem of one dimensional heat flow

$$\begin{aligned}
\frac{\partial}{\partial t}x(t,\eta) &= \frac{\partial^2}{\partial \eta^2}x(t,\eta) + b(\eta)u(t) \\
x(t,0) &= x(t,1) = 0, t > 0 \\
x(0,\eta) &= x_0(\eta), \eta \in [0,1] \\
y(x) &= \int_0^1 c(\eta)x(t,\eta)d\eta, x > 0.
\end{aligned}$$

Using a standard finite element approach based on the first order B-splines, we obtain the following ordinary differential equation

$$\left\{ \begin{array}{lcl} M\dot{\mathrm{x}}(t) & = & K\mathrm{x}(t) + Fu(t) \\ y(t) & = & C\mathrm{x}(t), \end{array} \right. \tag{20}$$

where the matrices M and K are given by:

$$M = \frac{1}{6n} \begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 4 \end{pmatrix}, K = -\alpha\, n \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

Using the semi-implicit Euler method, we get the following discrete dynamical system

$$(M - K)\, \dot{x}(t) = M\, x(t) + \; Fu_k.$$

We set $A = -(M - \Delta t K)^{-1}\, M$ and $B = \Delta t\, (M - \Delta t K)^{-1}\, F.$

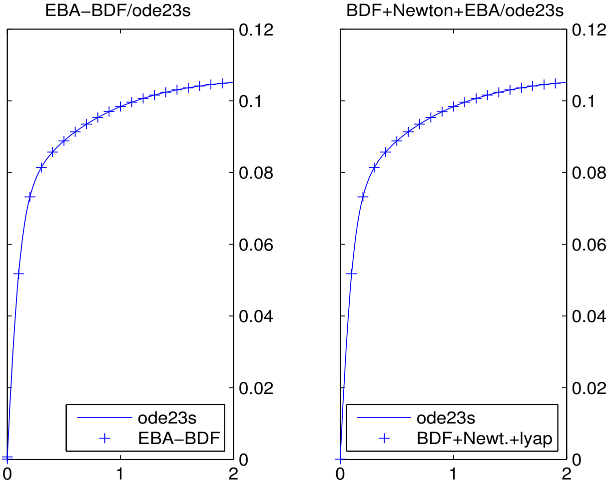

FIGURE – Computed values of $X_{11}(t)$ versus the time $t \in [0, 2]$ for ode23s and EBA-BDF (left) and for ode23s with BDF(2)+Newton+EBA (right)

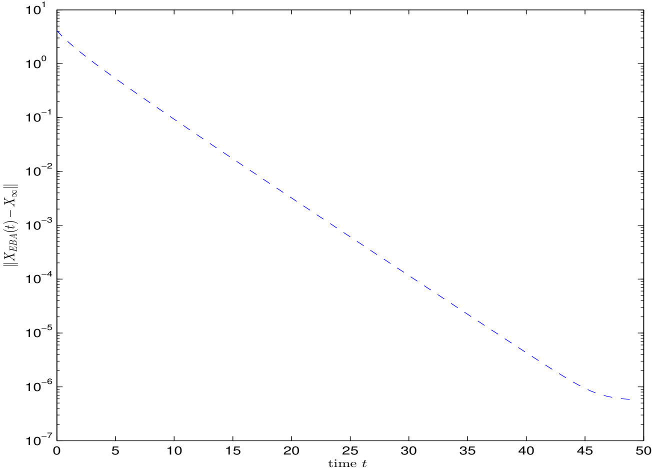

FIGURE – Error norms $\| X_{EBA-BDF}(t) - X_\infty \|$ for $t \in [0, 50]$.

size(A)	Times (s)	Residual norms	Iterations (m)
2500	$17s$	5.35×10^{-10}	11
4900	$64s$	7.62×10^{-10}	14
6400	$163s$	1.43×10^{-10}	19
10000	$457s$	2.35×10^{-10}	21

TABLE – EBA-BDF(2) method: times (s), residual norms, number of EBA iterations (m).

Example 4 For this experiment, we compared our EBA-BDF method with LRSOS (Stillfjord 2015) and BDF-LR-ADI (Benner-Mena-Saak 2015). We considered the RAIL problem with $n = 1357$ and $n = 5177$, from the IMTEK collection.

size n	EBA-BDF	LRSOS	BDF-LR-ADI
1357	130.4 s	145.5 s	259.2 s
5177	578 s	919.4 s	6482.3 s

TABLE – Execution times for the RAIL problem

We notice that for EBA-BDF, we have an expression of the residual norm (without computing the approximation) which is not the case for LRSOS and BDF-LR-ADI

A test for Diff. Lyapunov equations

Example 5 For this experiment, we compred the results obtained by EBA-BDF and EBA-Expon for differential Lyapunov matrix equations with $B = rand(n, 3)$ and for the test problems: fdm ($n = 14.400$) and Heat Flow ($n = 10.000$).

Test problem	EBA-BDF	EBA-EXP	Res. Norms
fdm 14400×14400	82s	19s	2.5×10^{-10}
Heat Flow 10000×10000	62s	11s	1.7×10^{-9}

TABLE – Times and residual norms for EBA-BDF and EBA-EXP

Some open problems

- More studies for LTV systems: Errors, Transfer functions,.....
- Time varying Second-order dynamical systems,
- Model-reduction for Large-scale nonlinear dynamical systems,
- Proper Orthogonal Decomposition (POD) vs Krylov subspaces.

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