Discrete group actions on 3-manifolds and embeddable Cayley complexes

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Joint with George Kontogeorgiou

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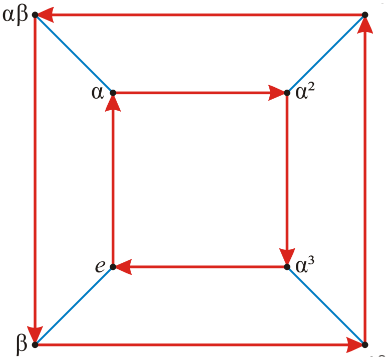
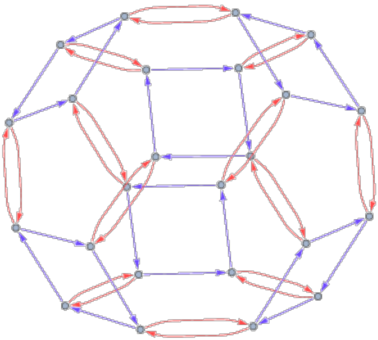
Most of the talk was on the blackboard. This is an extended version of the slides containing the main results appearing on the blackboard.

Planar Cayley graphs





Theorem (Folklore)

A finite group admits a faithful action (by homeomorphisms or isometries) on $\mathbb{S}^2$ if and only if it has a planar Cayley graph.

Whitney's unique embedding theorem

Theorem (Whitney '32)

Any two embeddings of a 3-connected, planar graph G into $\mathbb{S}^2$ coincide up to homeomorphism.

More precisely, for every two embeddings $\phi, \psi : G \to \mathbb{S}^2$, there is a homeomorphism $\alpha : \mathbb{S}^2 \to \mathbb{S}^2$ such that $\psi = \alpha \circ \phi$.

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Theorem (G & Kim '23)

Any two embeddings of a simply-connected, locally 3-connected, 2-complex into $\mathbb{S}^3$ coincide up to homeomorphism.

Theorem (G & Kontogeorgiou '23+)

For a finite group Γ the following are equivalent:

1. *Γ admits a faithful action by homeomorphisms/smooth maps/isometries on $\mathbb{S}^3$;*
2. *Γ admits a generalised Cayley complex X with a Γ-invariant planar rotation system;*
3. *Γ admits a generalised Cayley complex X with an embedding $\phi : X \to \mathbb{S}^3$ with Γ-invariant planar rotation system.*

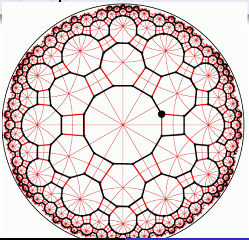
Theorem (Dinkelbach & Leeb '09, Pardon '21)

A finite group admits a faithful action by homeomorphisms/smooth maps/isometries on $\mathbb{S}^3$ if and only if it is isomorphic to a subgroup of the orthogonal group $O(4)$.

Theorem (Levinson & Maskit ’75, G ’20, Bowditch ’21)

For a finitely generated group Γ, the following are equivalent:

1. Γ *admits a faithful, properly discontinuous action by homeomorphisms on a planar surface;*
2. Γ *has a Cayley graph admitting a consistent embedding;*
3. Γ *has a Cayley multi-graph admitting a consistent embedding every facial path of which is finite;*
4. Γ *admits a faithful, properly discontinuous,* ***co-compact*** *action by homeomorphisms on the sphere, the plane $\mathbb{R}^2$, the open annulus, or the Cantor sphere.*



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Theorem (Svarc–Milnor lemma '68)

Let G be a group acting by isometries on a proper length space X such that the action is properly discontinuous and cocompact. Then G is quasi-isometric to X.

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Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

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Three-manifolds are greatly more complicated than surfaces, and I think it is fair to say that until recently there was little reason to expect any analogous theory for manifolds of dimension 3 (or more)—except perhaps for the fact that so many 3-manifolds are beautiful. The situation has changed, so that I feel fairly confident in proposing the

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In §2, I will describe some theorems which support the conjecture, but first some explanation of its meaning is in order.

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Theorem (Perelman '03)

It is true.

Theorem

Let M be a connected, simply connected, topological 3-manifold. Suppose M admits a properly-discontinuous, co-compact action by homeomorphisms. Then M is homeomorphic to one of the following four spaces: (i) $\mathbb{S}^3$, (ii) $\mathbb{R}^3$, (iii) $\mathbb{S}^2 \times \mathbb{R}$, or (iv) the Cantor 3-sphere.

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A finite graph is planar if and only if it does not contain a homeomorphic copy of K_5 or $K_{3,3}$.

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A finite, simply-connected 2-complex embeds into $\mathbb{S}^3$ if and only if it admits a planar rotation system.

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This is 'equivalent' to the Poincaré conjecture!