SHADOWS OF FINITE GROUPS

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e.g: for $n=3$ $\dim \rho_{(3)} = 1$, $\dim \rho_{(1^3)} = q^3$, $\dim \rho_{(21)} = q(q+1)$

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$$\Rightarrow \quad \left. \frac{|GL_n(q)|}{(q-1)^n} \right|_{q=1} = n! = \mathfrak{S}_n$$

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and $\dim \chi_\lambda = \dim \rho_\lambda \big|_{q=1}$

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* blocks ...

Classification of Weyl groups W

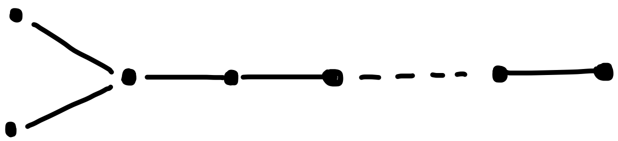
A_{n-1}

PGL_n, SL_n

B_n, C_n

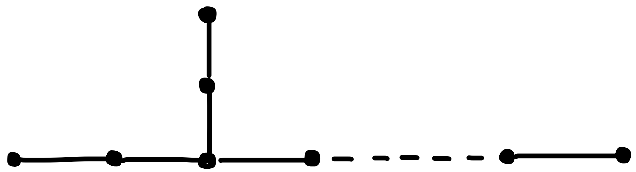
SO_{2n+1}, Sp_{2n}

D_n



SO_{2n}

E_n
$(n = 6, 7, 8)$



F_4

G_2

Another example: $G_2(q)$

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6 reflections : $s, t, tst, sts, tstst, ststs$

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10 unipotent characters with dimensions

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Rmk : $10 > 6 = \#\mathrm{Irr}\,W$ (of dimension $1,1,1,1,2,2$)

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[Lefschetz] Can be computed from $\# X(\mathbb{F}_{q^n})$

(more generally one would use $\# X^{gF_n}$ $g \in \mathbb{G}(\mathbb{F}_q)$)

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but the number $\# X^{F^m}$ does, as a polynomial in $\mathbb{Z}[q]$

(for most of the complex ref. groups)

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Using $\# X^{F^n}$ we find another one of dimension $\pm \frac{\zeta}{1-\zeta^2} q(q-1)$