An analogy between number theory and spectral geometry

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Historical roots in number theory

- Leopold Kronecker (1880): characterize polynomials by factorisation patterns modulo primes; i.e., number fields by splitting behaviour of primes, i.e., by their zeta function $\zeta_K(s) = \prod (1 - N(\mathfrak{p})^{-s})^{-1}$.
- Mihály Bauer (1903): if K, L Galois over $\mathbf{Q}$, then $K = L$ iff the same set of primes is totally split in K and L.
- Fritz Gassmann (1926): $\zeta_K = \zeta_L \iff \mathrm{Ind}_{G_K}^{G_\mathbf{Q}} \mathbf{1} \cong \mathrm{Ind}_{G_L}^{G_\mathbf{Q}} \mathbf{1}$, and this is strictly weaker than $K \cong L$ in general.



Manifold problem

Problem

Let M_1 and M_2 be closed Riemannian manifolds. If M_1 and M_2 are Laplace-isospectral, are M_1 and M_2 isometric?

John Milnor: **No** for 16-dimensional tori

$$M_i = \Gamma_i \backslash \mathbf{R}^{16} \text{ with } \Gamma_1 = E_8 \oplus E_8 \text{ and } \Gamma_2 = E_{16}$$

Marie-France Vignéras: **No** for compact Riemann surfaces
$\Gamma_i \backslash \mathbf{H}^2$ where Γ_i are the one-units in two different maximal orders in the quaternion algebra D over $\mathbf{Q}(\sqrt{10})$ ramified at $7, 11, 11+3\sqrt{10}$ and one real place (the unramified real place gives $\Gamma_i \hookrightarrow D \otimes \mathbf{R} \cong \mathrm{M}_2(\mathbf{R})$)

Toshikazu Sunada: **No** by weak conjugation
$M_i = H_i \backslash M$ with different homology groups, where M has a finite isometry group G with two non-conjugate subgroups H_i of fixed point free elements that are weakly conjugate: $\mathrm{Ind}_{H_1}^{G} 1 \cong \mathrm{Ind}_{H_2}^{G} 1$.

Milnor 1964, Vignéras 1980, Sunada 1985

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Main result

Theorem (C-Peyerimhoff 2020)

Suppose that M_1 and M_2 are closed connected oriented Riemannian manifolds such that

1. *there is a diagram*

   $$M_1 \xrightarrow{p_1} M_0 \xleftarrow{p_2} M_2$$

   where M_0 is a developable Riemannian orbifold.
2. *The action of the covering group of the normal closure of the diagram is homologically wide.*

Then M_1 and M_2 are isometric over M_0 if and only if the multiplicity of zero in the spectra (= dimension of kernel) of a finite number of specific twisted Laplacians on M_1 and M_2 is equal.

Example (Gaßmann 1926)

$H_1 = \langle (12)(34), (13)(24) \rangle$, $H_2 = \langle (12)(34), (12)(56) \rangle$ are weakly conjugate in $G = S_6$. M_0 compact $g = 2$ Riemann surface; need 24 equalities for 180-dim. reps.

What is a twisted Laplacian?

Data

- G a finite group
- A (fixed-point free) G-cover $M' \to M$
- A group representation: $\rho\colon \pi_1(M) \twoheadrightarrow G \to \mathrm{GL}_n(\mathbf{C})$

Definition

The *twisted Laplacian* Δ_ρ is the operator on

$$C^\infty_\rho(M', \mathbf{C}^n) := \{\vec{f} \in C^\infty(M', \mathbf{C}^n) \mid \vec{f}(\gamma x) = \rho(\gamma)\vec{f}(x), \forall\, x \in M', \gamma \in G\}.$$

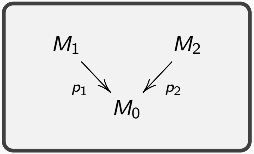
defined by

$$\Delta_\rho f = \Delta_{M'} \vec{f}.$$

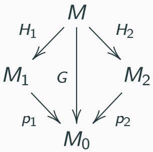
Example

$\chi\colon \mathbf{Z}/2\mathbf{Z} \to \{\pm 1\}$, $M' \xrightarrow{2:1} M$, $C^\infty(M') = C^\infty_{\mathrm{even}}(M') \oplus C^\infty_{\mathrm{odd}}(M') = C^\infty(M) \oplus C^\infty_{\mathrm{odd}}(M')$, $\Delta_{M'} = \Delta_M \oplus \Delta_\chi$.

Diagram condition



- Does not always exist.
- Implies commensurability: there exists a diagram



- Diagram exists for commensurable hyperbolic manifolds if and only if they are *not* arithmetic ($\Leftarrow$ Margulis)
- Milnor: true; Vignéras: false; Sunada: true.

Homological wideness condition

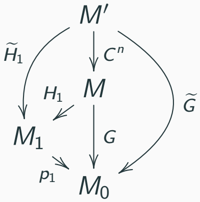
Definition

The action of G is homologically wide if $\Psi\colon G \to \mathrm{Aut}(\mathrm{H}_1(M, \mathbf{Q}))$ contains the regular representation.

- $\iff \exists c \in \mathrm{H}_1(M, \mathbf{Q})\colon \dim \mathrm{span}_{\mathbf{Q}}\{g(c)\colon g \in G\} = |G|$.
- M_0 2-manifold: $\iff \chi_{M_0} < 0$ (all groups) ($\Leftarrow$ Lefschetz)
- For any G, exists closed hyperbolic 3-manifold with action of G that is homologically wide or trivial ($\Leftarrow$ surgery).
- Only the trivial group acts homologically widely on a locally symmetric space of rank $\geqslant 2$ ($\Leftarrow$ Kazhdan property (T)).
- Milnor: no; Vignéras: yes; Sunada: depends.

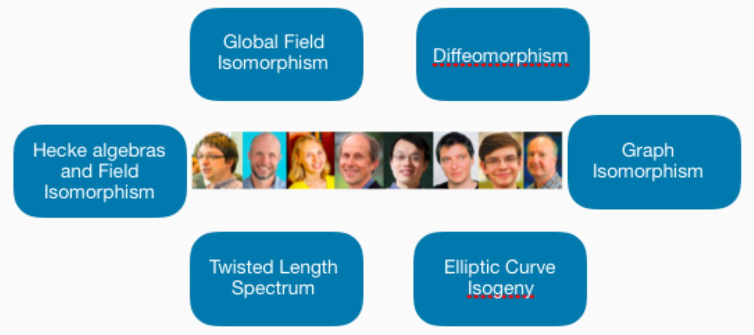
Steps in the proof

- For characters χ_i on H_i, replace $\mathrm{Ind}_{H_1}^G \chi_1 \cong \mathrm{Ind}_{H_2}^G \chi_2$ by a spectral condition (idea of Solomatin for global function fields): $\mathrm{mult}_0(\mathrm{Sp}_{M_i}(\Delta_{\overline{\chi}_i \otimes \mathrm{Res}_{H_i}^G \mathrm{Ind}_{H_j}^G \chi_j}))$ *is independent of* $i, j = 1, 2$.
- Homological wideness $\Rightarrow$ existence of a covering manifold of M realising a *wreath product* $\widetilde{G}$ of $C := \mathbf{Z}/\ell\mathbf{Z}$ with G for the action of G on the coset space G/H_1. Need ℓ coprime to $|H_1|$.



- On the wreath product, there exists a *solitary* character Ξ of order $\ell \geqslant 3$ for which $\mathrm{Ind}_{\widetilde{H}_1}^{\widetilde{G}} \chi_1 \cong \mathrm{Ind}_{\widetilde{H}_2}^{\widetilde{G}} \Xi \Rightarrow H_1$ and H_2 conjugate in $\widetilde{G}$ (construction of Bart de Smit).
- Find we need at most $2\ell|\mathrm{Hom}(H_2, \mathbf{C}^*)|$ equalities of $\mathrm{mult}_0(\Delta_\rho)$ for some $[G : H_1]$-dimensional representations ρ.

Let's twist again?



C-de Jong 2012, C 2013, C-Karemaker 2017
C-de Smit-Li-Marcolli-Smit 2019, Smit 2020, C-Groenland-Peyerimhoff 2023(?)

Final words...

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