Numerical methods for large-scale differential matrix Riccati equations in control theory

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Abstract: In thist talk, we consider the continuous-time differential Riccati equation (DRE in short) on the time interval $[0, T_f]$ of the form

$$\begin{cases} \dot{X}(t) = A^T X(t) + X(t) A - X(t) B B^T X(t) + C^T C \\ X(0) = X_0, \ t \in [0 \ T_f] \end{cases} \tag{1}$$

where X_0 is some given $n \times n$ matrix, $A \in \mathbb{R}^{n\times n}$ is assumed to be large, sparse and nonsingular, $B \in \mathbb{R}^{n\times s}$ and $C \in \mathbb{R}^{s\times n}$. The matrices B and C are assumed to have full rank with $s \ll n$; A, B, and C are also assumed time-independent. If $C = 0$ then the problem is reduced to differentail Lyapunov matrix equations (DLE). These equations appear in many problems such in control theory for finite horison or in model reduction for large scale time-dependent dynamical systems. The Linear Quadratic Regulator (LQR) problem is a well known design technique in the theory of optimal control. It can be described as follows.

For each initial state x_0, find the optimal cost $J(x_0, \hat{u})$ such that:

$$J(x_0, \hat{u}) = \inf_u \left\{ \int_0^{T_f} \left(y(t)^T y(t) + u(t)^T u(t) \right) dt \right\},$$

under the dynamic constrains

$$\begin{cases} \dot{x}(t) &= A\,x(t) + B\,u(t), \quad x(0) = x_0. \\ y(t) &= C\,x(t) \end{cases}$$

$x(t)$ is the state vector of dimension n, $u(t) \in \mathbb{R}^p$ the control vector, $y(t)$ the output vector of length s. (DRE) are generally solved by Backward Differentiation Formula (BDF) (or Rosenbrock) methods leading to large scale algebraic Lyapunov or Riccati equation which has to be solved for each timestep. However, these techniques are not effective for large problems because, at each timestep, one has to solve large scale algebraic Lyapunov or Riccati equations which could be very expensive. Here, we propose new approaches based on projection on small subspaces. For differential Lyapunov equations, we construct approximate solutions from the exponential expression of the exact solution using Krylov subspace methods to approximate exponential of a matrix times a block of vectors. For differential Riccati equations, we project the problem onto a small block Krylov or extended block Krylov subspace and then and obtain a low-dimentional differential algebraic Riccati equation. The latter matrix differential problem is solved by Backward Differentiation Formula (BDF) method and the obtained solution is used to reconstruct an approximate solution of the original problem. We give some theoretical results and simple expressions of the residual norms allowing the implementation of a stop test in order to limit the dimension of the projection spaces. Uppers bounds for the norm of the errors are also given. The proposed numerical experiments show the effectiveness of our approaches.

Key words: Extended block Krylov; Low rank approximation; Differential Riccati equations; LQR problem; Lyapunov equations