

# Normalised solutions for beginners

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14<sup>th</sup> November 2023

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## The problem

Let  $N \geq 2$ ,  $2 < p < \frac{2N}{N-2}$  with  $p \neq 2 + \frac{4}{N}$ ,  $\rho > 0$ , and consider the problem

$$-\Delta u + \lambda u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \quad (1)$$

where  $\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ , paired with the constraint

$$\int_{\mathbb{R}^N} u^2 \, dx = \rho^2, \quad (2)$$

with  $\lambda \in \mathbb{R}$  to be determined.

If  $(\lambda, u)$  solves (1)–(2), we call it a **normalised solution**.

The problem (1)–(2) appears when looking for solutions to

$$\begin{cases} i \frac{\partial \Phi}{\partial t} - \Delta \Phi = |\Phi|^{p-2} \Phi & \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} |\Phi(x, t)|^2 \, dx = \rho^2 & \text{(conserved in time)} \end{cases}$$

as **standing waves**, i.e.,

$$\Phi(x, t) = e^{-i\lambda t} u(x).$$

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# The settings

Let us define

$$H^1(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) \mid \frac{\partial u}{\partial x_j} \in L^2(\mathbb{R}^N) \forall j = 1, \dots, N \right\}.$$

What is  $\frac{\partial u}{\partial x_j}$  for  $u \in L^2(\mathbb{R}^N)$ ? It is defined via

$$\int_{\mathbb{R}^N} \frac{\partial u}{\partial x_j} \varphi \, dx = - \int_{\mathbb{R}^N} u \frac{\partial \varphi}{\partial x_j} \, dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N).$$

One can prove  $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$  for every  $2 \leq q \leq \frac{2N}{N-2}$  ( $2 \leq q < \infty$  if  $N = 2$ ).

Take  $v \in H^1(\mathbb{R}^N)$ . If we multiply (1) by  $v$  and integrate by parts, we get

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If  $I: H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$  and  $u, v \in H^1(\mathbb{R}^N)$ , we denote

$$I'(u)v := \lim_{t \rightarrow 0} \frac{I(u + tv) - I(u)}{t}.$$

Then  $I'(u) \in (H^1(\mathbb{R}^N))'$ .

We write  $I \in C^1(H^1(\mathbb{R}^N))$  iff  $u \mapsto I'(u)$  is continuous.

We say that  $u \in H^1(\mathbb{R}^N)$  is a critical point for  $I \in C^1(H^1(\mathbb{R}^N))$  iff  $I'(u) = 0$ , i.e.,  $I'(u)v = 0$  for every  $v \in H^1(\mathbb{R}^N)$ .

**Recall:** If  $A \subset H^1(\mathbb{R}^N)$  is open and  $u \in A$ , then  $I|_A'(u) = 0 \Leftrightarrow I'(u) = 0$ .

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Let  $\mathcal{Q} := \{ u \in H^1(\mathbb{R}^N) \mid G(u) = 0 \} \neq \emptyset$  for some  $G \in C^1(H^1(\mathbb{R}^N))$  such that  $G'(u) \neq 0$  for every  $u \in \mathcal{Q}$ .

If  $u \in \mathcal{Q}$ , then

$$I|_{\mathcal{Q}}'(u) = 0$$

if and only if

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$$I'(u) = \lambda G'(u) \text{ for some } \lambda \in \mathbb{R}.$$

Such  $\lambda$  is called a **Lagrange multiplier**.

If, moreover,  $I(u) = \min \{ I(v) \mid v \in H^1(\mathbb{R}^N) \text{ and } G(v) \leq 0 \}$ , then  $\lambda \leq 0$ . Finally,  $\lambda G'(u) = 0$  if  $G(u) < 0$ .

Equivalently,  $I'(u) + \lambda G'(u) = 0$  and  $\lambda \geq 0$ .



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$H_{\text{rad}}^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$  for every  $2 < q < \frac{2N}{N-2}$ , hence  $u_n \in \mathcal{S}$ ,  $u_n \rightarrow u$  in

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$$u \in \mathcal{S} \quad \text{and} \quad J'(u)v = 0 \quad \text{for every } v \in \mathcal{T}_u \mathcal{S},$$

then there exists  $\lambda \in \mathbb{R}$  such that  $(\lambda, u)$  is a solution to (1)–(2).

$H_{\text{rad}}^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$  for every  $2 < q < \frac{2N}{N-2}$ , hence  $u_n \in \mathcal{S}$ ,  $u_n \rightharpoonup u$  in

$H_{\text{rad}}^1(\mathbb{R}^N) \not\Rightarrow u \in \mathcal{S}$ . For this reason we consider

$$\mathcal{D} := \left\{ u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} u^2 \, dx \leq \rho^2 \right\}.$$

Clearly  $u_n \in \mathcal{D}$ ,  $u_n \rightharpoonup u$  in  $H^1(\mathbb{R}^N) \Rightarrow u \in \mathcal{D}$ .



# The proof ( $2 < p < 2 + 4/N$ )

## Theorem

There exists a solution  $(\lambda, u)$  to (1)–(2) such that  $J(u) < 0$  and  $\lambda > 0$ .

**Gagliardo–Nirenberg inequality:** for every  $2 < q < \frac{2N}{N-2}$  there exists  $C_{q,N} > 0$  such that for every  $u \in H^1(\mathbb{R}^N)$  there holds

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with  $\delta_q = N(\frac{1}{2} - \frac{1}{q}) \in ]0, 1[$ . In particular,  $q\delta_q < 2 \Leftrightarrow q < 2 + \frac{4}{N}$  (resp. ‘=’, ‘>’). We work in  $H_{\text{rad}}^1(\mathbb{R}^N)$ . The norm in  $H^1(\mathbb{R}^N)$  is given by

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$J|_{\mathcal{D}}$  is coercive (and bounded below).

Proof.

If  $u \in \mathcal{D}$ , then from the Gagliardo–Nirenberg inequality

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From Nehari + Pohožaev we obtain

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = N \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} |u|^p dx. \quad (4)$$

Define

$$\mathcal{M} := \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid (4) \text{ holds} \right\}$$

and note that for every  $u \in \mathcal{M}$

$$J(u) = \underbrace{\frac{1}{p} \left( \frac{N}{4}(p-2) - 1 \right)}_{C_0} \int_{\mathbb{R}^N} |u|^p dx > 0.$$

If  $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ , then  $u_r := u(r \cdot) \in \mathcal{M}$ , where

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$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = N \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} |u|^p dx. \quad (4)$$

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## Theorem

There exists a solution  $(\lambda, u)$  to (1)–(2) such that  $J(u) > 0$  and  $\lambda > 0$ .

## Lemma

$$\inf_{u \in \mathcal{M} \cap \mathcal{D}} |\nabla u|_2 > 0.$$

## Proof.

$$|\nabla u|_2^2 = N\left(\frac{1}{2} - \frac{1}{p}\right)|u|_p^p \lesssim |\nabla u|_2^{p\delta_p} |u|_2^{p(1-\delta_p)} \leq \rho^{p(1-\delta_p)} |\nabla u|_2^{p\delta_p} \text{ with } \rho\delta_p > 2. \quad \square$$

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$J$  is coercive on  $\mathcal{M} \cap \mathcal{D}$  and  $m := \inf_{\mathcal{M} \cap \mathcal{D}} J > 0$ .

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$m$  is attained.

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Let  $u_n \in \mathcal{M} \cap \mathcal{D}$  such that  $J(u_n) \rightarrow m$ . Then it is bounded, hence there exist  $u \in \mathcal{D}$  such that  $u_n \rightarrow u$  in  $H^1(\mathbb{R}^N)$  and  $u_n \rightarrow u$  in  $L^p(\mathbb{R}^N)$  (up to a subsequence).

Next,  $0 < m = \lim_n C_0 |u_n|_p^p = C_0 |u|_p^p$ , therefore  $u \neq 0$  and we can consider  $u_r = u(r \cdot) \in \mathcal{M}$ .

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Let  $u \in \mathcal{M} \cap \mathcal{D}$  such that  $J(u) = m > 0$ . There exist  $\lambda \geq 0$  and  $\sigma \in \mathbb{R}$  such that  $-(1 + 2\sigma)\Delta u + \lambda u = \left(1 + \sigma N \frac{p-2}{2}\right) |u|^{p-2} u$ . Then the Nehari identity  $(1 + 2\sigma)|\nabla u|_2^2 + \lambda|u|_2^2 = \left(1 + \sigma N \frac{p-2}{2}\right) |u|_p^p$  and the Pohožaev identity  $(1 + 2\sigma) \frac{N-2}{2N} |\nabla u|_2^2 + \frac{\lambda}{2} |u|_2^2 = \left(1 + \sigma N \frac{p-2}{2}\right) \frac{1}{p} |u|_p^p$  hold. From the two identities we obtain  $(1 + 2\sigma)|\nabla u|_2^2 = N \left(1 + \sigma N \frac{p-2}{2}\right) \left(\frac{1}{2} - \frac{1}{p}\right) |u|_p^p$ , and from  $u \in \mathcal{M}$  we obtain  $\sigma \left(\frac{1}{2} - \frac{1}{p}\right) (N(p-2) - 4) |u|_p^p = 0$ , whence  $\sigma = 0$  and  $-\Delta u + \lambda u = |u|^{p-2} u$ . If  $\lambda = 0$ , then the Nehari and Pohožaev identities read  $\frac{1}{p} |u|_p^p = \frac{N-2}{2N} |\nabla u|_2^2 = \frac{N-2}{2N} |u|_p^p$ , so  $\lambda > 0$  and  $u \in \mathcal{S}$ .  $\square$



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