Partial differential equations/Calculus of variations

A maximum principle for the system $\Delta u - \nabla W(u) = 0$

Un principe du maximum pour le système $\Delta u - \nabla W(u) = 0$

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Article history:
Received 5 August 2015
Accepted after revision 1 April 2016
Available online 14 April 2016

Presented by Haïm Brézis

A maximum principle is established for minimal solutions to the system $\Delta u - \nabla W(u) = 0$, with a potential $W$ vanishing at the boundary of a closed convex set $C_0 \subset \mathbb{R}^m$, which is either $C^2$ smooth or coincides with a point $\{a\}$.

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1. Introduction and statement of the maximum principle

We will consider the system

$$\Delta u - \nabla W(u) = 0, \quad u : A \to \mathbb{R}^m, \quad A \subset \mathbb{R}^n,$$

which is the Euler–Lagrange equation of the free energy functional $J(u; A) = \int_A \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) dx$. Assuming that the potential $W$ satisfies:

(a) $W \in C^1(\mathbb{R}^m; \mathbb{R})$, $W(a) = 0$ for some $a \in \mathbb{R}^m$, $W \geq 0$, 
(b) there exists $r_0 > 0$ such that for every $\xi \in \mathbb{R}^m$ with $|\xi| = 1$: $(0, r_0) \ni r \to W(a + r\xi)$ is strictly increasing,

the following maximum principle was proved in [1].
Theorem 1.1. Let $A \subset \mathbb{R}^n$ be an open, connected, bounded set, with $\partial A$ Lipschitz, and suppose that $v(\cdot) \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$ minimizes $J(u; A)$ subject to its boundary conditions on $\partial A$:

$$J(v; A) = \min \{ J(u; A) : u = v \text{ on } \partial A \}.$$

Then, if there holds $|v(x) - a| \leq r$ on $\partial A$, for some $r > 0$ with $0 < 2r \leq r_0$, there also holds $|v(x) - a| \leq r$ on $A$.

The scope of this paper is to establish a generalization of Theorem 1.1, by considering potentials $W$ that vanish at the boundary of a closed convex set $C_0 \subset \mathbb{R}^m$, which is either $C^2$ smooth or coincides with a point $\{a\}$. We mention that the convexity assumption is essential for similar problems involving systems of PDEs (cf. [6,12,13]). Assuming that

(i) $W \in C^1(\mathbb{R}; \mathbb{R})$ with $W \mid_{a \in C_0} = 0$, $W \geq 0$ on $\mathbb{R}^m \setminus C_0$,

(ii) there exists $r_0 > 0$ such that for every outer normal vector $\xi$ at the point $p \in \partial C_0$ ($|\xi| = 1$, if $C_0 = \{a\}$): $(0, r_0) \ni r \to W(p + r\xi)$ is increasing, and moreover $W(p + r\xi) > 0$,

we can state the following maximum principle.

Theorem 1.2. Let $A \subset \mathbb{R}^n$ be an open, connected, bounded set, with $\partial A$ Lipschitz, and $v \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$. Suppose that $v$ minimizes $J(u; A)$ subject to its boundary conditions on $\partial A$,

$$J(v; A) = \min \{ J(u; A) : u = v \text{ on } \partial A \}.$$

Then, if there holds

$$d(x) := d(v(x), C_0) \leq r \text{ on } \partial A, \text{ for some } r > 0 \text{ with } 0 < 2r \leq r_0,$$

where $d$ is the Euclidean distance, there also holds

$$d(x) \leq r \text{ on } A.$$

Moreover, the attainment of equality in (3) at an interior point of $A$:

$$d(\hat{x}) = r, \text{ for some } \hat{x} \in A,$$

implies that $d(x) = r, \forall x \in A$, and in addition if $C_0$ is strictly convex, $v(x) \equiv \text{constant}$ in $A$.

The following extension is also true:

Theorem 1.3. Let $A \subset \mathbb{R}^n$ be an open, connected, bounded set, with $\partial A$ Lipschitz, and let $\partial D \neq \emptyset$ be a Lipschitz portion of $\partial A$. Assume that $v(\cdot) \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$ minimizes $J(u; A)$ subject to its boundary conditions on $\partial D$: $J(v; A) = \min \{ J(u; A) : u = v \text{ on } \partial D \}$. Then, the condition $d(v(x), C_0) \leq r$ on $\partial D$, $0 < 2r \leq r_0$, implies the same conclusions as in Theorem 1.2 above.

Note that the strict monotonicity assumption (b) in Theorem 1.1 has been weakened. Theorem 1.2 is a corollary of the replacement result established in Lemma 3.2 below. For $u(\cdot) \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$ satisfying the boundary condition (2), it is shown that if $d(u(x), C_0) > r$ on a non-negligible subset, then there exists a map $\tilde{u}$ coinciding with $u$ on $\partial A$, and having less energy. To construct the competitor $\tilde{u}$, we utilize the following decomposition of $u$:

$$u(x) = p(x) + (u(x) - p(x)),$$

where $p(x) := p(u(x))$ is the projection of $u(x)$ onto $C_0$. Next, we define the distance function $d(x) := d(u(x), C_0) = |u(x) - p(x)|$, and consider a deformation of $u$ of the form:

$$\tilde{u}(x) = p(x) + g(d(x))(u(x) - p(x)),$$

where $g : \mathbb{R} \to [0, \infty)$ is a suitable locally Lipschitz function. We point out that the above expression of $\tilde{u}$ by-passes the difficulty encountered in [1], which utilizes the normal vector: $n(x) := \frac{u(x) - a}{|u(x) - a|}$ if $u(x) \neq a$, and $n(x) := 0$ if $u(x) = a$, and the polar representations of $u$ (respectively $\tilde{u}$): $u(x) = a + |u(x) - a|n(x)$ (resp. $\tilde{u}(x) = a + f(|u(x) - a|)n(x)$), with $f : \mathbb{R} \to [0, \infty)$, locally Lipschitz. Indeed, from [6], the computation of $|\nabla u|$ is elementary. In addition, the decomposition (5) allows us to deal in one shot with the point case (where $p(x) = a$, cf. Theorem 1.1), as well as with the case of smooth convex sets.

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1. We are indebted to A. Savas-Halilaj for suggesting the use of the usual maximum principle at this point. The strong maximum principle part of the theorem follows from the usual maximum principle as developed in [12] and [13].
2. Comparison with the usual maximum principle

The theorems above are different from the usual maximum principle in the following respects:

(a) $W$ is not convex, hence the usual maximum principle is not valid even for the scalar case $m = 1$;
(b) the monotonicity condition (ii) on $W$ is extremely mild, and allows applicability in situations where degeneracy is natural (cf. [3]);
(c) the usual maximum principle is a calculus fact and applies to all solutions. This is not true for the result above.

Consider the O.D.E.

$$u'' - W'(u) = 0, \quad u : \mathbb{R} \to \mathbb{R}, \quad W : \mathbb{R} \to \mathbb{R}, \quad W(u) = \frac{1}{4}(u^2 - 1)^2. \quad (7)$$

Notice that (7) has periodic solutions for $r > 0$ as small as desired such that $\min_{\mathbb{R}} u = -1 + r$ and $\max_{\mathbb{R}} u = 1 - r$. By choosing $A = (0, T)$ such that $u(0) = u(T) = 1 - r$, and $C_0 = \{1\}$, we see that Theorem 1.1 does not apply to these solutions. Similarly, for the potential $H(u) = 1 + \cos(\pi u)$, and for $C_0 = [1, 3]$, the previous periodic solution $u$ does not satisfy the maximum principle of Theorem 1.2.

The following example shows that Theorem 1.1 does not apply even to local minimizers (stable solutions to (1), defined in terms of the definiteness of sign of the second variation). Consider the scalar P.D.E.

$$\epsilon^2 \Delta u - W'(u) = 0 \quad (8)$$

where $W$ is as in (7) above and $\Omega$ is a dumbbell domain. It is well known (cf. [5,9,10]) that for $\epsilon > 0$ sufficiently small, (8) has a stable solution which on the left and right of the neck is as close to $-1$ and $+1$, respectively, as desired (by taking $\epsilon > 0$ sufficiently small). By choosing the set $A = \Omega \setminus B$, with $B \subseteq \Omega$ a closed ball located on the right of the neck, and $\partial_B A = \partial B$, we can secure that $|u(x) - 1| \leq r$ on $\partial_B A$, and therefore we see that Theorem 1.3 does not apply.

The solutions to (1) for which Theorems 1.2 and 1.3 apply are usually called minimal. They have the property that they minimize the free energy with respect to their Dirichlet values on the boundary of any open bounded subset of their domain of definition. This is reminiscent of a familiar property of minimal surfaces. That it is shared by the minimal solutions is not surprising, since the functional $J$ is linked to the perimeter functional if scaled appropriately. More precisely, it is well known that the Gamma limit under a blow-down of $J$ is the perimeter functional (cf. [11] for the scalar case, and [2] for the vector), and that the level sets of the rescaled minimizers converge to minimal surfaces.

3. Proof of the maximum principle

In what follows $C_0 \subset \mathbb{R}^m$ is a closed convex set, which is either $C^2$ smooth or coincides with a point $\{a\}$. We first compute $|\nabla \tilde{u}|^2$ for the map $\tilde{u}$ defined in (6), utilizing the properties of the projection $p(x)$.

**Proposition 3.1.** Let $A \subset \mathbb{R}^n$ open and bounded, with Lipschitz boundary, let $u(\cdot) \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$, and let $\tilde{u}(x) = p(x) + g(d(x))(u(x) - p(x))$, where $p(x) := p(u(x))$ is the projection of $u(x)$ onto $C_0$, $d(x) := d(u(x)), C_0 = |u(x) - p(x)|$, and $g : \mathbb{R} \to [0, \infty)$ is a locally Lipschitz function. Then, $\tilde{u}(\cdot) \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$, and

$$|\nabla \tilde{u}|^2 = |\nabla p|^2 + (f'(d))^2|\nabla d|^2 + (g(d))^2(|\nabla (u - p)|^2 - |\nabla d|^2) + 2g(d)\sum_{i=1}^n (p_{x_i}, u_{x_i} - p_{x_i}),$$

where we have set $f(s) := sg(s)$. In addition, if $|f'| \leq 1$ and $0 \leq g \leq 1$, then $|\nabla \tilde{u}|^2 \leq |\nabla u|^2$.

The proof of the maximum principle is based on the following cut-off lemma:

**Lemma 3.2.** Let $A \subset \mathbb{R}^n$, open, bounded, connected, with $\partial A$ Lipschitz, and let $W$ satisfy Hypotheses (i), (ii) above. Suppose that $u(\cdot) \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$. If the following two conditions hold,

1. $d(x) := d(u(x)), C_0 \leq r$ on $\partial A, 0 < 2r \leq r_0$.
2. $L^n(A \cap \{d(x) > r\}) > 0 (L^n(E), the n-dimensional Lebesgue measure),$

then, there is $\tilde{u}(\cdot) \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$ such that

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2 We want to thank Prof. N.D. Alikakos for providing the examples of this section.
Proof. Case 1. We will first establish the lemma under the additional hypothesis:

\[ d(x) \leq r_0. \]  \hfill (10)

The argument here is easy since \( u \) stays in the monotonicity region of \( W \) about \( C_0 \). Let \( f(s) = \min\{s, r\} = g(s)s \). Clearly, \( |f'| \leq 1 \) and \( 0 \leq g \leq 1 \). By Proposition 3.1, \( \tilde{u}(x) = p(x) + g(d(x))(u(x) - p(x)) \) satisfies

\[ \int_A |\nabla \tilde{u}(x)|^2dx \leq \int_A |\nabla u(x)|^2dx. \]  \hfill (11)

Having a closer look we see that in case of equality in (11), then

\[ 0 = \int_A |\nabla \tilde{u}|^2dx - \int_A |\nabla u|^2dx \leq \int_A |\nabla d|^2((f'(d))^2 - 1)dx \leq - \int_{A \cap \{d > r\}} |\nabla d|^2dx, \]

from which it follows that \( \nabla d = 0 \) a.e. on \( A \cap \{d > r\} \), and therefore \( \nabla (d - d) = 0 \) a.e. on \( A \), where \( \tilde{d}(x) = f(d(x)) \) via (6).

Since \( d - d \in W^{1,2}(A) \), we have by connectedness \( \tilde{d}(x) - d(x) = \text{constant} \) a.e. in \( A \), and from \( d - d = 0 \) on \( \partial A \) in the sense of trace, we obtain \( \tilde{d}(x) - d(x) = 0 \) a.e. in \( A \), in contradiction to assumption (II) in Lemma 3.2. Therefore we have strict inequality in (11).

On the other hand, since \( f(d) = g(d)d \leq d \leq r_0 \), we have by (ii):

\[ \int_A W(p(x) + g(d(x))(u(x) - p(x)))dx \leq \int_A W(p(x) + (u(x) - p(x)))dx, \]

hence Case 1 is settled. Notice that in this case the strictness in (9c) was obtained via the gradient term.

Case 2. Assume

\[ \mathcal{L}^n(A \cap \{d > r_0\}) > 0. \]  \hfill (12)

Consider the following cut-off functions:

\[ \alpha(s) := \begin{cases} 1 & \text{for } s \leq r \\ \frac{2r-s}{r} & \text{for } r \leq s \leq 2r \\ 0 & \text{for } s \geq 2r, \end{cases} \]

\( f(s) = \min\{s, r\} \alpha(s) = g(s)s \).

Again, it is clear that \( |f'| \leq 1 \) and \( 0 \leq g \leq 1 \), thus \( \tilde{u}(x) = p(x) + g(d(x))(u(x) - p(x)) \) satisfies thanks to Proposition 3.1:

\[ |\nabla \tilde{u}(x)|^2 \leq |\nabla u(x)|^2. \]  \hfill (13)

We note in passing that \( \tilde{u} \) is a reflection of \( u \) along \( d(u, C_0) = r \), and thus (13) is expected. Unlike in Case 1, the strictness of the inequality in (9c) will follow from the potential term. We will need the following proposition.

Proposition 3.3. (’Continuity’ of Sobolev functions, cf. [4].) Let \( A \subset \mathbb{R}^n \), open, bounded and connected, with Lipschitz boundary, and assume that \( f \in W^{1,2}(A; \mathbb{R}) \) satisfies

\[ f \leq \hat{r} \text{ on } \partial A \text{ (in the sense of trace) and } \mathcal{L}^n(A \cap \{\hat{s} < f\}) > 0 \text{ for some } \hat{r} < \hat{s}. \]  \hfill (14)

Then, \( \mathcal{L}^n(A \cap \{\hat{r} < f \leq \hat{s}\}) > 0 \).

Proof. Let \( \sigma, \tau : A \to \mathbb{R} \) be defined by

\[ \sigma(x) = \min\{f(x), \hat{s}\} = \begin{cases} f(x) & \text{for } x \in E_1 := A \cap \{f \leq \hat{r}\} \\ \hat{s} & \text{for } x \in E_3 := A \cap \{\hat{s} < f\}, \end{cases} \]

\[ \tau(x) = \max\{\sigma(x), \hat{r}\} = \begin{cases} \hat{r} & \text{for } x \in E_1 \\ \hat{r} & \text{for } x \in E_3. \end{cases} \]
Suppose, for the sake of contradiction, that $\mathcal{L}^n(A \cap \{ f < \hat{s} \}) = 0$. Therefore, $\tau$ is a step function. Thus $\nabla \tau = 0$ a.e. in $A$. On the other hand, $\sigma$, $r$ are in $W^{1,2}(A; \mathbb{R})$ (cf. [8, p. 130]). This and the connectedness of $A$ imply that $\tau \equiv \text{constant}$ (cf. [7, p. 307]). Hence $\tau \equiv \hat{s}$ since $\mathcal{L}^n(E_3) > 0$. It follows that $\mathcal{L}^n(E_1) = 0$ and $f > \hat{s}$ a.e. in $A$. Thus, $f \geq \hat{s}$ on $\partial A$ in the sense of trace, which is contradicting (14). The proof is complete. \hfill $\Box$

**Conclusion:** Let $\epsilon > 0$ such that $W(u) > 0$ on $r_0 \leq d(u, C_0) \leq r_0 + \epsilon$. We define the sets

$$E_1 := A \cap \{ d \leq r_0 \}, \quad E_2 := A \cap \{ r_0 < d \leq r_0 + \epsilon \}, \quad E_3 := A \cap \{ d > r_0 + \epsilon \}.$$ 

From (12), we obtain that in the event that $\mathcal{L}^n(E_2) = 0$, then necessarily $\mathcal{L}^n(E_3) > 0$. But $d \leq r < r_0$ on $\partial A$, hence by Proposition 3.3:

$$\mathcal{L}^n(E_2) > 0. \quad (15)$$

Therefore, (15) holds under any circumstances. On $A \cap \{ 0 \leq d \leq 2r \}$ we have:

$$W(\bar{u}(x)) = W(p(x) + g(d(x)(u(x) - p(x)))) \leq W(p(x) + (u(x) - p(x))) = W(u(x)),$$

since $g(d) = f(d) \leq d \leq 2r \leq r_0$. On the other hand, on $A \cap \{ d > 2r \}$ we have $0 = W(\bar{u}(x)) \leq W(u(x))$, while on $E_2$: $0 = W(\bar{u}(x)) < W(u(x))$. Therefore, we have $J(\bar{u}; A) < J(u; A)$ and the proof of Lemma 3.2 is complete. \hfill $\Box$

Now the proof of Theorem 1.2 is straightforward.

**Proof of Theorem 1.2.** We proceed by contradiction. So suppose that (3) does not hold, hence $\mathcal{L}^n(A \cap \{ d(v(x), C_0) > r \}) > 0$. But this contradicts the minimality of $v$ by Lemma 3.2. Thus (3) holds. Next, suppose (4) holds and notice that $d^2 = d^2(\nu, C_0)$ satisfies

$$\Delta(d^2)(x) = 2d(x)\nabla d(x), \quad \nabla W(d^2)(x)) + \sum_{i=1}^{n} D^2(d^2)(v(x)) (v_{x_i}(x), v_{x_i}(x)) \geq 0,$$

for every $x \in A$ such that $0 < d(v(x), C_0) \leq r_0$, thanks to Hypothesis (ii) on $W$, to the convexity of the function $u \to d^2(u, C_0)$ in the complement of $C_0$, and to the fact that the minimizer $\nu$ solves system (1). By the strong maximum principle, it follows that $d(v, C_0)$ is constant in $A$. In addition, if $C_0$ is strictly convex, then the function $u \to d^2(u, C_0)$ is strictly convex in the complement of $C_0$. This implies that

$$\Delta(d^2)(x) \geq \epsilon |\nabla v(x)|^2, \quad \forall x \in A : 0 < d(v(x), C_0) \leq r_0,$$

for some $\epsilon > 0$. Thus, $\nabla v \equiv 0$ since $d(v, C_0)$ is constant, and as a consequence $v$ is constant in $A$. \hfill $\Box$

**Proof of Theorem 1.3.** First we note that if in Lemma 3.2, specifically in condition (1), one replaces $\partial A$ with $\partial \hat{A}$, then the same conclusion (9) holds, where $\hat{A}$ is now replaced with $\hat{A}$. The argument is completely altered. Similarly, in Proposition 3.3, $\partial A$ is replaced with $\partial \hat{A}$, without change in the proof. The proof of Theorem 1.3 is complete. \hfill $\Box$

**Acknowledgements**

Panayotis Smyrnelis was partially supported by Fondo Basal CMM-Chile, FONDECYT postdoctoral grant No. 3160055, and through the project PDEGE Partial Differential Equations Motivated by Geometric Evolution, co-financed by the European Union European Social Fund (ESF) and national resources, in the framework of the program Aristeia of the Operational Program Education and Lifelong Learning of the National Strategic Reference Framework (NSRF).

**References**


