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Gradient estimates for semilinear elliptic systems and other related results

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A periodic connection is constructed for a double well potential defined in the plane. This solution violates Modica's estimate as well as the corresponding Liouville theorem for general phase transition potentials. Gradient estimates are also established for several kinds of elliptic systems. They allow us to prove the Liouville theorem in some particular cases. Finally, we give an alternative form of the stress–energy tensor for solutions defined in planar domains. As an application, we deduce a (strong) monotonicity formula.

Keywords: Modica's estimate; Liouville theorem; gradient estimates; entire solutions to elliptic systems; stress–energy tensor; monotonicity formula

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1. Introduction

In this paper we study the possibility of extending the Modica estimate (see [11]) to the vector case. The Modica estimate states that, for a non-negative potential $W \in C^2(\mathbb{R}, \mathbb{R})$ and for every bounded entire solution $u \in C^3(\mathbb{R}^n, \mathbb{R})$ of the equation

$$\Delta u = W'(u), \tag{1.1}$$

the following inequality holds:

$$\tfrac{1}{2}|\nabla u(x)|^2 \leqslant W(u(x)) \quad \forall x \in \mathbb{R}^n. \tag{1.2}$$

A particular case occurs when $n = 1$. Then, for the bounded solutions $u\colon \mathbb{R} \to \mathbb{R}$ of the ordinary differential equation (ODE)

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = W'(u), \tag{1.3}$$

the *Hamiltonian* $H = \frac{1}{2}|u_x|^2 - W(u)$ is a non-positive constant. This law, which expresses the conservation of mechanical energy, follows by integration of (1.3).

The Modica estimate has many corollaries (see [4, 11]). The two most important are the following.

(1) A Liouville-type theorem: if $u\colon \mathbb{R}^n \to \mathbb{R}$ is a bounded solution of (1.1) such that $W(u(x_0)) = 0$ for some $x_0 \in \mathbb{R}^n$, then u is a constant.

(2) The strong monotonicity formula: according to which, for every bounded solution $u\colon \mathbb{R}^n \to \mathbb{R}$ of (1.1) and every $x \in \mathbb{R}^n$, the quotient

$$\frac{1}{r^{n-1}} \int_{B(x,r)} [\tfrac{1}{2}|\nabla u|^2 + W(u)]\,\mathrm{d}x$$

is an increasing function of $r > 0$ ($B(x,r) \subset \mathbb{R}^n$ denotes the ball centred at x of radius r).

The hypothesis that the solutions are entire is essential for proving the Modica estimate. Other gradient bounds can be obtained for solutions of (1.1) defined in proper domains of $\mathbb{R}^n$ (see [8]).

In the vector case, for non-negative potentials $W \in C^2(\mathbb{R}^m, \mathbb{R})$, and for bounded entire solutions $u \in C^3(\mathbb{R}^n; \mathbb{R}^m)$ of the system

$$\Delta u = \nabla W(u), \tag{1.4}$$

the Modica estimate no longer holds. This is a well-known fact for the Ginzburg–Landau potential $W\colon \mathbb{R}^m \to \mathbb{R}$, $W(u) = \frac{1}{4}(|u|^2 - 1)^2$ (see [7, 10]). In the present paper we give a counterexample that violates both the Modica estimate and the Liouville-type theorem for a double well potential defined in the plane (see § 2). In § 3, we establish gradient estimates for several kinds of elliptic systems following the method of Caffarelli *et al*. [4]. Since there are no general estimates in the vector case, we show how to obtain gradient bounds in various situations. Our aim is to present a flexible technique that can easily be adapted to a more general context, or to study more specific problems. That is why, after stating several abstract theorems, we focus on the Ginzburg–Landau system (3.17), and give, in this particular case, a gradient bound that is sharp asymptotically. From these estimates, we can, under certain assumptions, deduce the Liouville-type theorem and the confinement of all bounded solutions in a determined region.

In § 4, for solutions to (1.4) defined in planar domains, we introduce a new tool (see [13]) that is equivalent to the stress–energy tensor (see [1,2]). More precisely, we associate to every solution $u\colon \mathbb{R}^2 \supset \Omega \to \mathbb{R}^m$ of (1.4) a function $U\colon \mathbb{R}^2 \supset \Omega \to \mathbb{R}$, which solves the equation $\Delta U = 4W(u)$. We show that the Modica estimate implies the convexity of U, and give as an application a (strong) monotonicity formula for all bounded solutions $u\colon \mathbb{R}^2 \to \mathbb{R}$ of (1.1).

2. Construction of a periodic connection for a double well potential in the plane

We shall construct a double well potential, $W\colon \mathbb{R}^2 \to \mathbb{R}$, such that

(i) $W(a^\pm) = 0$ with $a^\pm = (\pm 2, 0)$, $W(u) > 0$ for $u \neq a^\pm$,

(ii) $D^2W(a^\pm)$ is a positive definite matrix,

(iii) W is symmetric with respect to the coordinate axes,

and a solution $u\colon \mathbb{R} \to \mathbb{R}^2$ of the ODE $\mathrm{d}^2u/\mathrm{d}x^2 = \nabla W(u)$ such that

(i) for all $x \in \mathbb{R}$, $u(x + T) = u(x)$ for some $T > 0$ (that is, u is periodic),

(ii) $u(0) = a^+$ and $u(\frac{1}{2}T) = a^-$ (u connects the minima of W),

(iii) the derivative of u at $x = 0$ or $x = \frac{1}{2}T$ does not vanish.

Clearly, since $|(\mathrm{d}u/\mathrm{d}x)(0)|^2/2 > W(u(0)) = 0$, this solution violates the Modica estimate as well as the Liouville-type theorem (since $W(u(0)) = 0$, u is bounded and not constant). We point out that $(\mathrm{d}u/\mathrm{d}x)(0)$ cannot vanish, since otherwise u would be constant in view of the uniqueness result for ODEs. To construct the solution and the potential, we proceed step by step.

STEP 1. We consider first a C^∞-smooth closed curve Γ in the plane, which is symmetric with respect to the coordinate axes and such that $\{(\pm 2, u_2) \colon u_2 \in [-1, 1]\} \subset \Gamma$. Γ will be the trajectory of our solution. We denote by n the inward normal to Γ, and by (e_1, e_2) the canonical basis of $\mathbb{R}^2$.

STEP 2. In a neighbourhood of $a^\pm$, we define W as follows:

$$W(u) = 2\lambda\rho(|u - a^\pm|^2) \quad \text{for } |u_1 \mp 2| \leqslant 1,\ |u_2| \leqslant 1,$$

where $\lambda > 0$ is a constant to be chosen, and $\rho \colon [0, \infty) \to [0, \frac{1}{2}]$ is a smooth increasing function such that

$$\rho(\alpha) = \begin{cases} \alpha & \text{for } 0 \leqslant \alpha \leqslant \frac{1}{4}, \\ \frac{1}{2} & \text{for } \alpha \geqslant \frac{3}{4}. \end{cases}$$

STEP 3. Next, we define u to be the solution of $\mathrm{d}^2u/\mathrm{d}x^2 = \nabla W(u)$ with initial data $u(0) = a^+$, and $(\mathrm{d}u/\mathrm{d}x)(0) = \frac{1}{2}e_2$. Since the potential is radial, we easily see that $\nabla W(2, u_2) = (0, 4\lambda\rho'(u_2^2)u_2)$, and that $u = (2, u_2)$ with

$$\frac{\mathrm{d}^2u_2}{\mathrm{d}x^2}(x) = 4\lambda\rho'(u_2^2)u_2, \quad \frac{\mathrm{d}u_2}{\mathrm{d}x}(0) = \frac{1}{2}, \quad u_2(0) = 0.$$

In addition, we note that $u_2(x) > 0$ for $x > 0$. Indeed, if $u_2(t_0) = 0$ for some $t_0 > 0$ such that $u(x) > 0$ in the interval $(0, t_0)$, then we have $u_2(0) = u_2(t_0) = 0$, with u_2 convex and positive in $(0, t_0)$, which is a contradiction. As a consequence, u_2 and $\mathrm{d}u_2/\mathrm{d}x$ are increasing for $x > 0$. Let $0 < t_1 < t_2$ be the times when $u_2(t_1) = \sqrt{3/4}$ and $u_2(t_2) = 1$. Now, we choose the constant λ such that $(\mathrm{d}u/\mathrm{d}x)(t_1) = e_2$. Since the Hamiltonian $H = \frac{1}{2}|u_x|^2 - W(u)$ is constant along solutions, we take λ such that $H = \frac{1}{2}(\frac{1}{2})^2 = \frac{1}{2} - \lambda$. With this choice of λ, we still have $(\mathrm{d}u/\mathrm{d}x)(x) = e_2$ for $x \in [t_1, t_2]$, since, by assumption, W is constant on this portion of the curve.

STEP 4. To extend u for $x \geqslant t_2$, we parametrize by arc length the part of Γ starting at the point $a^+ + e_2$ and ending at the point $a^- + e_2$. Let $\gamma \colon [t_2, t_3] \to \mathbb{R}^2$ be such a parametrization. Then, we set $u(x) := \gamma(x)$ for $x \in [t_2, t_3]$. Clearly, $u \colon [0, t_3] \to \mathbb{R}^2$ is smooth, since in the interval $[t_1, t_2]$ u also parametrizes Γ by arc length. In addition, we have $(\mathrm{d}^2u/\mathrm{d}x^2)(x) \perp \Gamma$ for $x \in [t_1, t_3]$. To see this, just differentiate the equation $|(\mathrm{d}u/\mathrm{d}x)(x)|^2 = 1$ and note that $\mathrm{d}u/\mathrm{d}x$ is the tangent unit vector of Γ.

STEP 5. Now, we define W in a tubular neighbourhood of the part of Γ starting at the point $a^+ + e_2$ and ending at the point $a^- + e_2$ (see [5]). For $x \in [t_2, t_3]$ and $|\mu| \leqslant \varepsilon \ll 1$ we set

$$W(u(x) + \mu n_{u(x)}) := \lambda + \mu\left\langle \frac{\mathrm{d}^2u}{\mathrm{d}x^2}(x), n_{u(x)} \right\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product. By construction, for $x \in [t_2, t_3]$,

$$\nabla W(u(x)) = \left\langle \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(x), n_{u(x)} \right\rangle n_{u(x)}, \tag{2.1}$$

and W is smooth in a neighbourhood of the part of Γ between the points a^+ and $a^- + e_2$. Indeed, at the junction of the square $\{(u_1, u_2)\colon |u_1 - 2| \leqslant 1, |u_2| \leqslant 1\}$ and the tubular neighbourhood, $W(u) \equiv \lambda$. By (2.1), we also see that u satisfies the equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(x) = \nabla W(u(x)) \quad \text{for } x \in [0, t_3].$$

Clearly, if ε is small enough, we can ensure that, for $x \in [t_2, t_3]$ and $|\mu| \leqslant \varepsilon \ll 1$, $W(u(x) + \mu n) \geqslant \frac{1}{2}\lambda$.

Step 6. To extend u for $x \geqslant t_3$, we set

$$u(x) := (-2, u_2(t_2 + t_3 - x)) \quad \text{for } x \in [t_3, t_2 + t_3],$$

and check, as in step 3, that it is a solution of $\mathrm{d}^2 u/\mathrm{d}x^2 = \nabla W(u)$. Since in the interval $[t_3, t_2 + t_3 - t_1]$ u parametrizes Γ by arc length, this extension is smooth at $x = t_3$. Furthermore, at time $\frac{1}{2}T := t_2 + t_3$, we have $u(\frac{1}{2}T) = a^-$. Next, we extend W by symmetry for $u_2 < 0$ in a neighbourhood of the remaining portion of Γ, setting $W(u_1, u_2) = W(u_1, -u_2)$. Since W is also, by construction, symmetric with respect to the u_2 coordinate axis, that is, $W(u_1, u_2) = W(-u_1, u_2)$, we have $\nabla W(u) = -\nabla W(-u)$. Thus, setting $u(x) := -u(x - \frac{1}{2}T)$ for $x \in [\frac{1}{2}T, T]$, we define a solution of $\mathrm{d}^2 u/\mathrm{d}x^2 = \nabla W(u)$ on the whole period $[0, T]$. To complete the construction, we extend u periodically for all $x \in \mathbb{R}$, and W on the whole plane in such a way that $W(u) > 0$ if $u \neq a^\pm$.

Remark 2.1. Let $W\colon \mathbb{R}^2 \to \mathbb{R}$ be a non-negative potential satisfying for every $u \in \mathbb{R}^2$ such that $|u| = R > 0$:

$$W(u) = \lambda \quad \text{and} \quad \nabla W(u) = -\mu u, \quad \text{with constants } \lambda, \mu > 0. \tag{2.2}$$

Then, we check that $u\colon \mathbb{R} \to \mathbb{R}^2$, $u(x) = R\mathrm{e}^{\mathrm{i}\sqrt{\mu}x}$ is a solution of the ODE $\mathrm{d}^2 u/\mathrm{d}x^2 = \nabla W(u)$, and that $H = \frac{1}{2}|u_x|^2 - W(u) = \frac{1}{2}R^2\mu - \lambda$ may become positive and arbitrarily large. This situation occurs in the case of the Ginzburg–Landau potential $W(u) = \frac{1}{4}(|u|^2 - 1)^2$: for every R, $0 < R < 1$, we have a periodic solution, u_R, of the ODE, for which the corresponding parameters are $\lambda_R = \frac{1}{4}(R^2 - 1)^2$ and $\mu_R = 1 - R^2$. The constant $H_R = -\frac{1}{4}(3R^4 + 4R^2 - 1)$ is positive if and only if $\sqrt{1/3} < R < 1$. Note that condition (2.2) may also be satisfied by multiple well potentials.

Remark 2.2. The Modica estimate does not allow the existence of a periodic connection $u\colon \mathbb{R} \to \mathbb{R}$ for the scalar problem (1.3). Indeed, if $W(u(x_0)) = 0$ for some $x_0 \in \mathbb{R}$, and u is bounded, then $u_x(x_0) = 0$, and by the uniqueness result for ODEs u coincides with the constant solution $v \equiv u(x_0)$. However, for a double well potential with non-degenerate zeros a^- and a^+, there exists a solution $u\colon \mathbb{R} \to \mathbb{R}$ of (1.3) (the heteroclinic connection) such that $\lim_{x \to \pm\infty} u(x) = a^\pm$ (see [3] for the extension of this result to the vector case). In addition, this solution satisfies the equipartition relation $\frac{1}{2}|u_x|^2 = W(u)$, i.e. $H = 0$. Note that in the case of our counterexample, there also exists a heteroclinic connection that takes its values (by symmetry) onto the line segment $(-2, 2)$ of the u_1 coordinate axis.

3. Gradient estimates and applications

The proof of the Modica estimate (see [11]) is based on the use of the so-called P-functions (see [15]). Let us explain briefly how they are chosen and used. To every solution $u\colon \mathbb{R}^n \to \mathbb{R}$ of the scalar equation (1.1) is associated the P-function $P(u; x) := \frac{1}{2}|\nabla u(x)|^2 - W(u(x))$. This choice is relevant, since the function P satisfies the inequality

$$|\nabla u|^2 \Delta P \geqslant \tfrac{1}{2}|\nabla P|^2 + 2W'(u)\nabla u \cdot \nabla P \tag{3.1}$$

(without any additional assumptions on W or u). Then, the maximum principle is applied to show that $P(u; x) \leqslant 0$, for every bounded solution u and every $x \in \mathbb{R}^n$.

For system (1.4), inequality (3.1) no longer holds. However, it is possible under appropriate assumptions to construct other P-functions to which the maximum principle can be applied. More precisely, we obtain inequalities of the form $\Delta P \geqslant hP$, and use the properties satisfied by the system and the solutions to ensure that $h \geqslant 0$.

In this section, we establish gradient estimates for several kinds of elliptic systems following the method of Caffarelli *et al*. [4]. We present this technique in various situations, and point out that this approach is quite flexible and can easily be adjusted to another context. We begin with a system involving a diagonal matrix $D = \operatorname{diag}(\nu_1, \dots, \nu_m)$ (see (3.2)). The expression of the P-function (see (3.8)) is interesting in this case, since it contains the coefficients of D. We obtain a rough estimate (see (3.3)), which is nevertheless sufficient to prove that

- all bounded solutions $u\colon \mathbb{R}^n \to \mathbb{R}^m$ of (3.2) have their images in a determined region $\omega \subset \mathbb{R}^m$,
- if $u(x_0) \in \partial\omega$ for some $x_0 \in \mathbb{R}^n$, then the solution u is constant (Liouville-type theorem).

Next, in theorem 3.2, we consider the standard system (1.4) and establish a similar result under an appropriate monotonicity assumption on the potential. Since the estimates given by the two previous theorems are general and rough, it is relevant to improve them by studying an important particular case. In theorem 3.5, we focus on the Ginzburg–Landau system (3.17), and obtain an estimate that is sharp asymptotically (see (3.18)). Finally, we consider phase transition potentials W, taking advantage of their convexity near the wells. Assuming that $|\nabla u(x)|$ is small enough when $u(x)$ lies outside the convexity region of W, we show that the solution u satisfies a stronger estimate than Modica's (see theorem 3.8). For a double well potential $W\colon \mathbb{R} \to \mathbb{R}$, the periodic solutions of the ODE (1.3) that are near the equilibrium in the phase plane satisfy this assumption.

Theorem 3.1. *Let $D = \operatorname{diag}(\nu_1, \dots, \nu_m)$ be an $m \times m$ diagonal matrix with $\nu_i > 0$ for all $i = 1, \dots, m$. Let A be an $m \times m$ matrix such that*

(i) $\langle (D^{-1}A + AD^{-1})u, u\rangle \geqslant 0$ *for all $u \in \mathbb{R}^m$,*

(ii) $\langle Au, u\rangle \geqslant c|u|^2$ *for all* $u \in \mathbb{R}^m$ *and for a constant* $c > 0$*, where* $|\cdot|$ *and* $\langle\cdot,\cdot\rangle$ *denote the Euclidean norm and inner product, respectively.*

Assume that $u = (u^1, \dots, u^m) \in C^2(\mathbb{R}^n;\ \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^m)$ *is an entire solution of the system*[1]

$$D\Delta u + [1 - \langle Au, u\rangle]u = 0, \tag{3.2}$$

Then,

$$\sum_{j=1}^m \tfrac{1}{2}\nu_j |\nabla u^j(x)|^2 \leqslant C[1 - \langle Au(x), u(x)\rangle], \tag{3.3}$$

for a constant $C(A, D, \|u\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^m)}) > 0$*. In particular,* $\langle Au(x), u(x)\rangle \leqslant 1$ *for every* $x \in \mathbb{R}^n$*, and if* u *is not constant, then* $\langle Au(x), u(x)\rangle < 1$ *for every* $x \in \mathbb{R}^n$*.*

Proof. Fix $M > 0$ and define

$$\mathcal{F}_M = \{u \text{ is an entire solution of (3.2)} \mid \|u\|^2_{L^\infty(\mathbb{R}^n;\mathbb{R}^m)} \leqslant M\}.$$

Let $u \in \mathcal{F}_M$. For $j = 1, \dots, m$ and $i = 1, \dots, n$ we have

$$\begin{aligned} \nu_j \Delta u^j &= [\langle Au, u\rangle - 1]u^j, \\ \nu_j \Delta u^j_{x_i} &= (\langle Au_{x_i}, u\rangle + \langle Au, u_{x_i}\rangle)u^j + [\langle Au, u\rangle - 1]u^j_{x_i}, \end{aligned} \tag{3.4}$$

and

$$\Delta(\tfrac{1}{2}\nu_j|\nabla u^j|^2) = \nu_j B_j + \nu_j \sum_{i=1}^n \Delta u^j_{x_i} u^j_{x_i}, \quad \text{where } B_j := \sum_{i,k=1}^n |u^j_{x_i x_k}|^2.$$

Therefore, using (3.4), we obtain

$$\Delta(\tfrac{1}{2}\nu_j|\nabla u^j|^2) = \nu_j B_j + \sum_{i=1}^n (\langle Au_{x_i}, u\rangle + \langle Au, u_{x_i}\rangle)u^j u^j_{x_i} + [\langle Au, u\rangle - 1]|\nabla u^j|^2$$

and

$$\Delta\biggl(\sum_{j=1}^m \tfrac{1}{2}\nu_j|\nabla u^j|^2\biggr) \geqslant B + [\langle Au, u\rangle - 1 - 2amM]|\nabla u|^2, \tag{3.5}$$

where $a := \|A\|_{L(\mathbb{R}^n;\mathbb{R}^m)}$ and $B := \sum_{j=1}^m \nu_j B_j$.

On the other hand, we also compute

$$\begin{aligned} \Delta(\tfrac{1}{2}[\langle Au, u\rangle - 1]) &= \tfrac{1}{2}(\langle A\Delta u, u\rangle + \langle Au, \Delta u\rangle) + \sum_{i=1}^n \langle Au_{x_i}, u_{x_i}\rangle \\ &\geqslant \tfrac{1}{2}[\langle Au, u\rangle - 1]\langle (AD^{-1} + D^{-1}A)u, u\rangle + c|\nabla u|^2, \end{aligned} \tag{3.6}$$

since $\Delta u = [\langle Au, u\rangle - 1]D^{-1}u$ and D is symmetric.

[1] This system reduces to (1.4) only when $(A + A^{\mathrm{T}}) = \mu D^{-1}$ for some $\mu \in \mathbb{R}$.

Now, let $\nu := \max_j\{\nu_j\}$, and let $\lambda > 0$ be such that, for every $v \in \mathbb{R}^m$ with $|v|^2 \leqslant M$,

$$(\langle Av, v\rangle - 1 - 2amM + \lambda c) \geqslant \tfrac{1}{2}\nu\langle (AD^{-1} + D^{-1}A)v, v\rangle. \tag{3.7}$$

Then, for every $u \in \mathcal{F}_M$ we define

$$P(u;x) := \left(\sum_{j=1}^{m} \tfrac{1}{2}\nu_j|\nabla u^j(x)|^2 + \tfrac{1}{2}\lambda[\langle Au(x), u(x)\rangle - 1]\right), \tag{3.8}$$

and thanks to (3.5)–(3.7) the inequality

$$\Delta P(u;x) \geqslant B + \langle (AD^{-1} + D^{-1}A)u(x), u(x)\rangle P(u;x) \tag{3.9}$$

holds in $\mathbb{R}^n$. The remainder of the proof proceeds as in [4]. We consider

$$P_M := \sup\{P(u;x) \mid u \in \mathcal{F}_M,\ x \in \mathbb{R}^n\}$$

and suppose by contradiction that $P_M > 0$. Note that, for $u \in \mathcal{F}_M$, $|\nabla u|$ is uniformly bounded, since u and Δu are uniformly bounded (see [9, § 3.4, p. 37]), and thus P_M is finite. By the definition of P_M, there exist two sequences, (u_k) in $\mathcal{F}_M$ and (x_k) in $\mathbb{R}^n$, such that $P(u_k; x_k) \to P_M$ as $k \to \infty$. Setting $v_k(x) := u_k(x + x_k)$, one can see that the sequence (v_k) belongs to $\mathcal{F}_M$ (since (3.2) is translation invariant), and $P(v_k; 0) = P(u_k; x_k) \to P_M$ as $k \to \infty$. As the first derivatives of the solutions in $\mathcal{F}_M$ satisfy a uniform bound and are equicontinuous on bounded domains (see [4, theorem 3.1]; [9, corollary 6.3, p. 93]), one can apply the Ascoli–Arzela theorem and deduce via a diagonal argument the existence of a solution $v \in F_M$, such that $P(v; 0) = P_M$. Applying then the maximum principle to $P(v;x)$ (see (3.9) and hypothesis (i)), one can see that $P(v;x) \equiv P_M$. In addition, $B \equiv 0$ and $\langle (AD^{-1} + D^{-1}A)v(x), v(x)\rangle P_M \equiv 0$. As a consequence, $v \equiv v_0$ is constant, and since v is a solution it follows that $\langle Av_0, v_0\rangle = 1$ or $v_0 = 0$. Thus, $P_M \leqslant 0$, and we have proved that, for every $u \in \mathcal{F}_M$ and every $x \in \mathbb{R}^n$,

$$\sum_{j=1}^{m} \tfrac{1}{2}\nu_j|\nabla u^j(x)|^2 \leqslant \tfrac{1}{2}\lambda[1 - \langle Au(x), u(x)\rangle] \Longrightarrow \langle Au(x), u(x)\rangle \leqslant 1. \tag{3.10}$$

To finish the proof, suppose that $\langle Au(x_0), u(x_0)\rangle = 1$ for a solution $u \in \mathcal{F}_M$ and for some $x_0 \in \mathbb{R}^n$. According to the above, $\max_{x\in\mathbb{R}^n} P(u;x) = P(u;x_0) = 0$. Thus, by the maximum principle, we deduce as before that $P(u;x) \equiv 0$ and $B \equiv 0$, which implies that u is constant. □

Theorem 3.2. *Let $W \in C^{2,\alpha}(\mathbb{R}^m, \mathbb{R})$ (with $0 < \alpha < 1$) be such that,*[2] *for some constant $R > 0$,*

$$u \in \mathbb{R}^m, \quad |u| > R \Longrightarrow u \cdot \nabla W(u) > 0. \tag{3.11}$$

Then, if $u \in C^2(\mathbb{R}^n; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^m)$ is an entire solution of (1.4), we have

$$\tfrac{1}{2}|\nabla u(x)|^2 \leqslant C(R^2 - |u(x)|^2) \tag{3.12}$$

[2] This regularity assumption on W ensures that every classical solution u of (1.4) is $C^{3,\alpha}$ smooth (see [9, theorem 6.17, p. 109]). As a consequence, we can compute the second derivatives of the P-function defined below.

for a constant $C(W, \|u\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^m)}) > 0$. In particular, $|u(x)| \leqslant R$ for all $x \in \mathbb{R}^n$, and if u is not constant, then $|u(x)| < R$ for all $x \in \mathbb{R}^n$.

Proof. Following Caffarelli *et al*. let

$$\mathcal{F}_M = \{u \text{ is an entire solution of (1.4)} \mid \|u\|^2_{L^\infty(\mathbb{R}^n;\mathbb{R}^m)} \leqslant M\},$$

where $M > 0$ is an arbitrary constant. There exists a constant $\mu > 0$ such that

$$\forall u \in \mathbb{R}^m \text{ with } |u|^2 \leqslant M,\ \forall \xi \in \mathbb{R}^m, \quad D^2W(u)(\xi, \xi) \geqslant -\mu|\xi|^2. \tag{3.13}$$

We can also check that there exists a constant $\kappa > 0$ such that

$$\forall u \in \mathbb{R}^m \text{ with } |u| \leqslant R, \quad u \cdot \nabla W(u) \geqslant \kappa(|u|^2 - R^2). \tag{3.14}$$

For every $u \in \mathcal{F}_M$ we define

$$P(u; x) := \tfrac{1}{2}|\nabla u(x)|^2 + \tfrac{1}{2}(\kappa + \mu)(|u(x)|^2 - R^2).$$

We set $B := (\sum_{i,j=1}^n |u_{x_i x_j}|^2)$, and compute

$$\begin{aligned}\Delta P(u; x) &= B + \sum_{i=1}^n D^2W(u)(u_{x_i}, u_{x_i}) + (\kappa + \mu)(|\nabla u|^2 + u \cdot \nabla W(u)) \\ &\geqslant B + \kappa|\nabla u|^2 + (\kappa + \mu)u \cdot \nabla W(u) \quad \text{(see (3.13))}.\end{aligned}$$

Thus,

$$\Delta P(u; x) \geqslant \begin{cases} B \geqslant 0 & \text{if } |u(x)| \geqslant R \quad \text{(see (3.11))}, \\ B + 2\kappa P(u; x) & \text{if } |u(x)| < R \quad \text{(see (3.14))}, \end{cases}$$

and by setting

$$h(u; x) := \begin{cases} 0 & \text{if } |u(x)| \geqslant R, \\ 2\kappa & \text{if } |u(x)| < R, \end{cases}$$

one can see that

$$\Delta P(u; x) \geqslant B + h(u; x)P(u; x), \tag{3.15}$$

with $h(u; \cdot) \in L^\infty(\mathbb{R}^n, \mathbb{R})$ and non-negative. Next, we consider

$$P_M := \sup\{P(u; x) \mid u \in \mathcal{F}_M,\ x \in \mathbb{R}^n\}$$

and suppose by contradiction that $P_M > 0$. Proceeding as in [4] and in theorem 3.1, we prove the existence of a solution $v \in F_M$, such that $P(v; 0) = P_M$. Thanks to (3.15), we can apply the maximum principle to $P(v; x)$, and deduce successively that $P(v; x) \equiv P_M$, $B = 0$ and v is constant. Using (3.11), one can also see that $|v| \leqslant R$, and thus $P_M \leqslant 0$. This proves that, for every $u \in \mathcal{F}_M$ and every $x \in \mathbb{R}^n$,

$$\tfrac{1}{2}|\nabla u(x)|^2 \leqslant \tfrac{1}{2}(\kappa + \mu)(R^2 - |u(x)|^2) \Longrightarrow |u(x)| \leqslant R. \tag{3.16}$$

To finish the proof, suppose that $|u(x_0)| = R$ for a solution $u \in \mathcal{F}_M$ and for some $x_0 \in \mathbb{R}^n$. It follows that $\max_{x\in\mathbb{R}^n} P(u; x) = P(u; x_0) = 0$. Thus, by the maximum principle, we deduce as before that $P(u; x) \equiv 0$, $B \equiv 0$ and u is constant. $\square$

REMARK 3.3. Condition (3.11) is satisfied by the symmetric phase transition potentials $W\colon \mathbb{R}^2 \simeq \mathbb{C} \to \mathbb{R}$, $W(z) = |z^N - 1|^2$, with $z \in \mathbb{C}$ and $N \geqslant 2$. Indeed,

$$z \cdot \nabla W(z) = 2N \operatorname{Re}(z^N(\bar{z}^N - 1)) \geqslant 2N|z|^N(|z|^N - 1).$$

Clearly, theorem 3.2 or theorem 3.1 (with A and D the identity maps of $\mathbb{R}^m$) also applies for the Ginzburg–Landau potential $W\colon \mathbb{R}^m \to \mathbb{R}$, $W(u) = \frac{1}{4}(|u|^2 - 1)^2$. In these particular cases, the solutions $u \in C^2(\mathbb{R}^n; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^m)$ of (1.4) satisfy $|u(x)| \leqslant 1$ for all $x \in \mathbb{R}^n$. Furthermore, if u is not constant, then $|u(x)| < 1$ for all $x \in \mathbb{R}^n$, and thus the Liouville theorem holds: if $W(u(x_0)) = 0$ for some $x_0 \in \mathbb{R}^n$, then u is a constant. Note that for the Ginzburg–Landau system there is a stronger result: it is proved in [6] that any distributional solution without any boundedness assumption is necessarily bounded in modulus by 1.

REMARK 3.4. If we just want to prove the confinement of all bounded solutions in a determined region (without obtaining a gradient estimate), a simpler P-function can be chosen. Let us consider, for instance, the solutions $u \in C^2(\mathbb{R}^n; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^m)$ of the system $\Delta u = F(u)$, with $F \in C^\alpha(\mathbb{R}^m; \mathbb{R}^m)$, and let $P \in C^2(\mathbb{R}^m, \mathbb{R})$ be a function such that $P(u) > 0$ implies

(i) $\langle \nabla P(u), F(u) \rangle > 0$,

(ii) $D^2 P(u)(\xi, \xi) \geqslant 0$ for all $\xi \in \mathbb{R}^m$.

Then, $P(u(x)) \leqslant 0$ for all $x \in \mathbb{R}^n$. Indeed, reproducing the previous arguments, we construct in the corresponding class $\mathcal{F}_M$ a solution $v \in C^2(\mathbb{R}^n; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^m)$ of $\Delta v = F(v)$ such that $P(v(0)) = P_M := \sup\{P(u(x)) \mid u \in \mathcal{F}_M,\ x \in \mathbb{R}^n\}$. Thanks to (i) and (ii), we have

$$\begin{aligned}(\Delta P(v))(x) &= \langle \nabla P(v(x)), F(v(x)) \rangle + \sum_{i=1}^n D^2 P(v(x))(v_{x_i}(x), v_{x_i}(x)) \\ &> 0 \quad \text{if } P(v(x)) > 0,\end{aligned}$$

which implies that $P_M = P(v(0)) \leqslant 0$.

As an application, we can take for P the distance d to a convex and compact subset $K \subset \mathbb{R}^m$, with C^2 boundary. The distance is convex outside K, and can be extended smoothly in the interior of K in such a way that $d(u, K) \leqslant 0$ if and only if $u \in K$. With this choice of P, we deduce that if $\langle \nabla d(u), F(u) \rangle > 0$ for $u \notin K$, then $u(\mathbb{R}^n) \subset K$ for every solution $u \in C^2(\mathbb{R}^n; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^m)$ of $\Delta u = F(u)$.

Considering again the multiple well potential

$$W\colon \mathbb{R}^2 \to \mathbb{R}, \quad W(u) = \prod_{i=1}^N |u - a_i|^2 \quad (N \geqslant 3),$$

where the points $a_1, \dots, a_N$ define a convex and closed polygon $K \subset \mathbb{R}^2$, one can prove that $u(\mathbb{R}^n) \subset K$ for every solution $u \in C^2(\mathbb{R}^n; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^m)$ of (1.4). To see this, take $P(u) = \langle u - a_k, r \rangle$, where r is the outer unit normal vector to an edge of K containing the vertex a_k. Clearly, P is convex, and we easily check that

$$\langle \nabla W(u), r \rangle = 2 \sum_{i=1}^N \left(\langle u - a_i, r \rangle \prod_{j \neq i} |u - a_j|^2 \right) > 0$$

when $\langle u - a_k, r \rangle > 0$.

Now, we shall improve estimate (3.12) for system (1.4) with the Ginzburg–Landau potential $W\colon \mathbb{R}^m \to \mathbb{R}$, $W(u) = \frac{1}{4}(|u|^2-1)^2$:

$$\Delta u = (|u|^2 - 1)u. \tag{3.17}$$

Theorem 3.5 (A. Farina, personal communication). *For every non-constant solution $u \in C^2(\mathbb{R}^n;\mathbb{R}^m)$ of (3.17), we have $|u(x)| < 1$ for every $x \in \mathbb{R}^n$, and the following estimate holds:*

$$\tfrac{1}{2}|\nabla u(x)|^2 < \sqrt{W(u)} = \tfrac{1}{2}(1-|u|^2). \tag{3.18}$$

Proof. Setting $Q\colon \mathbb{R}^m \to \mathbb{R}$, $Q(u) = \frac{1}{2}|u|^2 - 1$, we check that, for all $u, \xi \in \mathbb{R}^m$,

$$\begin{aligned} |\nabla W(u)|^2 &= (|u|^2-1)^2|u|^2 = 4(Q(u))^2(2Q(u)+1), \\ u \cdot \nabla W(u) &= (|u|^2-1)|u|^2 = 2Q(u)(2Q(u)+1), \\ D^2W(u)(\xi,\xi) &= 2\langle \xi, u\rangle^2 + (|u|^2-1)|\xi|^2 \geqslant 2Q(u)|\xi|^2 \end{aligned}$$

(where the angled brackets and the centred dot denote the Euclidean inner product). Then, we proceed as before. Since the image of every solution u lies in the unit ball (see [6]), we consider

$$\mathcal{F}_1 = \{u \text{ is an entire solution of (3.17)} \mid \|u\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^m)} \leqslant 1\}$$

and define

$$P(u;x) = \tfrac{1}{2}|\nabla u(x)|^2 + Q(u(x)).$$

Let $P_1 := \sup\{P(u;x) \mid u \in \mathcal{F}_1,\ x \in \mathbb{R}^n\}$, and suppose by contradiction that $P_1 > 0$. We set $B := (\sum_{i,j=1}^n |u_{x_i x_j}|^2)$ and compute

$$\begin{aligned} \Delta P(u;x) &= B + \sum_{i=1}^n D^2W(u)(u_{x_i}, u_{x_i}) + |\nabla u|^2 + u \cdot \nabla W(u) \\ &\geqslant B + 2Q(u)|\nabla u|^2 + |\nabla u|^2 + (2Q(u)+1)2Q(u) \\ &\geqslant B + 2(2Q(u)+1)P(u;x) \geqslant 2|u|^2 P(u;x). \end{aligned} \tag{3.19}$$

Proceeding as in [4], we then prove the existence of a solution $v \in \mathcal{F}_1$ such that $P(v;0) = P_1$. Thanks to (3.19) we can apply the maximum principle to $P(v;x)$ and deduce successively that $P(v;x) \equiv P_1$, $B = 0$ and $P_1 \leqslant 0$. Thus, we have proved that, for every $u \in \mathcal{F}_1$ and every $x \in \mathbb{R}^n$,

$$\tfrac{1}{2}|\nabla u(x)|^2 \leqslant \sqrt{W(u)} = \tfrac{1}{2}(1-|u|^2).$$

By again applying the maximum principle, one can see that this inequality is strict, except for constant solutions $u \equiv u_0$ such that $|u_0| = 1$. □

In the particular case where $n = 1$, we give an even more precise result.

Theorem 3.6. *For every non-constant solution $u \in C^2(\mathbb{R};\mathbb{R}^m)$ of the ODE*

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = (|u|^2-1)u, \tag{3.20}$$

we have, for every $x \in \mathbb{R}^n$, $|u(x)| < 1$, and the following estimate holds:

$$\frac{1}{2}\left|\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right|^2 \leqslant \begin{cases} |u|^2\sqrt{W(u)} & \text{for } |u|^2 \geqslant \frac{2}{3}, \\ W(u) + \frac{1}{12} & \text{for } |u|^2 \leqslant \frac{2}{3}. \end{cases} \tag{3.21}$$

In other words, the Hamiltonian $H = \frac{1}{2}|u_x|^2 - W(u)$ of u is less than or equal to $\frac{1}{12}$, and if $S := \sup_{\mathbb{R}} |u(x)|^2 > \frac{2}{3}$, then $H \leqslant \frac{1}{4}(1-S)(3S-1)$.

Proof. We repeat the proof of theorem 3.5 with another choice of the P-function. We define

$$P(u;x) = \frac{1}{2}\left|\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right|^2 - W(u(x)) + \phi(Q(u(x))),$$

where $\phi \in C^2([-\frac{1}{2}, 0], \mathbb{R})$ is strictly increasing and convex. Next, we compute

$$\begin{aligned} \frac{\mathrm{d}^2 P}{\mathrm{d}x^2}(u;x) &= \phi''(Q(u))\langle u_x, u\rangle^2 + \phi'(Q(u))(|u_x|^2 + u \cdot \nabla W(u)) \\ &\geqslant 2\phi'(Q(u))\tfrac{1}{2}|u_x|^2 + 2\phi'(Q(u))(2Q^2(u) + Q(u)). \end{aligned} \tag{3.22}$$

If, in addition, the function ϕ satisfies $3s^2 + s \geqslant \phi(s)$ for all $s \in [-\frac{1}{2}, 0]$, then we have

$$\frac{\mathrm{d}^2 P}{\mathrm{d}x^2}(u;x) \geqslant h(u;x)P(u;x) \quad \text{with } h(u;x) := 2\phi'(Q(u(x))) > 0. \tag{3.23}$$

We construct a sequence of functions ϕ_ε as follows. First, we define for every $\varepsilon > 0$ an increasing function $\rho_\varepsilon \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\rho_\varepsilon(t) = \begin{cases} t & \text{for } t \geqslant 2\varepsilon, \\ \varepsilon & \text{for } t \leqslant 0, \end{cases}$$

and $\rho_\varepsilon(t) \geqslant t$ for all $t \in \mathbb{R}$. Then, we set

$$\phi_\varepsilon(s) := \int_0^s \rho_\varepsilon(6t+1)\,\mathrm{d}t$$

and check that this sequence has all the aforementioned properties. We also note that, as $\varepsilon \to 0$, ϕ_ε converges uniformly on the interval $[-\frac{1}{2}, 0]$ to the function

$$\phi(s) = \begin{cases} 3s^2 + s & \text{for } s \geqslant -\frac{1}{6}, \\ -\frac{1}{12} & \text{for } s \leqslant -\frac{1}{6}. \end{cases}$$

Proceeding as in theorem 3.5, we prove that

$$P_\varepsilon(u;x) := \frac{1}{2}\left|\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right|^2 - W(u(x)) + \phi_\varepsilon(Q(u(x))) \leqslant 0,$$

and letting $\varepsilon \to 0$ we obtain (3.21). □

Remark 3.7. With the help of the periodic solutions of the ODE (3.20) that we mentioned in remark 2.1, we shall check the sharpness of estimates (3.18) and (3.21). For every $0 < R < 1$,

$$u_R\colon \mathbb{R} \to \mathbb{C} \simeq \mathbb{R}^2 \subset \mathbb{R}^m \ (m \geqslant 2), \quad u_R(x) = R\exp(\mathrm{i}\sqrt{1-R^2}x),$$

is a solution of (3.20), and clearly

$$\left|\frac{\mathrm{d}u_R}{\mathrm{d}x}\right|^2 = |u_R(x)|^2(1 - |u_R(x)|^2).$$

Thus, estimate (3.21) is optimal for $|u|^2 \geqslant \frac{2}{3}$, and estimate (3.18) is sharp asymptotically, since

$$\frac{1}{2}\left|\frac{\mathrm{d}u_R}{\mathrm{d}x}\right|^2 \sim \sqrt{W(u_R)} \quad \text{as } R \to 1.$$

Also note that, due to the existence of a heteroclinic connection, we have the following lower bound:

$$\sup\left\{\frac{1}{2}\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 : u \text{ is a solution of the ODE (3.20)}\right\} \geqslant \begin{cases} |u|^2\sqrt{W(u)} & \text{for } |u|^2 \geqslant \frac{1}{3}, \\ W(u) & \text{for } |u|^2 \leqslant \frac{1}{3}. \end{cases} \tag{3.24}$$

The next theorem applies in the case of phase transition potentials with N non-degenerate zeros, since the potential is convex in a neighbourhood of each of these minima. Note that the Ginzburg–Landau potential $W(u) = \frac{1}{4}(|u|^2 - 1)^2$ that we considered before is nowhere convex inside the unit ball.

Theorem 3.8. *Let $W \in C^{2,\alpha}(\mathbb{R}^m, \mathbb{R})$ (with $0 < \alpha < 1$) be a non-negative potential that is convex in the closed set $F \subset \mathbb{R}^m$ (that is, $D^2W(u)(\xi,\xi) \geqslant 0$ for all $u \in F$ and all $\xi \in \mathbb{R}^m$). Let $u \in C^2(\mathbb{R}^n; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^m)$ be an entire solution of (1.4). We set*

$$\varepsilon := \inf_{\mathbb{R}^m \setminus F} W, \qquad S := \sup_{u^{-1}(\mathbb{R}^m \setminus F)} \|\nabla u\|^2.$$

Then, if $0 < S < 2\varepsilon/n$, the following estimate holds:

$$\tfrac{1}{2}n|\nabla u(x)|^2 \leqslant \frac{\varepsilon}{S}|\nabla u(x)|^2 \leqslant W(u(x)) \quad \forall x \in \mathbb{R}^n.$$

In addition, if $S = 0$ or $u(\mathbb{R}^n) \subset F$, then u is constant.

Proof. We set $\lambda := 2\varepsilon/S$ and assume that $\lambda > n$ and $S > 0$. We define, for every bounded solution $v\colon \mathbb{R}^n \to \mathbb{R}^m$ of (1.4), the function

$$P(v; x) := \tfrac{1}{2}\lambda|\nabla v(x)|^2 - W(v(x)).$$

Following Caffarelli *et al.* let $P_u := \sup_{x\in\mathbb{R}^n} P(u; x)$ and suppose by contradiction that $P_u > 0$. By definition of P_u, there exists a sequence (x_k) in $\mathbb{R}^n$ such that $P(u; x_k) \to P_u$ as $k \to \infty$. Setting $v_k(x) := u(x + x_k)$, one can see that

(i) the sequence (v_k) is uniformly bounded in $\mathbb{R}^n$,

(ii) all the v_k solve (1.4) (since (1.4) is translation invariant),

(iii) $\sup_{v_k^{-1}(\mathbb{R}^m\setminus F)} \|\nabla v_k\|^2 \leqslant S$,

(iv) $P(v_k;0) = P(u;x_k) \to P_u$ as $k \to \infty$.

Owing to the fact that the first derivatives of the sequence (v_k) satisfy a uniform bound and are equicontinuous on bounded domains (see [4, theorem 3.1]), one can apply the Ascoli–Arzela theorem and deduce via a diagonal argument the existence of a bounded solution $v\colon \mathbb{R}^n \to \mathbb{R}^m$ of (1.4), such that $P(v;0) = P_u$. Furthermore, since $v_k \to v$ and $\nabla v_k \to \nabla v$ uniformly on compact sets, we still have

$$\sup_{v^{-1}(\mathbb{R}^m\setminus F)} \|\nabla v\|^2 \leqslant S \quad \text{and} \quad P_u = \sup_{x\in\mathbb{R}^n} P(v;x) = P(v;0). \tag{3.25}$$

Now, we set

$$B := \biggl(\sum_{i,j=1}^{n} |v_{x_i x_j}|^2\biggr), \qquad A := \biggl|\sum_{i=1}^{n} v_{x_i x_i}\biggr|^2$$

and compute

$$\begin{aligned} \Delta P(v;x) &= \lambda B + (\lambda-1)\sum_{i=1}^{n} D^2W(v)(v_{x_i}, v_{x_i}) - A \\ &\geqslant (\lambda-n)B + (\lambda-1)\sum_{i=1}^{n} D^2W(v)(v_{x_i}, v_{x_i}) \quad (\text{since } nB \geqslant A) \\ &\geqslant (\lambda-n)B \geqslant 0 \quad \text{if } v(x) \in F. \end{aligned} \tag{3.26}$$

Using (3.25), we see that if $P(v;x) = P_u$, the two situations below are impossible:

(i) $v(x) \in \mathbb{R}^m \setminus F$,

(ii) $v(x) \in \partial F$, and $v(\omega) \cap (\mathbb{R}^m \setminus F) \neq \emptyset$ for every neighbourhood $\omega \subset \mathbb{R}^n$ of x.

Thus, there exists a neighbourhood $\omega \subset \mathbb{R}^n$ of x such that $v(\omega) \subset F$, and inequality (3.26) holds in ω. Applying the maximum principle, we deduce that

$$P(v;\cdot) \equiv P_u \quad \text{in } \omega$$

and, by connectedness,

$$P(v;\cdot) \equiv P_u \quad \text{in all } \mathbb{R}^n.$$

Due to (3.26), this implies that $B \equiv 0$, v is constant and $P_u \leqslant 0$. Therefore, we have proved that, for every $x \in \mathbb{R}^n$,

$$\tfrac{1}{2}\lambda|\nabla u(x)|^2 \leqslant W(u(x)).$$

In the case where $S = 0$, taking $\lambda \to \infty$, we see that u is constant. Finally, in the case where $u(\mathbb{R}^n) \subset F$, we take an arbitrary $\lambda > n$, and omit from the proof the arguments involving the set $u^{-1}(\mathbb{R}^m \setminus F)$. □

4. An alternative form of the stress–energy tensor in the plane

We first recall the definition of the stress–energy tensor used in [1] to establish various properties of the solutions to (1.4), among them the weak monotonicity formula. To every solution $u\colon \mathbb{R}^n \supset \Omega \to \mathbb{R}^m$ to system (1.4) is associated the stress–energy tensor, T, which is the following $n \times n$ symmetric matrix

$$T(u) := \frac{1}{2}\begin{pmatrix} |u_{x_1}|^2 - \sum_{i\neq 1}^{n} |u_{x_i}|^2 - 2W(u) & 2u_{x_1}\cdot u_{x_2} & \cdots & 2u_{x_1}\cdot u_{x_n} \\ 2u_{x_2}\cdot u_{x_1} & |u_{x_2}|^2 - \sum_{i\neq 2}^{n} |u_{x_i}|^2 - 2W(u) & \cdots & 2u_{x_2}\cdot u_{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ 2u_{x_n}\cdot u_{x_1} & 2u_{x_n}\cdot u_{x_2} & \cdots & |u_{x_n}|^2 - \sum_{i\neq n}^{n} |u_{x_i}|^2 - 2W(u) \end{pmatrix}, \tag{4.1}$$

whose elements are invariant under rotations of the coordinate system. Note that $T(u)$ can also be written as the sum of a scalar and a symmetric matrix:

$$T(u) = -(\tfrac{1}{2}|\nabla u|^2 + W(u))I_n + (u_{x_i}\cdot u_{x_j})_{1\leqslant i,j\leqslant n},$$

where I_n denotes the identity matrix of $\mathbb{R}^n$. Setting $T = (T_1, \ldots, T_n)^{\mathrm{T}}$ and $\operatorname{div} T = (\operatorname{div} T_1, \ldots, \operatorname{div} T_n)^{\mathrm{T}}$, the tensor has the remarkable property that $\operatorname{div} T = 0$ for every solution to (1.4).

In this section, we give an alternative form of the stress–energy tensor T in the plane. Let $\Omega \subset \mathbb{R}^2$ be an open and simply connected domain of the plane. We associate to every solution $u\colon \mathbb{R}^2 \supset \Omega \to \mathbb{R}^m$ of (1.4) (where $W\colon \mathbb{R}^m \to \mathbb{R}$ is at least C^1 smooth) a function U that solves the equation $\Delta U = 4W(u)$. Indeed, if $u \in C^2(\Omega, \mathbb{R}^m)$ is a solution to (1.4) in Ω, the equations $\operatorname{div} T_1 = 0$ and $\operatorname{div} T_2 = 0$ can be interpreted as the compatibility conditions

$$\left.\begin{aligned} [|u_{x_1}|^2 - |u_{x_2}|^2 + 2W(u)]_{x_2} &= [2u_{x_1}\cdot u_{x_2}]_{x_1}, \\ [|u_{x_2}|^2 - |u_{x_1}|^2 + 2W(u)]_{x_1} &= [2u_{x_1}\cdot u_{x_2}]_{x_2}, \end{aligned}\right\} \tag{4.2}$$

which ensure the existence of a function $U \in C^3(\Omega, \mathbb{R})$, defined modulo an affine function, whose Hessian matrix is

$$D^2U = \begin{pmatrix} |u_{x_1}|^2 - |u_{x_2}|^2 + 2W(u) & 2u_{x_1}\cdot u_{x_2} \\ 2u_{x_1}\cdot u_{x_2} & |u_{x_2}|^2 - |u_{x_1}|^2 + 2W(u) \end{pmatrix}. \tag{4.3}$$

We note that $D^2U \equiv 0$ if and only if $W(u) \equiv 0$, $|u_{x_1}| \equiv |u_{x_2}|$ and $u_{x_1}\cdot u_{x_2} \equiv 0$. In particular, when $W \equiv 0$, the Hessian matrix D^2U of the function U is related to the Hopf differential (see [12]):

$$\Phi := \tfrac{1}{4}([|u_{x_1}|^2 - |u_{x_2}|^2] - 2\mathrm{i}\langle u_{x_1}, u_{x_2}\rangle)\,\mathrm{d}z \otimes \mathrm{d}z, \quad \text{where } z := x_1 + \mathrm{i}x_2.$$

Both are two-dimensional objects that vanish if and only if the solution u is conformal.

In the next proposition, we give a boundary condition for solutions of (1.4) to be conformal. It is interesting to compare this result with the corresponding ones for harmonic maps (see [14]).

PROPOSITION 4.1. *We assume that the potential $W \in C^1(\mathbb{R}^m, \mathbb{R})$ is non-negative. Let $B \subset \mathbb{R}^2$ be a ball of radius R, and let $u \in C^1(\bar{B}, \mathbb{R}^m) \cap C^2(B, \mathbb{R}^m)$ be a solution of (1.4) satisfying on ∂B the boundary condition:*

$$|u_\tau|^2 - |u_\nu|^2 + 2W(u) \leqslant 0, \tag{4.4}$$

where ν is the outer unit normal vector to ∂B, τ is the tangential one, $u_\tau := \nabla u \cdot \tau$ and $u_\nu := \nabla u \cdot \nu$. Then, u is a harmonic map that is also conformal in B.

Proof. Without loss of generality, we assume that B is centred at the origin. We consider the polar coordinates (r, θ) and the corresponding positively oriented orthonormal basis $(\nu = x/|x|, \tau)$. Applying Green's formula to the function U, we first prove that

$$\int_B 4W(u)\,\mathrm{d}x = \int_{\partial B} U_\nu = R\int_{\partial B} (|u_\tau|^2 - |u_\nu|^2 + 2W(u))\,\mathrm{d}\sigma(x), \tag{4.5}$$

since $U_\nu(R,\theta) = RU_{\tau\tau}(R,\theta) - U_{\theta\theta}(R,\theta)/R$ and $U_{\tau\tau} := D^2U(x)(\tau,\tau) = |u_\tau|^2 - |u_\nu|^2 + 2W(u)$. Next, using the boundary condition (4.4), we deduce that $W(u) \equiv 0$ in B. Thus, u is harmonic and, moreover, satisfies $|u_\tau|^2 - |u_\nu|^2 = 0$ on ∂B. To conclude we apply a result for harmonic maps established in [14]. $\square$

When the solution u is defined and bounded in all $\mathbb{R}^2$, it is known that its first derivatives are also bounded (see [9, § 3.4, p. 37]). In this case, the corresponding function U is a solution of the equation $\Delta U = 4W(u)$ in $\mathbb{R}^2$, with bounded second derivatives. According to the following proposition, U is the unique function, modulo a harmonic polynomial of degree 2, satisfying these properties.

PROPOSITION 4.2. *Let $u \in C^2(\mathbb{R}^2, \mathbb{R}^m)$ be a bounded solution to (1.4) in $\mathbb{R}^2$. Then, every solution V of the equation $\Delta V = 4W(u)$ in $\mathbb{R}^2$ with bounded second derivatives can be written as*

$$V = U + \lambda(x_1^2 - x_2^2) + \mu x_1 x_2 + \alpha x_1 + \beta x_2 + \gamma$$

for constants λ, μ, α, β and γ.

Proof. Let V be a solution of the equation $\Delta V = 4W(u)$ in $\mathbb{R}^2$ with bounded second derivatives. We define the harmonic function $h := V - U$ in $\mathbb{R}^2$. Since the second derivatives of V are bounded, we deduce by Liouville's theorem that the second derivatives of h are constants. Thus, h is a harmonic polynomial of degree 2. $\square$

We now give a geometric interpretation of the Modica estimate.

PROPOSITION 4.3. *If the potential $W\colon \mathbb{R}^m \to \mathbb{R}$ is non-negative and $u \in C^2(\mathbb{R}^2, \mathbb{R}^m)$ is a solution to (1.4), then the corresponding function U is convex if and only if*

$$(|u_{x_1}|^2 - |u_{x_2}|^2)^2 + 4(u_{x_1} \cdot u_{x_2})^2 \leqslant 4(W(u))^2 \quad \forall x \in \mathbb{R}^2. \tag{4.6}$$

Moreover,

- *Modica's inequality (see (1.2)) implies the convexity of the function U,*
- *when $m = 1$, Modica's inequality is equivalent to the convexity of the function U. In particular, U is convex if u is bounded.*

Proof. The function U is convex if and only if

$$\det(D^2U) \geqslant 0 \Longleftrightarrow (|u_{x_1}|^2 - |u_{x_2}|^2)^2 + 4(u_{x_1} \cdot u_{x_2})^2 \leqslant 4(W(u))^2 \quad \forall x \in \mathbb{R}^2.$$

Modica's inequality implies the last inequality for every $m \geqslant 1$, and is equivalent to it when $m = 1$. To see this, just check that

$$\begin{aligned} |\nabla u|^4 &\geqslant (|u_{x_1}|^2 - |u_{x_2}|^2)^2 + 4(u_{x_1} \cdot u_{x_2})^2 \quad \text{for every } m \geqslant 1, \\ |\nabla u|^4 &= (|u_{x_1}|^2 - |u_{x_2}|^2)^2 + 4(u_{x_1} \cdot u_{x_2})^2 \quad \text{for } m = 1. \end{aligned}$$

□

REMARK 4.4. Unfortunately, the convexity of U cannot substitute for the Modica estimate when $m \geqslant 2$. We shall give a counterexample showing that in general the function U is not convex. We consider a bounded solution $u\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the Ginzburg–Landau system (3.17), mentioned in [10], and having the following two properties:

$$|u(x)| = 1 - \frac{d^2}{2|x|^2} + o\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty \text{ with } d \geqslant 1, \tag{4.7}$$

$$\int_{\mathbb{R}} [|u_{x_1}|^2 - |u_{x_2}|^2 + 2W(u(x))]\,\mathrm{d}x_1 = 0 \quad \forall x_2 \in \mathbb{R}. \tag{4.8}$$

From (4.7) and (4.8), it follows that the inequality

$$|u_{x_1}|^2 - |u_{x_2}|^2 + 2W(u) \geqslant 0 \tag{4.9}$$

is not satisfied in all $\mathbb{R}^2$, and as a consequence U is not convex. Indeed, if (4.9) holds in $\mathbb{R}^2$, then (4.8) implies that $U_{x_1x_1} = |u_{x_1}|^2 - |u_{x_2}|^2 + 2W(u) \equiv 0$, and, integrating, we find that $U(x_1, x_2) = f(x_2)x_1 + g(x_2)$, where $f, g\colon \mathbb{R} \to \mathbb{R}$ are two smooth functions. Since $4W(u) = \Delta U = f''(x_2)x_1 + g''(x_2)$ is bounded, we deduce that $f'' \equiv 0$ and $4W(u) = g''(x_2)$. Finally, from the latter equation and (4.7), it follows that $g'' \equiv 0$, $W(u) \equiv 0$ and $|u| \equiv 1$, which contradicts (4.7).

Also note that a simpler counterexample invalidating the convexity of U is provided by the solutions of the Ginzburg–Landau system: $u_R\colon \mathbb{R}^2 \to \mathbb{R}^2 \simeq \mathbb{C}$, $u_R(x_1, x_2) = R\exp(\mathrm{i}\sqrt{1 - R^2}x_1)$.

As an application of the function U, we shall prove a (strong) monotonicity formula involving only the term with the potential. We need first to establish the following lemma.

LEMMA 4.5. *Let $V \in C^2(\mathbb{R}^n, \mathbb{R})$ be a convex function. Then*

$$r \to \frac{1}{r^{n-1}} \int_{B(x,r)} \Delta V(x) \, \mathrm{d}x$$

is an increasing function of $r > 0$ ($B(x,r) \subset \mathbb{R}^n$ denotes the ball of radius r centred at x).

Proof. Without loss of generality we suppose that $x = 0$. Since every $x \neq 0$ can be written $x = \rho n$ with $\rho = |x|$ and $n = x/|x|$, we have

$$\int_{B(0,r)} \Delta V(x) \, \mathrm{d}x = \int_{\partial B(0,r)} \frac{\partial V}{\partial n}(x) \, \mathrm{d}\sigma(x) = r^{n-1} \int_{\partial B(0,1)} \frac{\partial V}{\partial n}(rn) \, \mathrm{d}\sigma(n). \tag{4.10}$$

Using the convexity of V, we see that

$$r_1 \leqslant r_2 \Longrightarrow \frac{\partial V}{\partial n}(r_1 n) \leqslant \frac{\partial V}{\partial n}(r_2 n)$$

for every $n \in \mathbb{R}^n$ such that $|n| = 1$. Thus, we deduce the desired result from (4.10). □

THEOREM 4.6. *Let $W \in C^2(\mathbb{R}^m, \mathbb{R})$ be a non-negative potential, and let $u \in C^2(\mathbb{R}^2, \mathbb{R}^m)$ be a solution to (1.4) satisfying (4.6). Then,*

$$r \to \frac{1}{r} \int_{B(x,r)} W(u(x)) \, \mathrm{d}x$$

is an increasing function of $r > 0$ ($B(x,r) \subset \mathbb{R}^n$ denotes the ball centred at x of radius r). In particular, for every bounded solution $u \in C^3(\mathbb{R}^2, \mathbb{R})$ of (1.1), the previous monotonicity formula holds.

Proof. It is a straightforward consequence of proposition 4.3 and lemma 4.5. For bounded solutions $u \in C^3(\mathbb{R}^2, \mathbb{R})$ of (1.1), Modica's estimate holds, and thus the corresponding function U is convex. □

REMARK 4.7. It is remarkable that an integral property such as the monotonicity formula in theorem 4.6 follows from a differential inequality (see (4.6)). Note that the monotonicity formula mentioned in § 1 also holds for vector solutions to (1.4), satisfying the Modica inequality (see [1]).

REMARK 4.8. Let us also give another application of lemma 4.5. If $u \colon \mathbb{R}^n \to \mathbb{R}^m$ is a harmonic map such that $|u|^2$ is convex, then

$$r \to \frac{1}{r^{n-1}} \int_{B(x,r)} |\nabla u(x)|^2 \, \mathrm{d}x$$

is an increasing function of $r > 0$ (since $\Delta |u|^2 = 2|\nabla u|^2$).

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References

1 N. D. Alikakos. Some basic facts on the system $\Delta u - \nabla W(u) = 0$. *Proc. Am. Math. Soc.* **139** (2011), 153–162.

2 N. D. Alikakos and A. C. Faliagas. The stress–energy tensor and Pohozaev's identity for systems. *Acta Math. Sci.* **32** (2012), 433–439.

3 N. D. Alikakos and G. Fusco. On the connection problem for potentials with several global minima. *Indiana Univ. Math. J.* **57** (2008), 1871–1906.

4 L. Caffarelli, N. Garofalo and F. Segala. A gradient bound for entire solutions of quasi-linear equations and its consequences. *Commun. Pure Appl. Math.* **47** (1994), 1457–1473.

5 M. P. Do Carmo. *Differential geometry of curves and surfaces* (Upper Saddle River, NJ: Prentice Hall, 1976).

6 A. Farina. Finite-energy solutions, quantization effects and Liouville-type results for a variant of the Ginzburg–Landau systems in $\mathbb{R}^K$. *Diff. Integ. Eqns* **11** (1998), 875–893.

7 A. Farina. Two results on entire solutions of Ginzburg–Landau system in higher dimensions. *J. Funct. Analysis* **214** (2004), 386–395.

8 A. Farina and E. Valdinoci. A pointwise gradient estimate in possibly unbounded domains with nonnegative mean curvature. *Adv. Math.* **225** (2010), 2808–2827.

9 D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics (Springer, 2001).

10 C. Gui. Hamiltonian identities for partial differential equations. *J. Funct. Analysis* **254** (2008), 904–933.

11 L. Modica. A gradient bound and a Liouville theorem for nonlinear Poisson equations. *Commun. Pure Appl. Math.* **38** (1985), 679–684.

12 R. Schoen and S.-T. Yau. *Lectures on differential geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology, vol. II (Boston, MA: International Press, 1994).

13 P. Smyrnelis. Solutions to elliptic systems with mixed boundary conditions. PhD thesis, University of Thessaloniki, 2012.

14 P. Smyrnelis. The harmonic map problem with mixed boundary conditions. *Proc. Am. Math. Soc.* **143** (2015), 1299–1313.

15 R. Sperb. *Maximum principles and their applications*. Mathematics in Science and Engineering, vol. 157 (New York: Academic, 1981).