*Advanced Nonlinear Studies* **13** (2013), 01–11

# Entire Solutions with Six-fold Junctions to Elliptic Gradient Systems with Triangle Symmetry

( To Klaus Schmitt on the occasion of his retirement. With enthusiasm and patience Klaus Schmitt was one of the professors at the University of Utah who taught the first author the beauty and power of nonlinear functional analysis and changed the course of his life. )

**Peter Bates***

*Department of Mathematics*

*Michigan State University, East Lansing, MI 48824*

*e-mail: bates@math.msu.edu*

**Giorgio Fusco**†

*Department of Mathematics*

*University L'Aquila, L'Aquila, Italy*

*e-mail: fusco@dm.univaq.it*

**Panayotis Smyrnelis**

*Department of Mathematics*

*University of Athens, Athens, Greece*

*e-mail: psmyrnelis@gmail.com*

*Communicated by Stephen Cantrell*

**Abstract**

We prove the existence of solutions to three-fold symmetric elliptic systems in $\mathbb{R}^2$ which have six-fold symmetry, asymptotically approaching each of three minima of the potential as $|x| \to \infty$ in two antipodal sectors of angle $\pi/3$.

*1991 Mathematics Subject Classification.* 35J47 35B06 35B08 35J50.

*Key words.* Equivariant system, triple junction, invariant region.

*This author acknowledges support through NSF 0908348

†This author acknowledges support through PRIN 2009 protocollo n.2009TA2595.02

# 1 Introduction

Let $W : \mathbb{R}^2 \to \mathbb{R}$ be a smooth, nonnegative potential which is invariant under the equilateral triangle reflection group $G$, having exactly three zeros, these being nondegenerate and located at $\mathbf{a}_1 = (1, 0)$, $\mathbf{a}_2 = (-1/2, \sqrt{3}/2)$, and $\mathbf{a}_3 = (-1/2, -\sqrt{3}/2)$. We denote the gradient of $W$ by $W_u$. We are interested in solutions to the elliptic system:

$$\Delta \mathbf{u} - W_u(\mathbf{u}) = 0, \tag{1.1}$$

being the equation for stationary states of

$$\mathbf{u}_t = \Delta \mathbf{u} - W_u(\mathbf{u}), \tag{1.2}$$

where $\mathbf{u} : \mathbb{R}^2 \to \mathbb{R}^2$.

Equation (1.2) is a special case of the Cahn-Morral system ([16], [8], [9], [13], [14], [15]) which was designed to model the spinodal decomposition and evolution of phase boundaries in multi-component alloys.

In that case $\mathbf{u}$ represents the vector of local mass-fractions of the various elements of the alloy; the mass fraction of one component being determined by the others, it is omitted. The function $W(\mathbf{u})$ gives the bulk energy density of an alloy with species having relative concentrations $\mathbf{u}$, and the interfacial energy density is represented by $\frac{1}{2}|\nabla \mathbf{u}|^2$. Thus, the free energy $J$ is given by

$$J_\Omega(\mathbf{u}) = \int_\Omega (\frac{1}{2}|\nabla \mathbf{u}|^2 + W(\mathbf{u}))dx, \tag{1.3}$$

where $\Omega$ is the region occupied by the alloy.

From this point of view, (1.2) is the statement that the local concentrations evolve in time according to $L^2$-steepest descent,

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{\delta J}{\delta \mathbf{u}}. \tag{1.4}$$

The natural boundary condition associated with (1.4) is the homogeneous Neumann condition

$$\frac{\partial \mathbf{u}}{\partial n} = 0 \text{ on } \partial\Omega. \tag{1.5}$$

Triple junctions in thin films of three component mixtures (therefore $\mathbf{u} \in \mathbb{R}^2$) are quite common as they are somewhat stable and there have been several significant results describing solutions of this type to (1.1) and (1.2) under various assumptions on $W$ (see, for instance, [6], [8], [21], [17], [5], [1], [7], [4], and [19]).

Here we establish the existence of solutions that have a six-fold junction with the plane divided into six equal sectors and each of the three 'phases' occurring in two antipodal sectors. The precise statement appears below but first we state the hypotheses.

**(W)**
Assume that $W : \mathbb{R}^2 \to \mathbb{R}$ is a nonnegative, $C^3$ potential which is symmetric under the reflection group generated by any two of the reflections in the lines $\pi_1 \equiv \{(u_1, u_2) : u_2 = 0\}$, $\pi_2 \equiv \{(u_1, u_2) :$

$u_2 = -\sqrt{3}u_1\}$, and $\pi_3 \equiv \{(u_1, u_2) : u_2 = \sqrt{3}u_1\}$. Denote these reflections by $\gamma_1, \gamma_2$, and $\gamma_3$, respectively.

It is assumed that $W$ has exactly three zeros, these being nondegenerate zeros of $W_u$, and located at $\mathbf{a}_1 = (1, 0) \in \pi_1$, $\mathbf{a}_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \in \pi_2$, and $\mathbf{a}_3 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}) \in \pi_3$.

Finally, it is assumed that there exists $M > 1$ such that $W(s\mathbf{u}) \geq W(\mathbf{u})$ for $s \geq 1$ and $|\mathbf{u}| = M$. An example of such a potential is $W(\mathbf{u}) \equiv (|\mathbf{u} - \mathbf{a}_1||\mathbf{u} - \mathbf{a}_2||\mathbf{u} - \mathbf{a}_3|)^2$.

**Theorem 1.1** *Assume $W$ satisfies (W). Then there exists an entire classical bounded solution $\mathbf{u} : \mathbb{R}^2 \to \mathbb{R}^2$ to system (1.1) such that*

*(i)* $\mathbf{u}$ *is equivariant with respect to G, i.e.,* $\mathbf{u}(gx) = g\mathbf{u}(x)$ *for all* $g \in G$.

*(ii)* $\mathbf{u}(-x) = \mathbf{u}(x)$ *for all* $x \in \mathbb{R}^2$.

*(iii) If* $D_1 \equiv \{x = (x_1, x_2) : |x_2| > \sqrt{3}|x_1|\}$, $D_2 \equiv \gamma_3 D_1$, *and* $D_3 \equiv \gamma_2 D_1$, *then, with d as distance, there exist positive constants c and C such that*

$$|\mathbf{u}(x) - \mathbf{a}_i| \leq C exp(-cd(x, \partial D_i)) \text{ for } x \in D_i.$$

*(iv)*

$$\mathbf{u}(\overline{D_1}) \subset \overline{S_1} \equiv \{\mathbf{u} = (u_1, u_2) : u_1 \geq \frac{1}{\sqrt{3}}|u_2|\},$$

*and by equivariance,*

$$\mathbf{u}(\overline{\gamma_2 D_1}) = \gamma_2 \mathbf{u}(\overline{D_1}),\ \mathbf{u}(\overline{\gamma_3 D_1}) = \gamma_3 \mathbf{u}(\overline{D_1}).$$

*In particular, at $x = 0$, $\mathbf{u}$ exhibits a 6-junction, each of the three phases ocurring twice as one divides the plane into six $\pi/3$ sectors with vertex at 0 (see Figures 1 and 2).*

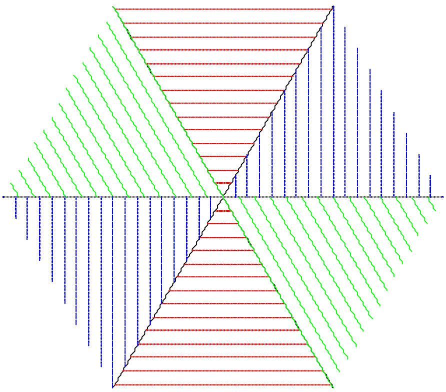

Figure 1: The **x**-plane showing the six-junction.

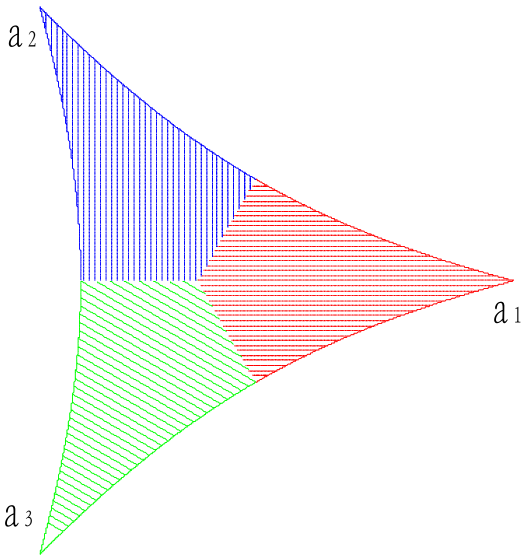


Figure 2: The **u**-plane indicating the images of the six segments of the **x**-plane.

Solutions to (1.1) with $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^m$ and with more general reflection groups, $G$, have been obtained in [5], [12], [4], [3], [10], and references therein. We take the results from these as our starting point and in particular use some of their techniques to prove the result above. We use variational arguments but note that the solution we seek has infinte energy when $\Omega = \mathbb{R}^2$ and so we first obtain a solution on $\Omega = B_R \equiv B(0, R)$, the ball centered at 0 of radius $R$, imposing homogeneous Neumann conditions at the boundary. To show the existence of a solution with the structure described in Theorem 1.1 (iii), we minimize in a set of maps that satisfy a sort of positivity condition in the sense that they map each $\pi/3$ sector into a corresponding $2\pi/3$ sector. Based on this and using results of [10], we show that the solution is nontrivial, satisfying the exponential estimate in (iii) above.

Observe that the solution given by Theorem 1.1 is different from the one constructed in [5] and that Theorem 1.1 implies nonuniqueness in the class of entire equivariant solutions which connect the minima of $W$, this being a local minimum of the energy with the added symmetry about 0 but a saddle when that symmetry is not imposed.

We also point out that similar results to that above may be obtained in other setting, in particular, for the case of $G$ being the reflection group of the pentagon in the plane and for the tetrahedon group in $\mathbb{R}^3$, being different from that obtained in [12]. We also may use similar techniques to obtain periodic solutions with respect to a hexagonal lattice in the plane. These extensions will be the subject of another paper.

# 2 Minimizers in Balls

In (1.3) we take $\Omega = B_R$ and denote the energy by $J_R \equiv J_{B_R}$. Let $W^{1,2}_E(B_R)$ be the Sobolev space of $\mathbb{R}^2$-valued functions defined on $B_R$ with one weak derivative in $L^2$ and which are both $G-$equivariant and also symmetric about 0, in the sense that $\mathbf{u}(-x) = \mathbf{u}(x)$. Define

$$\mathcal{A}^R \equiv \{\mathbf{u} \in W^{1,2}_E(B_R) : \mathbf{u}(\overline{D_1 \cap B_R}) \subset \overline{S}_1\},$$

Note that $\mathcal{A}^R \neq \emptyset$, since the following function belongs to it: Identify $\mathbf{x} \in \mathbb{R}^2$ with $z = re^{i\theta} \in \mathbb{C}$ and consider the function $\mathbf{u}$ defined by

$$re^{i\theta} \to -\frac{r^2}{1 + r^2} e^{-2i\theta}, \text{ for } \pi/3 \leq \theta \leq 2\pi/3,$$

and extend it equivariantly to $\theta \in [0, \pi]$ using the reflection group. The symmetry about 0 is automatic due to the fact that this function is even in $r$. This function is easily seen to be smooth in the sector $\theta \in (\pi/3, 2\pi/3)$ and Lipschitz continuous across the boundaries of these sectors.

We consider the problem (for each $R > 0$)

$$\min_{\mathbf{u} \in \mathcal{A}^R} J_R(\mathbf{u}). \tag{2.1}$$

Minimizing in the symmetry class does not provide a constraint (see [18]) but the mapping property in the definition of $\mathcal{A}^R$ could be seen as such, which would complicate the Euler-Lagrange equation. The following lemma shows that this is not the case.

Take the unit normals to the lines $\pi_1, \pi_2$, and $\pi_3$ as $\mathbf{n}_1 \equiv (0, 1), \mathbf{n}_2 \equiv (\frac{\sqrt{3}}{2}, \frac{1}{2})$, and $\mathbf{n}_3 \equiv (\frac{\sqrt{3}}{2}, -\frac{1}{2})$, respectively.

It is useful to consider the overlapping quarter-disks

$$O_{11} \equiv \{\mathbf{x} : \mathbf{x} \cdot \mathbf{n}_2 > 0,\ \mathbf{x} \cdot \mathbf{a}_2 > 0\} \cap B_R$$

and

$$O_{12} \equiv \{\mathbf{x} : \mathbf{x} \cdot \mathbf{n}_3 < 0,\ \mathbf{x} \cdot \mathbf{a}_3 < 0\} \cap B_R.$$

**Lemma 2.1** *Let $\mathbf{u}(x,t)$ be the solution to the parabolic system (1.2) on $\Omega = B_R$ satisfying (1.5) and with initial data $\mathbf{u}_0 \in C(B_R)$ which is G-equivariant, $\mathbf{u}_0(-x) = \mathbf{u}_0(x)$ and satisfies the mapping condition*

$$\mathbf{n}_2 \cdot \mathbf{u}_0(x) > 0 \textit{ for } \ x \in O_{11}, \tag{2.2}$$

*and*

$$\mathbf{n}_3 \cdot \mathbf{u}_0(x) > 0 \textit{ for } \ x \in O_{12}. \tag{2.3}$$

*Then, for all $t > 0$, $\mathbf{u}(D_1 \cap B_R, t) \subset S_1$.*

*Proof.* Let

$$D_{11} \equiv \{x = (x_1, x_2) : x_2 > \sqrt{3}|x_1|\} \cap B_R = O_{11} \cap O_{12}.$$

Then the conditions on $\mathbf{u}_0$ imply $\mathbf{u}_0(D_{11}) \subset S_1$ and because of the symmetry about 0, $\mathbf{u}_0(D_1 \cap B_R) \subset S_1$. Similarly, it is sufficient to show that

$$\mathbf{u}(D_{11}, t) \subset S_1.$$

Define

$$\eta \equiv \mathbf{n}_2 \cdot \mathbf{u}(x,t)$$

and note that $\eta$ satisfies

$$\eta_t = \Delta\eta - c\eta \text{ in } B_R,\ \ t > 0, \tag{2.4}$$

$$\eta(x,0) = \mathbf{n}_2 \cdot \mathbf{u}_0(x), \tag{2.5}$$

$$\frac{\partial\eta}{\partial n} = 0, \text{ for } |x| = R, \tag{2.6}$$

where

$$c(x,t) \equiv \frac{\mathbf{n}_2 \cdot W_u(\mathbf{u}(x,t))}{\mathbf{n}_2 \cdot \mathbf{u}(x,t)} \quad \text{provided } \eta \neq 0.$$

Since $\mathbf{u}(\cdot,t)$ and $W_u$ are $G$-equivariant, $\mathbf{u}(\cdot,t), W_u(\mathbf{u}(\cdot,t)) : \pi_i \to \pi_i$ and hence,

$$\mathbf{n}_i \cdot \mathbf{u}(x,t) = 0$$

and

$$\mathbf{n}_i \cdot W_u(\mathbf{u}(x,t)) = 0$$

for $x \in \pi_i$, and each i=1, 2, 3.

Furthermore, since $\mathbf{u}(-x,t) = \mathbf{u}(x,t)$ and since $\mathbf{u}(\gamma_i x,t) = \gamma_i \mathbf{u}(x,t)$, it follows that for $x \in \operatorname{span}\{\mathbf{n}_i\}$,

$$\mathbf{u}(x,t) \in \pi_i \text{ and again } \mathbf{n}_i \cdot \mathbf{u}(x,t) = \mathbf{n}_i \cdot W_u(\mathbf{u}(x,t)) = 0.$$

These observations show that

$$\eta \equiv 0 \text{ on } \partial O_{11} \cap B_R.$$

We also have, $\eta(x,0) > 0$ for all $x \in O_{11}$ by (2.2).

It should be noted that $\mathbf{n}_i \cdot \mathbf{u}(x,t) = 0$ implies $\mathbf{n}_i \cdot W_u(\mathbf{u}(x,t)) = 0$ and since $W$ is smooth $c(x,t)$, defined above, can be extended continuously.

By restricting our attention to $O_{11}$, the Maximum Principle for parabolic equations implies that $\eta(x,t) > 0$ for all $x \in O_{11}$ and $t > 0$.

A similar argument applied to $\xi(x,t) \equiv \mathbf{n}_3 \cdot \mathbf{u}(x,t)$ restricted to $O_{12}$ shows that $\mathbf{n}_3 \cdot \mathbf{u}(x,t) > 0$ for all $x \in O_{12}$ and $t > 0$. Thus, we have shown that for each $t > 0$, $\mathbf{u}(\cdot,t)$ satisfies (2.2) and (2.3), giving the conclusion.

One may check that if $\mathbf{u}$ is $G$-equivariant and symmetric about 0 then $\mathbf{u}(D_1 \cap B_R) \subset S_1$ is equivalent to the two conditions (2.2) and (2.3). Taking the minimizer, $\mathbf{u}_R$, in (2.1) as initial data and recognizing that $J_R$ is a Lyapunov functional, we find that $\mathbf{u}_R$ is a stationary solution for the parabolic equation and hence satisfies (1.1).

A simple cut-off comparison function with the growth condition on $W$ shows that the minimizer is pointwise bounded, uniformly in $R$:

$$|\mathbf{u}_R(x)| \le M \text{ for } x \in B_R. \tag{2.7}$$

It follows that $\mathbf{u}_R \in C^{4+\alpha}$ for some $\alpha \in (0,1)$ and that there is a constant, $M'$ independent of $R$, such that

$$\|\mathbf{u}_R\|_{C^{4+\alpha}} \le M'. \tag{2.8}$$

# 3 Replacement lemmas and energy estimates

Here we reproduce the analysis in [10] used to establish bounds on the measure of the set on which $\mathbf{u}_R$ is far from $\mathbf{a}_1$ which are enough to obtain (iii) of Theorem 1.1.

For $\mathbf{u} \in \mathbb{R}^2 \backslash \{\mathbf{a}_1\}$ we write $\mathbf{u} = q\nu + \mathbf{a}_1$ where $q = |\mathbf{u} - \mathbf{a}_1|$ and $\nu = \frac{\mathbf{u}-\mathbf{a}_1}{|\mathbf{u}-\mathbf{a}_1|} \in \mathbb{S}^1$. For any $\nu \in \mathbb{S}^1$ define

$$\bar{Q}_\nu \equiv \sup\{q : q\nu + \mathbf{a}_1 \in S_1 \cap B_M\},$$

where $M$ is that from the condition on $W$. Let $\bar{Q} = \max\{\bar{Q}_\nu : \nu \in \mathbb{S}^1\}$.

Define $V : [0,\infty) \times \mathbb{S}^1 \to \mathbb{R}$ by

$$V(q,\nu) \equiv W(q\nu + \mathbf{a}_1). \tag{3.1}$$

If $\mathbf{u} \in \mathcal{A}^R$ set

$$q^{\mathbf{u}}(x) \equiv |\mathbf{u} - \mathbf{a}_1| \text{ for } x \in \bar{D}_1 \cap B_R,$$

$$\nu^{\mathbf{u}}(x) \equiv \frac{\mathbf{u} - \mathbf{a}_1}{|\mathbf{u} - \mathbf{a}_1|}, \text{ for } x \in \bar{D}_1 \cap B_R \backslash \{x : q^{\mathbf{u}}(x) = 0\}.$$

Then one may calculate

$$J_R(\mathbf{u}) = 3\int_{A_R}(\frac{1}{2}|\nabla q^{\mathbf{u}}|^2 + (q^{\mathbf{u}})^2(\nu_{x_1}\cdot\nu_{x_1} + \nu_{x_2}\cdot\nu_{x_2}) + V(q^{\mathbf{u}},\nu^{\mathbf{u}}))dx, \tag{3.2}$$

where $A_R = \bar{D}_1 \cap B_R\backslash\{x : q^{\mathbf{u}}(x) = 0\}$. The nondegeneracy and coercivity assumptions on $W$ allow one to establish the following elementary lemma, whose proof is given in [10]:

**Lemma 3.1** *There exist positive constants $c, \bar{q}$ and $\bar{V}$ such that*

$$V_{qq}(q,\nu) \geq c^2 \ \ for \ \ (q,\nu) \in (0,\bar{q}) \times \mathbb{S}^1, \tag{3.3}$$

$$V(q,\nu) \geq \tilde{V}(q_0,q,\nu) \equiv V(q_0,\nu) + V_q(q_0,\nu)(q-q_0), \ \ for \ \ (q_0,q,\nu) \in (0,\bar{q}) \times (q_0,\bar{q}] \times \mathbb{S}^1, \tag{3.4}$$

$$\tilde{V}_{q_0}(q_0,q,\nu) \geq 0 \ for \ \ (q_0,q,\nu) \in (0,\bar{q}) \times (q_0,\bar{q}] \times \mathbb{S}^1, \tag{3.5}$$

$$V(\bar{q},\nu) \leq V(q,\nu) \leq \bar{V} \ for \ \ (q,\nu) \in [\bar{q},\bar{Q}_\nu] \times \mathbb{S}^1. \tag{3.6}$$

Since $|\mathbf{u}_R(x)| \leq M$ it follows that

$$q^{\mathbf{u}_R} \leq \bar{Q}.$$

We will prove that $\mathbf{u}_R$ has a certain behavior by showing that otherwise, using the representation (3.2), we can make modifications to $\mathbf{u}_R$ that reduce $J_R$. The modification is done in $D_R \equiv D_{11}$ and then extended equivariantly. The strategy to obtain the exponential estimate in the theorem is to obtain a good bound on the measure of the set on which $\mathbf{u}_R$ is far from $\mathbf{a}_1$.

Let $\bar{q}' \in (0,\bar{q})$ and set

$$A_{\bar{q}'}^{\mathbf{u}_R} \equiv \{x \in D_R : q^{\mathbf{u}_R}(x) > \bar{q}'\}.$$

For a subset $S \subset D_R$ define

$$S^{\triangleright} \equiv S \cup \gamma_1' S,$$

the symmetrized version of $S$ about span$\{\mathbf{n}_1\}$, where $\gamma_1'$ is the reflection in that line.

We state the following lemmas, which are also from [10] where their proofs may be found.

**Lemma 3.2** *Let $\lambda > 0$ be fixed and assume that $B(x_0, l+\lambda) \subset D_R$ for some $l > 0$. Assume*

$$S \equiv A_{\bar{q}'}^{\mathbf{u}_R} \cap (B^{\triangleright}(x_0,l+\lambda)\backslash B^{\triangleright}(x_0,l)) \neq \emptyset. \tag{3.7}$$

*Then there is a constant $K$, independent of $R$ and $l > 0$, and a Lipschitz continuous map $\mathbf{v} \in \mathcal{A}^R$ such that*

*(i)*

$$\begin{cases} \nu^{\mathbf{v}} = \nu^{\mathbf{u}_R} \ for \ \ x \in D_R, \\ \mathbf{v} = \mathbf{u}_R \ for \ \ x \in D_R\backslash S, \\ q^{\mathbf{v}} = \bar{q}' \ for \ \ x \in A_{\bar{q}'}^{\mathbf{u}_R} \cap B^{\triangleright}(x_0, l+\lambda/2), \end{cases} \tag{3.8}$$

*(ii)*

$$J_{D_R}(\mathbf{v}) - J_{D_R}(\mathbf{u}_R) = J_S(\mathbf{v}) - J_S(\mathbf{u}_R) \leq K|S|,$$

*where $|S|$ is the two-dimensional measure of $S$.*

Let $c$ and $\bar{q}$ be as in Lemma 3.1 and $\bar{q}' \in (0, \bar{q})$ as above. For a function $\phi$ defined on $B(x_0, L) \subset D_R$, let $\phi^{\triangleright}$ be the symmetric extension to $B(x_0, L)^{\triangleright}$, that is, the map defined by

$$\phi^{\triangleright}(\mathbf{x}) = \begin{cases} \phi(\mathbf{x}) \text{ if } (\mathbf{x} \cdot \mathbf{a}_1)(\mathbf{x}_0 \cdot \mathbf{a}_1) \geq 0, \\ \phi(\gamma' \mathbf{x}) \text{ if } (\mathbf{x} \cdot \mathbf{a}_1)(\mathbf{x}_0 \cdot \mathbf{a}_1) < 0. \end{cases}$$

**Lemma 3.3** *Assume that $B(x_0, L) \subset D_R$ and let $\phi : B(x_0, L) \to \mathbb{R}$ be the solution to*

$$\begin{cases} \Delta\phi &= c^2\phi \ \ in \ \ B(x_0, L), \\ \phi &= \bar{q}' \ \ on \ \ \partial B(x_0, L). \end{cases} \tag{3.9}$$

*Let $\mathbf{u} \in \mathcal{A}^R$ be continuous and satisfy*

$$q^{\mathbf{u}} \leq \bar{q}' \ for \ \ x \in \bar{B}(x_0, L). \tag{3.10}$$

*Then, there exists a map $\mathbf{w} \in \mathcal{A}^R$ such that*

$$q^{\mathbf{w}} \leq \phi, for \ \ x \in \bar{B}(x_0, L)$$

*and*

$$\begin{aligned} J_{D_R}(\mathbf{u}) - J_{D_R}(\mathbf{w}) &\geq J_{B(x_0,L)^{\triangleright}}(\mathbf{u}) - J_{B(x_0,L)^{\triangleright}}(\mathbf{w}) \\ &\geq \int_{B(x_0,L)^{\triangleright} \cap \{q^{\mathbf{u}} > \phi^{\triangleright}\}} (V(q^{\mathbf{u}}, \nu^{\mathbf{u}}) - V(\phi^{\triangleright}, \nu^{\mathbf{v}}) - V_q(\phi^{\triangleright}, \nu^{\mathbf{u}})(q^{\mathbf{u}} - \phi^{\triangleright}))dx. \end{aligned}$$

This is proved using careful estimates on the local energy, written in the form of (3.2), using the nondegeneracy of $W_{uu}(\mathbf{a}_1)$ so that $\phi$ can be used as a local pointwise comparison function that has exponential decay. This result may then be used to obtain, again from [10], the following:

**Corollary 3.1** *Fix $\bar{l} > 0$ and $\lambda > 0$, Let $l > \bar{l}$ and assume that $B(x_0, l + \lambda) \subset D_R$. Let $\mathbf{v} \in \mathcal{A}^R$ be the map constructed in Lemma 3.2. Then there is a constant $k > 0$, independent of $l > \bar{l}$ and $R$ and a map $\hat{\mathbf{v}} \in \mathcal{A}^R$ such that*

$$J_{D_R}(\mathbf{v}) - J_{D_R}(\hat{\mathbf{v}}) \geq J_{B(x_0,l+\lambda/2)^{\triangleright}}(\mathbf{v}) - J_{B(x_0,l+\lambda/2)^{\triangleright}}(\hat{\mathbf{v}}) \geq k|A_{\bar{q}'}^{\mathbf{u}_R} \cap B(x_0, l)^{\triangleright}|. \tag{3.11}$$

# 4 Proof of the main theorem

Estimate (3.11) and the estimate in (ii) of Lemma 3.2 imply a bound on the size of the subset of $D_R$ on which $\mathbf{u}_R$ is far from $\mathbf{a}_1$. Indeed, we have:

**Proposition 4.1** *Let $\bar{q}$ be as in Lemma 3.1. Then there exists $l_0 > 0$ independent of $R$ and such that, if $\mathbf{u}_R \in \mathcal{A}^R$ is a minimizer of $J_R$ on that set, we have*

$$if \ x \in D_R \ and \ d(x, \partial D_R) \geq l_0, \ \ then \ q^{\mathbf{u}_R}(x) < \bar{q}. \tag{4.1}$$

*Proof.* The proof is also from [10] but since it is short and basic for the proof of the exponential estimate in Theorem 1.1 we include it here. The map $\mathbf{v}$ constructed in Lemma 3.2 satisfies the assumptions on $\mathbf{u}$ in Lemma 3.3 with $L = l + \frac{\lambda}{2}$. Let $\hat{\mathbf{v}}$ be the map $\mathbf{w}$ given by Lemma 3.3 for $\mathbf{u} = \mathbf{v}$ and $L = l + \frac{\lambda}{2}$. Then Lemma 3.2, Corollary 3.1 and the minimality of $\mathbf{u}_R$ imply

$$0 \geq J_R(\mathbf{u}_R) - J_R(\hat{\mathbf{v}}) = J_R(\mathbf{u}_R) - J_R(\mathbf{v}) + J_R(\mathbf{v}) - J_R(\hat{\mathbf{v}}) \geq -K(\sigma_{l+\lambda} - \sigma_l) + k\sigma_l, \tag{4.2}$$

where $\sigma_L \equiv |A^{\mathbf{u}_R}_{\bar{q}'} \cap B(x, L)|$.

From (4.2) we derive

$$\sigma_{l+\lambda} - \sigma_l \geq \frac{k}{K}\sigma_l$$

and

$$\sigma_{l+\lambda} \geq (1 + \frac{k}{K})\sigma_l. \tag{4.3}$$

Assume $q^{\mathbf{u}_R} \geq \bar{q}$. Then (2.8) implies $q^{\mathbf{u}_R} \geq \bar{q}'$ on $B(x, \bar{l})$, where $\bar{l} \equiv \frac{\bar{q}-\bar{q}'}{M'}$. Therefore we have $\sigma_{\bar{l}} = |B(x, \bar{l})|$ and, if we set

$$\sigma_j = \sigma_{\bar{l}+j\lambda}, \ \ j = 0, 1, \ldots, j_0,$$

where $j_0$ is defined later, then from (4.3) we obtain

$$\sigma_j \geq (1 + \frac{k}{K})^j\sigma_{\bar{l}},$$

and

$$\sigma_{j+1} - \sigma_j \geq \frac{k}{K}\sigma_j \geq \frac{k}{K}(1 + \frac{k}{K})^j\sigma_{\bar{l}}. \tag{4.4}$$

Since $\sigma_{j+1} - \sigma_j \leq 2\pi(2(\bar{l} + j\lambda) + \lambda)$ grows linearly in $j$, there is a minimum value of $j$ such that inequality (4.4) is violated, in contradiction with the minimality of $\mathbf{u}_R$. We let $j_0$ be this minimum value and define $l_0 \equiv \bar{l} + j_0\lambda$. This concludes the proof.

We are now able to prove Theorem 1.1, following that of [10]. Take $l_0$ as above.

Using the Arzela-Ascoli theorem, (2.8) allows us to pass to the limit as $R \to \infty$ along a subsequence in equation (1.1) satisfied by $\mathbf{u}_R$, obtaining a classical solution $\mathbf{u}$ to (1.1) on $\mathbb{R}^2$ and mapping $D_1$ into $S_1$.

Take $R > R_0 \equiv \min\{R > 0 : \exists x \in D_R \text{ such that } B(x_0, l_0) \subset D_R\}$. Then (4.1) of Proposition 4.1 allows us to apply Lemma 3.3 with $\bar{q}' = \bar{q}$, $\mathbf{u} = \mathbf{u}_R$, and for any $x = x_0$ with $d(x_0, \partial D_R) = l_0 + L$ with $L > 0$. This gives

$$|\mathbf{u}_R(x) - \mathbf{a}_1| = q^{\mathbf{u}_R} \leq \phi(0, L). \tag{4.5}$$

Now Lemma 2.4 of [11] gives an estimate for $\phi$:

$$\phi(0, L) \leq \bar{q}e^{-kL} = \bar{q}e^{kl_0}e^{-kd(x,\partial D_R)}, \tag{4.6}$$

for some $k > 0$ independent of $L \in [1, \infty)$. From (2.7), (4.1), and (4.5), we obtain

$$|\mathbf{u}_R(x) - \mathbf{a}_1| \leq Ke^{-kd(x,\partial D_R)} \ \text{ for } \ x \in D_R, \tag{4.7}$$

where $k, K > 0$ are independent of $R$. Passing to the limit as $R \to \infty$ in (4.7) yields (ii) of Theorem 1.1 and completes the proof.

# References


[1] S. Alama, L. Bronsard, and C. Gui, *Stationary layered solutions in* $\mathbb{R}^2$ *for an Allen-Cahn system with multiple well potential*, Calc. Var. Partial Differential Equations **5** No. 4 (1997), 359-390,

[2] N. D. Alikakos *Some basic facts on the system u  Wu(u) = 0*, Proc. Amer. Math. Soc. **139** No. 1 (2011), 153-162.

[3] N. D. Alikakos, *A new proof for the existence of an equivariant entire solution connecting the minima of the potential for the system* $\Delta u W_u(u) = 0$, preprint.

[4] N. D. Alikakos and G. Fusco, *Entire solutions to equivariant elliptic systems with variational structure*, Arch. Rat. Mech. Analysis **202** No. 2 (2011), 567-597.

[5] L. Bronsard, C. Gui, and M. Schatzman, *A three-layered minimizer in R2 for a variational problem with a symmetric three-well potential*, Comm. Pure. Appl. Math. **49** No. 7 (1996), 677-715.

[6] L. Bronsard and F. Reitich, *On three-phase boundary motion and the singular limit of a vector-valued Ginzburg-Landau equation*, Arch. Rat. Mech. Analysis **124** No. 4 (1993), 355-379.

[7] C. M. Elliott and S. Luckhaus, *A Generalized Diffusion Equation for Phase Separation of a Multi-component Mixture with Interfacial Free Energy,* Preprint 195, Sonderforschungsbereich **256**, Bonn, 1991.

[8] D. J. Eyre, *Systems of CahnHilliard equations,* SIAM J. Appl. Math., **53** (1993), 1686-1712.

[9] D. J. Eyre, *Cascades of spinodal decompositions in the ternary Cahn-Hilliard equations*, in Mathematics of Microstructure Evolution, L. Q. Chen, B. Fultz, J. W. Cahn, J. R. Manning, J. E. Morral, and J. A. Simmons, eds., The Minerals, Metals and Materials Society, Warrendale, PA, 1996, 367-378.

[10] G. Fusco, *Equivariant entire solutions to the elliptic system* $\Delta u = W_u(u)$ *for general G-invariant potentials*, preprint, 2012.

[11] G. Fusco, F. Leonetti, and C. Pignotti, *A uniform estimate for positive solutions of semilinear elliptic equations*, Trans. Amer. Math. Soc. **363** No. 8 (2011), 4285– 4307.

[12] C. Gui and M. Schatzman, *Symmetric quadruple phase transitions*, Ind. Univ. Math. J. **57** no.2 (2008), 781–836.

[13] J. J. Hoyt, *Spinodal decomposition in ternary alloys,* Acta Metallurgica, **37** (1989), 24892497.

[14] J. J. Hoyt, *The continuum theory of nucleation in multicomponent systems,* Acta Metallurgica et Materialia, **38** (1990), 1405-1412.

[15] J. J. Hoyt, *Linear spinodal decomposition in a regular ternary alloy,* Acta Metallurgica et Materialia, **38** (1990), 227-231.

[16] J. E. Morral and J. W. Cahn, *Spinodal decomposition in ternary systems,* Acta Metall., **19** (1971), 1037-1045.

[17] A. Novick-Cohen, *Triple-junction motion for an Allen-Cahn/Cahn-Hilliard system*, Phys. D **137** no. 1–2 (2000), 1-24,

[18] R. Palais, *The principle of symmetric criticality*, Commun. Math. Phys. **69** no.1 (1979) 19–30.

[19] M. Saez-Trumper, *Existence of a solution to a vector-valued Allen-Cahn equation with a three well potential*, Indiana Univ. Math. J. **58** no. 1 (2009), 213-267.

[20] P. Sterneberg, *Vector-valued local minimizers of nonconvex variational problems,* Rocky Mount. Math. J. **21** (1991), 799-807.

[21] P. Sternberg and W. Ziemer, *Local minimizers of a three-phase partition problem with triple junction*, Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), 1059-1073.